TAIWANESE JOURNAL OF MATHEMATICS
Vol. 9, No. 1, pp. 67-72, March 2005
This paper is available online at http://www.math.nthu.edu.tw/tjm/

ZEROS OF FINITE WAVELET SUMS

Noli N. Reyes

Abstract. For certain analytic functions ψ, a lower Riesz bound for a finite wavelet system generated by ψ, yields an upper bound for the number of zeros on a bounded interval of the corresponding wavelet sums. In particular, we show that if the Fourier transform of ψ is compactly supported, say on $[-\Omega, \Omega]$, and if $B > 2e\Omega$, then any finite sum $\sum_{|k|\leq \alpha/2} a_k\psi(x-k)$ cannot have more than $B\alpha$ zeros in $[-\alpha, \alpha]$ for $\alpha > 0$ sufficiently large.

1. Introduction and Notation

In this note, we obtain upper bounds on the number of zeros of finite wavelet sums on bounded intervals. More precisely, we show that for a class of analytic functions ψ such that a finite collection of wavelets

$$\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k), \quad (j,k) \in I,$$

is linearly independent, given $\alpha > 0$ sufficiently large, there exists a positive integer $N(\alpha)$ such that any sum $\sum_{(j,k)\in I} a_{j,k}\psi_{j,k}$ will have at most $N(\alpha)$ zeros in $[-\alpha, \alpha]$. In particular, we show that if the Fourier transform of ψ is compactly supported, say on $[-\Omega, \Omega]$, and if $B > 2e\Omega$, then any finite sum

$$\sum_{|k|\leq \alpha/2} a_k\psi(x-k)$$

cannot have more than $B\alpha$ zeros in $[-\alpha, \alpha]$ for $\alpha > 0$ sufficiently large.

Our starting point in obtaining such upper bounds is a lower Riesz bound; i.e., a finite positive number C_0 such that

$$\sum_{(j,k)\in I} |a_{j,k}|^2 \;\leq\; C_0^2 || \sum_{(j,k)\in I} a_{j,k}\psi_{j,k}||_2^2$$

Received April 8, 2003; Revised June 11, 2003.
Communicated by Szi-Bi Hsu.
2000 *Mathematics Subject Classification*: 41A, 42A.
Key words and phrases: Fourier transform, wavelet, Riesz basis, lower Riesz bound, zeros.

for any finite collection $\{a_{j,k} : (j,k) \in I\}$ of complex numbers. Several authors ([1-4]) have investigated the question of linear independence of Gabor and wavelet systems and have also obtained estimates for lower Riesz bounds.

In [3], Christensen and Lindner state without proof that if the support of the Fourier transform of $\psi \in L^2$ is contained in $(-\infty, p]$ where $p > 0$, and there is a non-degenerate interval E contained in $[p/2, p]$ such that $\hat{\psi}(x) \neq 0$, for $x \in E$, then any finite family of wavelets $\psi_{j,k},\ (j,k) \in I$, is linearly independent. This can be proven using an argument similar to that of the Remark in the next section. They also obtain lower Riesz bounds, which is a more delicate question.

We shall define the Fourier transform by

$$\hat{f}(\omega) \ = \ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx$$

for integrable functions f. With this convention, the inversion formula becomes

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\xi) e^{-i\xi x} d\xi,$$

valid under various conditions. For $1 \leq p < \infty$, we adopt the usual notations

$$||f||_p^p \ = \ \int_{-\infty}^{\infty} |f(x)|^p dx,$$

while $||f||_\infty$ denotes the essential supremum of $|f|$. For a function $\psi \in L^2(\mathbf{R})$ and with $\lambda = (j,k) \in \mathbf{Z} \times \mathbf{Z}$, we let

$$\psi_\lambda(x) \ = \ 2^{j/2} \psi(2^j x - k).$$

2. General Estimate for Number of Zeros

In Lemma 1 below, $\psi : \mathbf{R} \longrightarrow \mathbf{C}$ is an infinitely differentiable function in $L^2(\mathbf{R})$ and I denotes a finite subset of $\mathbf{Z} \times \mathbf{Z}$. Suppose that for some constant C_0 (possibly depending on I),

$$\sum_{\lambda \in I} |a_\lambda|^2 \ \leq \ C_0^2 || \sum_{\lambda \in I} a_\lambda \psi_\lambda ||_2^2 \qquad (1)$$

for any finite collection of complex numbers $a_\lambda,\ \lambda \in I$. We let

$$M = \max\{j : (j,k) \in I\} \quad \text{and} \quad m = \min\{j : (j,k) \in I\}. \qquad (2)$$

Lemma 1. *Let $\alpha > 0$ such that*

$$2C_0^2|I| \int_{|x|\geq 2^{m-1}\alpha} |\psi(x)|^2 dx \;\leq\; 1 \tag{3}$$

and $|k|2^{-j} < \alpha/2$ whenever $(j,k) \in I$. If a finite sum $\sum_{\lambda\in I} a_\lambda\psi_\lambda$ has n zeros in $[-\alpha,\alpha]$, then

$$n! \;\leq\; C_1\sqrt{\alpha}(2^{M+1}\alpha)^n ||\psi^{(n)}||_\infty \tag{4}$$

where $C_1 = 4C_0(2^M|I|)^{1/2}$.

Proof of Lemma 1. Let $f = \sum_{\lambda\in I} a_\lambda\psi_\lambda$ have n zeros in $[-\alpha,\alpha]$. If $(j,k)\in I$, then

$$\int_{|x|\geq\alpha} |\psi_{j,k}(x)|^2 dx \;\leq\; \int_{|y|\geq 2^{j-1}\alpha} |\psi(y)|^2 dy$$

since $|k|2^{-j} < \alpha/2$.

Combining the above estimate with (1), (3) and the Cauchy-Schwartz inequality, we obtain

$$||f||_2^2 \leq 2\int_{|x|\leq\alpha} |f(x)|^2 dx. \tag{5}$$

Suppose $x_1,\cdots,x_n$ are zeros of f in $[-\alpha,\alpha]$. Then

$$|f(x)| \;\leq\; \frac{2}{n!}||f^{(n)}||_\infty|(x-x_1)\cdots(x-x_n)| \;\leq\; \frac{2(2\alpha)^n||f^{(n)}||_\infty}{n!} \tag{6}$$

for any real number x with $|x|\leq\alpha$. To see this, we consider the real and imaginary parts of f. Suppose u is the real or imaginary part of f and x is a fixed real number in $[-\alpha,\alpha]$. The function

$$u_x(t) = (x-x_1)\cdots(x-x_n)u(t) - u(x)(t-x_1)\cdots(t-x_n)$$

has $n+1$ zeros in $[-\alpha,\alpha]$. Therefore, there is a point ξ in $[-\alpha,\alpha]$ such that $u_x^{(n)}(\xi) = 0$. This implies (6).

Integrating (6) over the interval $[-\alpha,\alpha]$ leads to

$$\int_{-\alpha}^{\alpha} |f(x)|^2 dx \;\leq\; \frac{4(2\alpha)^{2n+1}||f^{(n)}||_\infty^2}{(n!)^2}.$$

In view of (5), we conclude that

$$||f||_2^2 \;\leq\; \frac{8(2\alpha)^{2n+1}||f^{(n)}||_\infty^2}{(n!)^2} \tag{7}$$

Meanwhile, we differentiate f n times, apply the Cauchy-Schwartz inequality, and use the lower Riesz bound given in (1). From this, we obtain

$$||f^{(n)}||_\infty \leq C_0|I|^{1/2}2^{M(n+1/2)}||f||_2||\psi^{(n)}||_\infty.$$

Combining this with (7) gives the desired inequality (4). ∎

Remark We point out that any finite family $\psi_{j,k},\ (j,k) \in I$, will also be linearly independent if for some $p > 0$, $\hat{\psi}(x) = 0$ for $0 \leq x \leq p$ and there exists a non-degenerate interval E contained in $[p, 2p]$ such that $\hat{\psi}(x) \neq 0$ for $x \in E$. The proof is quite straightforward. Assuming

$$\sum_{j=J_1}^{J_2} \sum_{k=m_j}^{n_j} a_{j,k}\psi_{j,k} \ = \ 0 \quad \text{in} \quad L^2(\mathbf{R}),$$

passing to the fourier transform, we obtain $\sum_{j=J_1}^{J_2} P_j(2^{-j}\xi)\hat{\psi}(2^{-j}\xi) = 0$ almost everywhere, where the P_j's are trigonometric polynomials. However,

$$\sum_{j=J_1+1}^{J_2} P_j(2^{-j}\xi)\hat{\psi}(2^{-j}\xi) \ = \ 0$$

for a.e. $\xi \in [0, 2^{J_1+1}p]$. This implies $P_{J_1}(\omega)\hat{\psi}(\omega) \ = \ 0$ for $0 \leq \omega \leq 2p$. From the hypothesis, we conclude that $P_{J_1}(\omega) \ = \ 0$ for $\omega \in E$. Thus, P_{J_1} must be identically zero. Iterating this argument, we deduce that all of the P_j's must be identically zero. ∎

Concrete Examples

In this section, we shall apply the general estimate of Lemma 1 to two concrete cases. In Theorem 1 below, we obtain a rough upper bound for the number of zeros of finite wavelet sums where the Fourier transform of the "mother" wavelet ψ is exponentially decaying. Theorem 2 focuses on sums of translates of ψ such that $\hat{\psi}$ is compactly supported. Up to a constant factor, the result of Theorem 2 is optimal.

We assume the same conditions as in section 1. Suppose $\psi : \mathbf{R} \longrightarrow \mathbf{C}$ is an infinitely differentiable function in $L^2(\mathbf{R})$ and I denotes a finite subset of $\mathbf{Z} \times \mathbf{Z}$. Moreover, there is a constant C_0 such that

(8) $$\sum_{\lambda \in I} |a_\lambda|^2 \ \leq \ C_0^2 || \sum_{\lambda \in I} a_\lambda \psi_\lambda ||_2^2,$$

for any finite collection of complex numbers $a_\lambda,\ \lambda \in I$. We let

(9) $$M = \max\{j : (j,k) \in I\} \quad \text{and} \quad m = \min\{j : (j,k) \in I\}.$$

Theorem 1. *Suppose that for some constants* $B > 0$ *and* $\beta > 1$,

$$|\hat{\psi}(\xi)| \;\leq\; \exp(-B|\xi|^{\beta})|$$

for all real numbers ξ. *Let* $\alpha > 0$ *such that* (3) *holds and* $|k|2^{-j} < \alpha/2$ *whenever* $(j,k) \in I$. *Then any finite sum* $\sum_{\lambda\in I} a_\lambda\psi_\lambda$ *cannot have more than* N *zeros in* $[-\alpha,\alpha]$ *where*

$$N \;=\; \frac{C_1 A\sqrt{\alpha}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} |\xi|\exp(A|\xi| - B|\xi|^{\beta})d\xi \;,$$

$A = 2^{M+1}\alpha$ *and* C_1 *is given in the statement of Lemma 1.*

Proof of Theorem 1. Fix a finite sum $\sum_{\lambda\in I} a_\lambda\psi_\lambda$ having n zeros in $[-\alpha,\alpha]$. By the inversion formula,

$$||\psi^{(n)}||_\infty \leq \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} |\xi|^n \exp(-B|\xi|^{\beta})d\xi.$$

Combining this with Lemma 1, we obtain

$$n! \;\leq\; \frac{C_1\sqrt{\alpha}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} |A\xi|^n \exp(-B|\xi|^{\beta})d\xi.$$

Applying the estimate $e^u > u^k/k!$ with $k = n-1$, we obtain the desired result. ■

Theorem 2 *Suppose* ψ *and* $x\psi(x)$ *belong to* $L^2(\mathbf{R})$ *and satisfies* (8) *for any finite collection of complex numbers* $a_\lambda,\ \lambda \in I$. *Furthermore, assume that* $\hat{\psi}$ *is compactly supported:*

$$\hat{\psi}(\omega) \;=\; 0 \quad \textit{if} \;\; |\omega| \geq \Omega. \tag{10}$$

If $B > 2e\Omega$, *any finite sum*

$$\sum_{|k|\leq\alpha/2} a_k\psi(x-k)$$

cannot have $\lfloor B\alpha\rfloor$ *zeros in* $[-\alpha,\alpha]$ *for* $\alpha > 0$ *sufficiently large.*

Here, $\lfloor x\rfloor$ denotes the greatest integer less than or equal to x. The exponent 1 of α is clearly optimal as shown by the example $\psi(x) = x^{-k}\sin^k x$.

Proof of Theorem 2. Suppose there exists a sequence $\{\alpha_m\}_{m=1}^{\infty}$ in $[1,\infty)$ tending to infinity such that for each m, there exists a function

$$f_m(x) \;=\; \sum_{|k|\leq\alpha_m/2} a_{m,k}\psi(x-k)$$

with $\lfloor B\alpha_m \rfloor$ zeros in $[-\alpha_m, \alpha_m]$. Since $x\psi(x) \in L^2$, we may assume that

$$4C_0^2\alpha_m \int_{\{|x|\geq \alpha_m/2\}} |\psi(x)|^2 dx \ < \ 1$$

for each m. Therefore, we may apply Lemma 1 with I taken as

$$I_m = \{(0,n) : n \in \mathbf{Z}, |n| \leq \alpha_m/2\}.$$

In this context, $m = M = 0$ and $C_1 \leq 4C_0(2\alpha_m)^{1/2}$.

Therefore, (4) implies that $n! \leq C(2\alpha_m)^{n+1}\Omega^n$ with $n = \lfloor B\alpha_m \rfloor$. Here and in what follows, C denotes a positive constant, possibly different at each occurrence, and depending only on ψ. Since $n! \geq n^n e^{-n} e$,

$$C^{1/\alpha_m} \ \leq \ \alpha_m^{1/\alpha_m} \left(\frac{2e\Omega\alpha_m}{B\alpha_m - 1} \right)^B$$

for each positive integer m. Finally, letting m tend to infinity, we obtain

$$1 \leq \left(\frac{2e\Omega}{B} \right)^B .$$

Therefore $B < 2e\Omega$. ■

ACKNOWLEDGEMENT

The author acknowledges support from the Research and Creative Scholarship Program of the University of the Philippines.

REFERENCES

1. P. Casazza, and O. Christensen, Hilbert space frames containing a Riesz basis and Banach spaces which have no subspace isomorphic to c_0, *J. Math. Anal. Appl.* **202** (1996), 940-950.
2. O. Christensen, Frames, Riesz Bases, and Discrete Gabor/Wavelet expansions, *Bull. Amer. math. Soc.* **38** (2001), 273-291.
3. O. Christensen and A. Lindner, Lower bounds for finite wavelet and Gabor systems, preprint (2001).
4. C. Heil, J. Ramanathan, and P. Topiwala, Linear independence of time-frequency translates, *Proc. Amer. Math. Soc.* **124** (1996), 2787-2795.

Noli N. Reyes
Department of Mathematics,
College of Science,
University of the Philippines,
Quezon City 1101,
Philippines
E-mail: noli@math.upd.edu.ph