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ON THE DIFFERENTIAL EQUATION $u'' - u^p = 0$

Meng-Rong Li

Abstract. In this paper we work with the ordinary differential equation $u''-u^p$ = 0 for some well-defined functions u^p. We obtain some interesting phenomena concerning blow-up, blow-up rate, life-span, stability, instability for solutions.

0. Introduction

In our papers [Li 1, 2] we studied the semi-linear wave equation $\Box u + f(u) = 0$ under some conditions, and we found some interesting results on blow-up, blow-up rate and estimates for the life-span of solutions, but no information on the singular set. So we would like to deal with particular cases in lower dimensional wave equations. We hope that this will help us understand the singular sets of the solutions for the semi-linear wave equations later.

In this work we denote u^p by the well-defined functions. We say p is odd (even, respectively) if $p = r/s,\ r,\ s \in 2\mathbb{N}+1,\ (r,s) = 1$ (common factor) and r is odd $(even,$ respectively$)$. By direct computation one sees that the following initial value problem for the ordinary differential equation

$$\begin{cases} u'' - u^p = 0,\ p \in (0,1), \\ u(0) = 0 = u'(0) \end{cases}$$

has at least two solutions, for instance, $u(t)=0$ and $u(t)=\left((1-p)\,t/\sqrt{2+2p}\right)^{\frac{2}{1-p}}$, so the solutions to the above initial value problem are not unique, in general. These functions u^p, $p \geq 1$ are locally Lipschitz; hence by the standard theory, the local existence and uniqueness of classical solutions of the equation

$$\begin{cases} u'' - u^p = 0,\ p \in (1,\infty), \\ u(0) = u_0,\ u'(0) = u_1, \end{cases} \tag{0.1}$$

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can be obtained.

In section 1 we deal with estimates for the life-span of the solutions to problem (0.1), in section 2 with blow-up rates and blow-up constants, in section 3 with critical points, in section 4 with the zeros and triviality, in section 5 with stability and instability.

Notation and Fundamental Lemmas

For a given function u in this work we use the following abbreviations

$$a_u(t) = u(t)^2, E_u(0) = u_1^2 - \frac{2}{p+1} u_0^{p+1},$$
$$J_u(t) = a_u(t)^{-\frac{p-1}{4}}, H2 := C^2[0,T].$$

Definition. A function $g : \mathbb{R} \to \mathbb{R}$ has a blow-up rate q means that there exist a finite number T^* and a non-zero $\beta \in \mathbb{R}$ such that

(0.2) $$\lim_{t \to T^*} g(t)^{-1} = 0,$$

(0.3) $$\lim_{t \to T^*} (T^* - t)^q g(t) = \beta,$$

in this case β is called the blow-up constant of g.

According to the uniqueness of solutions to the equation (0.1), we rewrite $a_u(t) = a(t), J_u(t) = J(t)$ and $E_u(t) = E(t)$ for convenience. After some elementary calculations we obtain the following lemma 1.

Lemma 1. *Suppose that* $u \in H2$ *is the solution of* (0.1), *then*

(0.4) $$E(t) = u'(t)^2 - \frac{2}{p+1} u(t)^{p+1} = E(0),$$

(0.5) $$(p+3) u'(t)^2 = (p+1) E(0) + a''(t),$$

(0.6) $$J''(t) = \frac{p^2-1}{4} E(0) J(t)^{\frac{p+3}{p-1}}$$

and

(0.7) $$J'(t)^2 = J'(0)^2 - \frac{(p-1)^2}{4} E(0) J(0)^{\frac{2(p+1)}{p-1}} + \frac{(p-1)^2}{4} E(0) J(t)^{\frac{2(p+1)}{p-1}}.$$

The following lemmas are easy to prove, so we omit their arguments.

Lemma 2. *Suppose that k_1 and $k_2$are real constants and $u \in C^2(\mathbb{R})$ satisfies*

$$u'' + k_1 u' + k_2 u \le 0, \quad u \ge 0,$$

$$u(0) = 0, u'(0) = 0,$$

then u must be trivial, that is, $u \equiv 0$.

Lemma 3. *If $g(t)$ and $h(t,r)$ are continuous with respect to their variables and the limit $\lim_{t\to T} \int_0^{g(t)} h(t,r)\,dr$ exists, then*

$$\lim_{t\to T} \int_0^{g(t)} h(t,r)\,dr = \int_0^{g(T)} h(T,r)\,dr.$$

1. Estimates for the Life-spans

To estimate the life-span of the solution to the equation (0.1), we separate this section into three parts, $E(0) < 0,\ E(0) = 0$ and $E(0) > 0$.

Here the life-span T^* of u means that u is the solution of problem (0.1) and u exists only in $[0, T^*)$ so that the problem (0.1) possesses the solution $u \in H2$ for $T < T^*$.

1.1.1 $E(0) \le 0$

In this subsection we study the cases $E(0) < 0$ and $E(0) = 0,\ a'(0) > 0$. The case that $E(0) = 0$ and $a'(0) \le 0$ will be considered in section 3 and section 4. We have the following result.

Theorem 4. *If T^* is the life-span of u and $u \in H2$ is the solution of the problem (0.1) with $E(0) < 0$, then T^* is finite, that is, u is only a local solution of (0.1). Further, for $a'(0) \ge 0$ we have the estimate*

$$T^* \le T_1^*(u_0, u_1, p) = \frac{2}{p-1} \int\limits_0^{J(0)} \frac{dr}{\sqrt{k_1 + E(0)\, r^{k_2}}}. \tag{1.1.1}$$

for $a'(0) < 0$, we have

$$\begin{aligned} T^* &\le T_2^*(u_0, u_1, p) \\ &= \frac{2}{p-1} \left(\int\limits_0^{k} \frac{dr}{\sqrt{k_1 + E(0)\, r^{k_2}}} + \int\limits_{J(0)}^{k} \frac{dr}{\sqrt{k_1 + E(0)\, r^{k_2}}} \right), \end{aligned} \tag{1.1.2}$$

where $k_1 := \frac{2}{p+1}, k_2 := \frac{2p+2}{p-1}$ and $k := \left(\frac{2}{p+1}\frac{-1}{E(0)}\right)^{\frac{p-1}{2p+2}}$.

Furthermore, if $E(0) = 0$ *and* $a'(0) > 0$, *then*

$$T^* \leq T_3^*(u_0, u_1, p) := \frac{4}{p-1}\frac{a(0)}{a'(0)}. \tag{1.1.3}$$

Proof. Under the condition, $E(0) < 0$, we know $a(0) > 0$; otherwise $u_0 = 0$, and then $E(0) = u_1^2 \geq 0$, yet this is contrary to $E(0) < 0$. The proof is divided into two cases, $a'(0) \geq 0$ and $a'(0) < 0$.

(i) $a'(0) \geq 0$. By (0.5), we find

$$\begin{cases} a'(t) \geq a'(0) - (p+1)E(0)t \quad \forall t \geq 0, \\ a(t) \geq a(0) + a'(0)t - \frac{p+1}{2}E(0)t^2 \quad \forall t \geq 0. \end{cases} \tag{1.1.4}$$

From (0.7), $a'(0) \geq 0$ and $J'(t) = -\frac{p-1}{4}a(t)^{-\frac{p+3}{4}}a'(t) < 0$, it follows that

$$J'(t) = -\frac{p-1}{2}\sqrt{k_1 + E(0)J(t)^{k_2}} \leq J'(0) \quad \forall t \geq 0 \tag{1.1.5}$$

and

$$J(t) \leq a(0)^{-\frac{p-1}{4}} - \frac{p-1}{4}a(0)^{-\frac{p+3}{4}}a'(0)t \quad \forall t \geq 0,$$

where $k_1 = \frac{1}{4}a(0)^{-\frac{p+3}{2}}a'(0)^2 - E(0)a(0)^{-\frac{p+1}{2}} = \frac{2}{p+1}$.

Thus, there exists a finite number $T_1^*(u_0, u_1, p) \leq \frac{4}{p-1}\frac{a(0)}{a'(0)}$ such that

$$J(T_1^*(u_0, u_1, p)) = 0$$

and

$$a(t) \to \infty \text{ for } t \to T_1^*(u_0, u_1, p).$$

This means that the life-span T^* of u is finite and $T^* \leq T_1^*(u_0, u_1, p)$.

Now we estimate this life-span $T_1^*(u_0, u_1, p)$. By $(1.1.5)$ and $J(T_1^*(u_0, u_1, p)) = 0$ we find

$$\int_{J(t)}^{J(0)} \frac{dr}{\sqrt{k_1 + E(0)r^{k_2}}} = \frac{p-1}{2}t \quad \forall t \geq 0 \tag{1.1.6}$$

and hence we obtain the estimate $(1.1.1)$.

(ii) $a'(0) < 0$. Using $(1.1.4)$ and $a'(0) < 0$ we find a unique finite number $t_0(u_0, u_1, p)$ such that

$$\text{(1.1.7)} \qquad \begin{cases} a'(t) < 0 \quad \text{for} \quad t \in (0, t_0(u_0, u_1, p)), \\ a'(t_0(u_0, u_1, p)) = 0, \\ a'(t) > 0 \quad \text{for} \quad t > t_0(u_0, u_1, p), \end{cases}$$

and $a(t_0(u_0, u_1, p)) > 0$. If not, then $u(t_0(u_0, u_1, p)) = 0$ and

$$E(0) = E(t_0(u_0, u_1, p)) = u'(t_0(u_0, u_1, p))^2 \geq 0;$$

yet this is in contradiction with $E(0) < 0$.
In this way it is easy to see that $a(t) > 0 \quad \forall t \geq 0$. Hence we get $u'(t_0(u_0, u_1, p)) = 0$ and

$$E(0) = -\frac{2}{p+1} u(t_0(u_0, u_1, p))^{p+1},$$
$$J(t_0(u_0, u_1, p))^{k_2} = \frac{2}{p+1} \frac{-1}{E(0)}.$$

After arguments similar to those in step (i), there exists a $T_2^*(u_0, u_1, p)$ such that the life-span T^* of u is bounded by $T_2^*(u_0, u_1, p)$, that is, $T^* \leq T_2^*(u_0, u_1, p)$. Analogously, by $(1.1.7)$ and (0.7) we obtain

$$\text{(1.1.8.1)} \qquad J'(t) = -\frac{p-1}{2} \sqrt{k_1 + E(0) J(t)^{k_2}} \quad \forall t \geq t_0(u_0, u_1, p),$$

$$\text{(1.1.8.2)} \qquad J'(t) = \frac{p-1}{2} \sqrt{k_1 + E(0) J(t)^{k_2}} \quad \forall t \in [0, t_0(u_0, u_1, p)]$$

and

$$\text{(1.1.9.1)} \qquad \int_{J(t)}^{J(t_0(u_0,u_1,p))} \frac{dr}{\sqrt{k_1 + E(0) r^{k_2}}} = \frac{p-1}{2} (t - t_0(u_0, u_1, p)) \quad \forall t \geq t_0(u_0, u_1, p),$$

$$\text{(1.1.9.2)} \qquad \int_{J(0)}^{J(t_0(u_0,u_1,p))} \frac{dr}{\sqrt{k_1 + E(0) r^{k_2}}} = \frac{p-1}{2} t_0(u_0, u_1, p).$$

Using $(1.1.9)$ and $J\left(t_0\left(u_0,u_1,p\right)\right)^{k_2}=\dfrac{2}{p+1}\dfrac{-1}{E\left(0\right)}$, $J\left(T_2^*\left(u_0,u_1,p\right)\right)=0$, it results

$$T_2^*\left(u_0,u_1,p\right)=t_0\left(u_0,u_1,p\right)+\frac{2}{p-1}\int_0^k\frac{dr}{\sqrt{k_1+E\left(0\right)r^{k_2}}}. \tag{1.1.10}$$

This estimate $(1.1.10)$ is equivalent to $(1.1.2)$.

(iii) $E\left(0\right)=0$. Now we prove $(1.1.3)$. By (0.6) and $E\left(0\right)=0$ we get $J''\left(t\right)=0\quad\forall t\geq 0$. From the positiveness of $a'\left(0\right)$, it follows that $J'\left(0\right)<0$ and

$$J\left(t\right)=a\left(0\right)^{-\frac{p-1}{4}}-\frac{p-1}{4}a\left(0\right)^{-\frac{p+3}{4}}a'\left(0\right)t\quad\forall t\geq 0$$

and also

$$a\left(t\right)=a\left(0\right)^{\frac{p+3}{p-1}}\left(a\left(0\right)-\frac{p-1}{4}a'\left(0\right)t\right)^{-\frac{4}{p-1}}\qquad\forall t\geq 0. \tag{1.1.11}$$

Therewith we conclude the estimate (1.1.3).

1.1.2. Properties of $T_1^*\left(u_0,u_1,p\right)$

In principle, $T_1^*\left(u_0,u_1,p\right)$ depends on three variables u_0,u_1 and p. Set $c_{k,p}:=\frac{(p+1)u_1^2}{2u_0^{p+1}}$, then

$$T_1^*\left(u_0,u_1,p\right)=\frac{\sqrt{2p+2}}{p-1}u_0^{-\frac{p-1}{2}}\left(1-c_{k,p}\right)^{-\frac{p-1}{2p+2}}\int_0^{\left(1-c_{k,p}\right)^{\frac{p-1}{2p+2}}}\frac{dr}{\sqrt{1-r^{\frac{2p+2}{p-1}}}}$$

and

$$\lim_{p\to\infty}T_1^*\left(u_0,u_1,p\right)=0,\ \lim_{p\to\infty}T_1^*\left(u_0,u_1,p\right)=\infty.$$

For convenience, we consider the case $u_1=0$,

$$T_1^*\left(u_0,0,p\right)=\frac{\sqrt{\pi}}{\sqrt{2p+2}}u_0^{-\dfrac{p-1}{2}}\frac{\Gamma\left(\dfrac{p-1}{2p+2}\right)}{\Gamma\left(\dfrac{p}{p+1}\right)}.$$

Using Maple we obtain the graphs of $T_1^*\left(u_0,0,p\right)$ below:

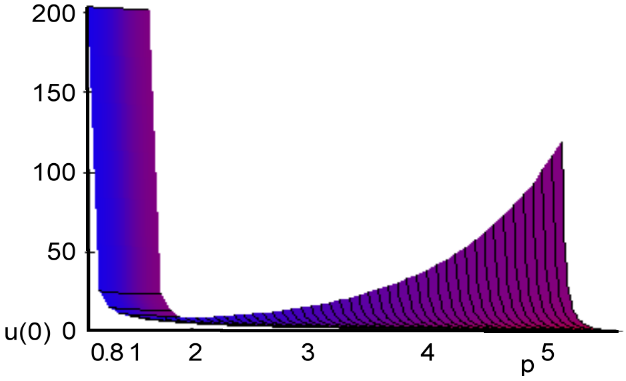

Fig. 1. Graph of $T_1^* (u_0, 0, p)$

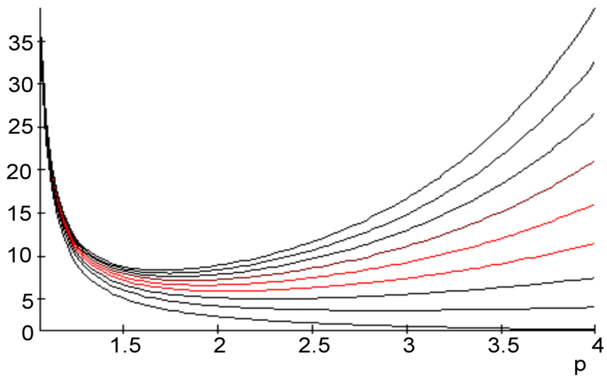

Fig. 2. Graphs of $T_1^* (u_0, 0, p)$, $u_0 \leq 1$

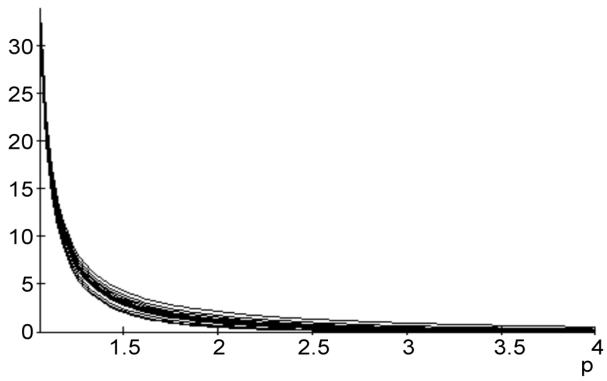

Fig. 3. Graphs of $T_1^* (u_0, 0, p)$, $u_0 > 1$

The above pictures show the properties of $T_1^*(u_0, 0, p)$:

(1) there exists a constant u_0^* such that $T_1^*(u_0, 0, p)$ is monotone decreasing in p for $u_0 \in [u_0^*, 1)$;

(2) there is a p_0 such that $T_1^*(u_0, 0, p)$ is decreasing in $(1, p_0)$ and increasing in (p_0, ∞) provided $u_0 \in [0, u_0^*)$;

(3) $T_1^*(u_0, 0, p)$ is differentiable in its variables and

(4) for $u_0 > 1$ the life-span $T_1^*(u_0, 0, p)$ is decreasing in p.

We now show the validity of statements (3) and (4) using the monotonicity of $T_1^*(1, 0, p)$ for $u_0 \neq 0$. To prove (1) and (2) we must show the existence of u_0^* with $\frac{\partial}{\partial p} T_1^*(u_0, 0, p) \leq 0$ for $1 > u_0 \geq u_0^*$, that is,

$$0 \leq \frac{p-1}{p+1}(p+3)\int_0^1 \left(1-r^{2\frac{p+1}{p-1}}\right)^{-1/2} dr + 4\int_0^1 \left(1-r^{2\frac{p+1}{p-1}}\right)^{-3/2} r^{2\frac{p+1}{p-1}} \ln r \, dr$$

$$+ (p-1)^2 (\ln u_0) \int_0^1 \left(1 - r^{2\frac{p+1}{p-1}}\right)^{-1/2} dr,$$

thus the existence of u_0^* can be obtained provided

$$\frac{p-1}{p+1}(p+3)\left(r^{2\frac{p+1}{p-1}} - 1\right) - 4\ln r > 0 \;\; \forall r > 1.$$

After some calculations it is easy to get the above assertion.

To grasp the property of the life-span $T_1^*(u_0, u_1, p)$ is very difficult, but for fixed initial data we want to know how the life-span varies with p, so now we consider the life-span $T_1^*(0.6, 0.2, p)$ and list the following tables as below.

p	$T_1^*(0.6, 0.2, p)$	p	$T_1^*(0.6, 0.2, p)$
1.001	2001. 5	2	3. 4135
1.004	501. 42	2.5	2. 7698
1.008	251. 42	3	2. 4659
1.012	168. 08	3.6497	2. 2644

After some computations we get

$$T_1^*(u_0, u_1, p)$$

$$= \frac{\sqrt{2p+2}}{p-1}\left(u_0^{p+1} - \frac{p+1}{2}u_1^2\right)^{-\frac{p-1}{2p+2}} \int_0^{\left(1-\frac{p+1}{2u_0^{p+1}}u_1^2\right)^{\frac{p-1}{2p+2}}} \frac{dr}{\sqrt{1-r^{\frac{2p+2}{p-1}}}}.$$

By the experience in studying the life span $T_1^*(u_0, 0, p)$, we consider the properties of the life-span $T_1^*(u_0, u_1, p)$ with $u_0 u_1 \geq 0$ in three cases:

Case 1. $0 < u_0^{p+1} - (p+1) u_1^2/2 < 1$. In this situation we find that

(i) for fixed u_1,

(5) there exists a constant u_0^* depending on u_1 such that $T_1^*(u_0, u_1, p)$ is monotone decreasing in p for $u_0 \geq u_0^*$,

(6) there is a p_0 so that $T_1^*(u_0, u_1, p)$ decreases in $(1, p_0)$ and increases in (p_0, ∞) provided $u_0 \in [0, u_0^*)$;

(ii) for fixed u_0, the life-span $T_1^*(u_0, u_1, p)$ decreases in u_1^2.

Case 2. $u_0^{p+1} - (p+1) u_1^2/2 > 1$. The life-span $T_1^*(u_0, u_1, p)$ decreases in p.

Case 3. $u_0^{p+1} - (p+1) u_1^2/2 = 1$. On the surface

$$\left\{(u_0, u_1, p) \in \mathbb{R}^3 \,\middle|\, u_0^{p+1} - (p+1) u_1^2/2 = 1,\ p > 1\right\}$$

we find that

$$T_1^*(u_0, u_1, p) = T_1^*(u_0, p) = \frac{\sqrt{2p+2}}{p-1} \int_0^{u_0^{-(p-1)/2}} \frac{1}{\sqrt{1 - r^{2(p+1)/(p-1)}}} dr$$

and $T_1^*(u_0, p)$ is monotone decreasing in u_0 and in p.

1.2. $E(0) > 0, a'(0)^2 \geq 4a(0) E(0)$

In this subsection we consider two cases

- $E(0) > 0,\ a'(0)^2 > 4a(0) E(0)$

and

- $E(0) > 0,\ a'(0)^2 = 4a(0) E(0),\ u_1 > 0.$

The case that $E(0) > 0,\ a'(0)^2 < 4a(0) E(0)$ will be considered in section 3 and section 4. The case that $E(0) > 0,\ a'(0)^2 = 4a(0) E(0),\ u_1 < 0$ will be postponed to section 3. For $E(0) > 0$ and $a'(0)^2 \geq 4a(0) E(0)$ we have the following blow-up result.

Theorem 5. *If T^* is the life-span of u and $u \in H2$ is the solution of the problem (0.1) with $E(0) > 0$, then T^* is finite, that is, u is only a local solution of (0.1), if one of the following is valid*

(i) $a'(0)^2 > 4a(0)E(0)$

or

(ii) $a'(0)^2 = 4a(0)E(0)$ *and* $u_1 > 0$

or

(iii) $a'(0)^2 = 4a(0)E(0)$, $u_1 < 0$ *and* p *is odd.*

Further, in the case of (i) *we have the estimate*

$$T^* \le T_4^*(u_0, u_1, p) = \frac{2}{p-1} \int_0^{J(0)} \frac{dr}{\sqrt{k_1 + E(0)\, r^{k_2}}} \tag{1.2.1}$$

and also

$$a'(0) \ge 0. \tag{1.2.2}$$

In the case of (ii) *we have*

$$T^* \le T_5^*(u_0, u_1, p) = \frac{2}{p-1} \int_0^{\infty} \frac{dr}{\sqrt{k_1 + E(0)\, r^{k_2}}}. \tag{1.2.1.1}$$

In the case of (iii) *we have*

$$T^* \le T_6^*(u_0, u_1, p) = \frac{2}{p-1} \int_0^{\infty} \frac{dr}{\sqrt{k_1 + E(0)\, r^{k_2}}}. \tag{1.2.1.2}$$

Proof. (i) $a'(0)^2 > 4a(0)E(0)$. By (0.6) we find

$$\begin{cases} k_3(u_0, u_1, p)\, J''(t) = (k_3(u_0, u_1, p)\, J(t))^q, \\ k_3(u_0, u_1, p)\, J(0) = k_3(u_0, u_1, p)\, a(0)^{-\frac{p-1}{4}}, \\ k_3(u_0, u_1, p)\, J'(0) = \dfrac{1-p}{4} k_3(u_0, u_1, p)\, a(0)^{-\frac{p+3}{4}}\, a'(0), \end{cases} \tag{1.2.3}$$

where $k_3(u_0, u_1, p) := \left(\dfrac{p^2-1}{4} E(0)\right)^{\frac{p-1}{4}}$ and $q := \dfrac{p+3}{p-1}$.

Now we set

$$\tilde{E}(t) := k_3(u_0, u_1, p)^2\, J'(t)^2 - \frac{2}{q+1} (k_3(u_0, u_1, p)\, J(t))^{q+1},$$

after some calculations we see that $\tilde{E}(t)$ is a constant and

$$\tilde{E}(t) = \tilde{E}(0) \tag{1.2.4}$$

$$= \left(\frac{p-1}{4}\right)^2 k_3(u_0, u_1, p)^2 a(0)^{-\frac{p+3}{2}} \left(a'(0)^2 - 4E(0)a(0)\right).$$

From the condition (i) and (0.4) follows

$$0 < \tilde{E}(t) = \frac{(p-1)^2}{2(p+1)} k_3(u_0, u_1, p)^2 a(t)^{-\frac{p+3}{2}} u(t)^{p+3}$$

$$= \frac{(p-1)^2}{2(p+1)} k_3(u_0, u_1, p)^2,$$

thus

$$u(t)^{p+1} > 0 \quad \forall t \geq 0. \tag{1.2.5}$$

By (0.5) we find

$$a'(t) = a'(0) + 2E(0)t + 2\frac{p+3}{p+1}\int_0^t u(r)^{p+1} dr \quad \forall t \geq 0 \tag{1.2.6}$$

and then

$$a'(t) \geq a'(0) + 2E(0)t \quad \forall t \geq 0. \tag{1.2.7}$$

Thus, for $a'(0) \geq 0$, using the same arguments as in the proof of theorem 4 we get the conclusions $(1.2.1)$.

Now let us show $(1.2.2)$. For $a'(0) < 0$, from $(1.2.7)$ it follows that $a'(t) \geq 0$ for large t. Suppose that $\bar{t}_0$ is the first number such that $a'(t) = 0$. Using (0.5) we get for $t \geq \bar{t}_0$

$$a'(t) = 2E(0)(t - \bar{t}_0) + 2\frac{p+3}{p+1}\int_{\bar{t}_0}^t u(r)^{p+1} dr \geq 0. \tag{1.2.6.1}$$

Hence,

$$\begin{cases} a'(t) < 0 \quad for \quad t \in (0, \bar{t}_0), \\ a'(\bar{t}_0) = 0, \\ a'(t) > 0 \quad for \quad t > \bar{t}_0, \end{cases} \tag{1.2.8}$$

and $a(\bar{t}_0) > 0$; if not, then $u(\bar{t}_0) = 0$; this is contrary to $(1.2.5)$. Thus,

$$u'(\bar{t}_0) = 0. \tag{1.2.9}$$

Therefore, by $(1.2.5)$,

$$(p+1)\,E(0) = -2u(\bar{t}_0)^{p+1} < 0. \tag{1.2.10}$$

The identity $(1.2.10)$ and the condition $E(0) > 0$ contradict each other; thus the existence of $\bar{t}_0$ is false, therefore $(1.2.2)$ is obtained.

(ii) By condition (ii) and $(1.2.6)$ we find

$$a'(t) = 2E(0)\,t + 2\frac{p+3}{p+1}\int_0^t u(r)^{p+1}\,dr \quad \forall t \geq 0. \tag{1.2.11}$$

We claim that $a'(t) > 0$ for every $t > 0$. If not, then there exists $\tilde{t} > 0$ such that $a'(\tilde{t}) = 0$. Let $\tilde{T}$ be the first non-zero value so that $a'\left(\tilde{T}\right) = 0$, then $u(t) > 0$ in $\left(0, \tilde{T}\right)$. By $(1.2.6)$ we get

$$0 = a'\left(\tilde{T}\right) = 2E(0)\,\tilde{T} + 2\frac{p+3}{p+1}\int_0^{\tilde{T}} u(r)^{p+1}\,dr.$$

This is therefore in contradiction with $E(0) > 0$; hence $a'(t) > 0$ $\forall t > 0$ and $J'(t) < 0$ $\forall t > 0$. Using (0.6) for each $t \geq \check{t} > 0$ we obtain

(1.2.12)

$$J'(t) = -\sqrt{J'(\check{t})^2 - \frac{(p-1)^2}{4}E(0)\,J(\check{t})^{\frac{2p+2}{p-1}} + \frac{(p-1)^2}{4}E(0)\,J(t)^{\frac{2p+2}{p-1}}}$$

and

$$\lim_{\check{t}\to 0} J'(\check{t})^2 - \frac{(p-1)^2}{4}u_1^2 J(\check{t})^{\frac{2p+2}{p-1}} = \frac{(p-1)^2}{2(p+1)},$$

thus from $(1.2.12)$, the estimate $(1.2.1.1)$ follows.

(iii) To see $(1.2.1.2)$, we use the fact that $u_0 = 0$ and $a'(0) = 2u_0u_1 = 0$ and (1.2.6), we can also get the identity (1.2.11), thus (1.2.1.2) follows.

2. Blow-up Rate and Blow-up Constant

In this section we study the blow-up rate and blow-up constant for a, a' and a'' under the conditions in section 1. We have the following results.

Theorem 6. *If $u \in H2$ is the solution of the problem (0.1) with one of the following properties that*

(i) $E(0) < 0$

(ii) $E(0) = 0, a'(0) > 0$

or

(iii) $E(0) > 0, a'(0)^2 > 4a(0)E(0)$

or

(iv) $E(0) > 0, a'(0)^2 = 4a(0)E(0), u_1 > 0$

or

(v) $E(0) > 0, a'(0)^2 = 4a(0)E(0), u_1 < 0$ *and p is odd.*

Then the blow-up rate of a is $4/(p-1)$, and the blow-up constant of a is $\sqrt[p-1]{4(p-1)^{-4}(p+1)^2}$, that is, for $m \in \{1,2,3,4,5,6\}$

$$\lim_{t \to T_m^*(u_0,u_1,p)^-} (T_m^*(u_0,u_1,p) - t)^{\frac{4}{p-1}} a(t) = 2^{\frac{2}{p-1}} (p+1)^{\frac{2}{p-1}} (p-1)^{-\frac{4}{p-1}}. \tag{2.1.1}$$

The blow-up rate of a' is $(p+3)/(p-1)$, and the blow-up constant of a' is $2^{\frac{2p}{p-1}} (p+1)^{\frac{2}{p-1}} (p-1)^{-\frac{p+3}{p-1}}$, that is, for $m \in \{1,2,3,4,5,6\}$

$$\lim_{t \to T_m^*(u_0,u_1,p)^-} (T_m^*(u_0,u_1,p) - t)^{\frac{p+3}{p-1}} a'(t) = 2^{\frac{2p}{p-1}} (p+1)^{\frac{2}{p-1}} (p-1)^{-\frac{p+3}{p-1}}. \tag{2.1.2}$$

The blow-up rate of a'' is $(2p+2)/(p-1)$, and the blow-up constant of a'' is $2^{\frac{2p}{p-1}} (p+1)^{\frac{2}{p-1}} (p-1)^{-\frac{2p+2}{p-1}} (p+3)$, that is, for $m \in \{1,2,3,4,5,6\}$

$$\lim_{t \to T_m^*(u_0,u_1,p)^-} a''(t) (T_m^*(u_0,u_1,p) - t)^{\frac{2p+2}{p-1}} = 2^{\frac{2p}{p-1}} (p+3)(p+1)^{\frac{2}{p-1}} (p-1)^{-\frac{2p+2}{p-1}}. \tag{2.1.3}$$

Proof. (i) Under this condition, $E(0) < 0, a'(0) \geq 0$ by $(1.1.1)$ and $(1.1.6)$ we get

$$\int_0^{J(t)} \frac{1}{T_1^*(u_0,u_1,p) - t} \frac{dr}{\sqrt{k_1 + E(0) r^{k_2}}} = \frac{p-1}{2} \quad \forall t \geq 0. \tag{2.1.4}$$

By lemma 4 and $(2.1.4)$ we obtain

(2.1.5) $$\lim_{t\to T_1^*(u_0,u_1,p)^-}\frac{1}{\sqrt{k_1}}\frac{J\left(t\right)}{T_1^*\left(u_0,u_1,p\right)-t}=\frac{p-1}{2}.$$

This identity $(2.1.5)$ is equivalent to $(2.1.1)$ for $m=1$.

For $E\left(0\right)<0, a'\left(0\right)<0$, using $(1.1.9)$ we have also

(2.1.6) $$\int_0^{J(t)}\frac{dr}{\sqrt{k_1+E\left(0\right)r^{k_2}}}=\frac{p-1}{2}\left(T_2^*\left(u_0,u_1,p\right)-t\right)\quad \forall t\geq t_0.$$

From lemma 4 and $(2.1.6)$, the estimate $(2.1.1)$ for $m=2$ follows.

By $(1.1.5)$ and $(1.1.8)$, for $m=1,2$, we find

(2.1.7) $$\lim_{t\to T_m^*(u_0,u_1,p)^-}J'\left(t\right)=-\frac{p-1}{\sqrt{2p+2}}$$

and

(2.1.8) $$\lim_{t\to T_m^*(u_0,u_1,p)^-}a'\left(t\right)\left(T_m^*\left(u_0,u_1,p\right)-t\right)^{\frac{p+3}{p-1}}$$
$$=2^{\frac{2p}{p-1}}\left(p+1\right)^{\frac{2}{p-1}}\left(p-1\right)^{-\frac{p+3}{p-1}}$$

and thus,

(2.1.9) $$\lim_{t\to T_m^*(u_0,u_1,p)^-}u'\left(t\right)^2\left(T_m^*\left(u_0,u_1,p\right)-t\right)^{\frac{2p+2}{p-1}}$$
$$=2^{\frac{2p}{p-1}}\left(p+1\right)^{\frac{2}{p-1}}\left(p-1\right)^{-\frac{2p+2}{p-1}},\qquad m=1,2.$$

Through (0.5) and $(2.1.9)$ we obtain

(2.1.10) $$\lim_{t\to T_m^*(u_0,u_1,p)^-}a''\left(t\right)\left(T_m^*\left(u_0,u_1,p\right)-t\right)^{\frac{2p+2}{p-1}}$$
$$=\left(p+3\right)\lim_{t\to T_m^*(u_0,u_1,p)^-}u'\left(t\right)^2\left(T_m^*\left(u_0,u_1,p\right)-t\right)^{\frac{2p+2}{p-1}},\qquad m=1,2.$$

This estimates $(2.1.10)$ and $(2.1.3)$ are equivalent for $m=1,2$.

(ii) For $E\left(0\right)=0, a'\left(0\right)>0$, for $m=3$, using $(1.1.11)$ we get

$$(2.1.11) \quad a(t) = a(0)^{\frac{p+3}{p-1}} \left(\frac{p-1}{4} a'(0) \right)^{-\frac{4}{p-1}} \left(T_m^* (u_0, u_1, p) - t \right)^{-\frac{4}{p-1}} \ \forall t \geq 0.$$

Therefore $(2.1.1), (2.1.2)$ and $(2.1.3)$ for $m = 3$ follow.

(iii) The proof of estimates $(2.1.1), (2.1.2)$ and $(2.1.3)$ for $m = 4, 5, 6$ are similar to the above arguments (i) in the proof of this theorem.

Now we consider the property of the blow-up constants K_1, K_2 and K_3. We have

$$K_1(p) = 2^{\frac{2}{p-1}} (p+1)^{\frac{2}{p-1}} (p-1)^{-\frac{4}{p-1}},$$

$$K_2(p) = 2^{\frac{2p}{p-1}} (p+1)^{\frac{2}{p-1}} (p-1)^{-\frac{p+3}{p-1}},$$

$$K_3(p) = 2^{\frac{2p}{p-1}} (p+3)(p+1)^{\frac{2}{p-1}} (p-1)^{-\frac{2p+2}{p-1}}.$$

Using Maple we have the graphs of K_1, K_2 and K_3 below.

We see that the graphs, $K_i(p), i = 1, 2, 3$ are all decreasing in p, and $K_i(p)$ tends to zero, as p tends to infinity. The monotonicity of these functions can be obtained after showing the following inequalities

$$\frac{p-1}{p+1} - 2 \leq \ln(2p+2) - 2\ln(p-1) \ \forall p > 1,$$

$$\frac{2p-2}{p+1} + 4\ln(p-1) \leq 2\ln 2 + 2\ln(p+1) + p + 3 \ \forall p > 1,$$

$$\frac{(p-1)^2}{p+3} + \frac{2p-2}{p+1} + 4\ln(p-1) \leq 2(\ln 2) + 2\ln(p+1) + 2p + 2 \ \forall p > 1.$$

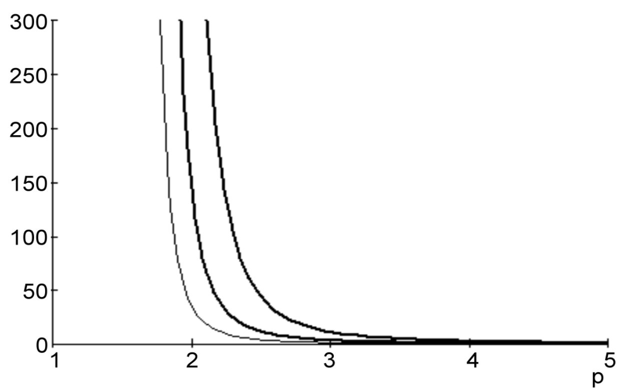

Fig. 4. graphs of $K_1(p)$ in thin, $K_2(p)$ in medium, $K_3(p)$in thick.

These inequalities are easy to check, so we omit the arguments.

3. Critical Pointt

In this section we study the following cases:

(i) $E(0) = 0, a'(0) < 0.$

(ii) $E(0) > 0, a'(0)^2 < 4a(0)E(0).$

(iii) $E(0) > 0, a'(0)^2 = 4a(0)E(0), u_1 < 0$ and p is even.

Under the condition (i) it is easily to see that

$$a(t) = a(0)^{\frac{p+3}{p-1}}\left(a(0) - \frac{p-1}{4}a'(0)t\right)^{-\frac{4}{p-1}} \quad \forall t \in (0,T).$$

Hence we find the limit $\lim_{t\to\infty} a(t) = 0$ and

$$\lim_{t\to\infty} t^{\frac{4}{p-1}} a(t) = a(0)^{\frac{p+3}{p-1}}\left(\frac{p-1}{-4}a'(0)\right)^{-\frac{4}{p-1}}.$$

For the convenience we consider the critical points and critical values of a and u of the solution (0.1), and we will prove the existence of critical points in section IV.

3.1 Estimate of the Critical Points

Under (ii) and (iii) we have the critical result.

Theorem 7. *Suppose that $u \in H2$ is the solution of the problem (0.1) with the property (ii) or (iii) and $T_0(u_0, u_1, p)$ is the critical point of u, that is,*

$$u'(T_0(u_0, u_1, p)) = 0,$$

then $u_0 \le 0$ and $T_0(u_0, u_1, p)$ is given by

$$T_0(u_0, u_1, p) = \int_{-u_0}^{-u(T_0(u_0,u_1,p))} \frac{dr}{\sqrt{E(0) - 2r^{p+1}/(p+1)}}, \tag{3.1}$$

where $-u(T_0(u_0, u_1, p)) = ((p+1)E(0)/2)^{1/(p+1)}$. Further under condition (ii) u_0 must be negative and p must be even.

Proof. By some computations one can find the non-positiveness of u_0 under (ii) or (iii); and in case of (ii), then p must be even. For $E(0) > 0$ and $a'(0)^2 < 4a$

$(0)\,E\,(0)$, using $(1.2.4)$ we find

$$\begin{aligned} \tilde{E}\,(t) &= k_3\,(u_0, u_1, p)^2\, J'\,(t)^2 - \frac{p-1}{p+1}\,(k_3\,(u_0, u_1, p)\, J\,(t))^{\frac{2p+2}{p-1}} \\ &= \tilde{E}\,(0) \qquad\qquad (3.2)\\ &= \left(\frac{p-1}{4}\right)^2 k_3\,(u_0, u_1, p)^2\, a\,(t)^{-\frac{p+3}{2}} \left(a'\,(t)^2 - 4E\,(0)\, a\,(t)\right) \ \forall t \in [0, T)\,, \end{aligned}$$

for all t in $[0, T)$, where $k_3\,(u_0, u_1, p) = \left(\frac{p^2-1}{4} E\,(0)\right)^{\frac{p-1}{4}}$.

By (3.2) and condition (ii), $E\,(0) > 0, a'\,(0)^2 < 4a\,(0)\,E\,(0)$, it is easy to see that $u_0 \neq 0$ and therefore we find that

$$0 > \tilde{E}\,(t) = \tilde{E}\,(0) = \frac{(p-1)^2}{2p+2} k_3\,(u_0, u_1, p)^2 \left| u_0^{-p-3} \right| u_0^{p+3};$$

thus $u_0^{p+3} < 0$, therefore we obtain that $u_0 < 0$, p is even and

$$\tilde{E}\,(t) = \tilde{E}\,(0) = -\frac{(p-1)^2}{2p+2} k_3\,(u_0, u_1, p)^2\,. \tag{3.3}$$

Since $u\,(t) < 0$ in a neighborhood of $t = 0$, $\tilde{E}\,(t)$ can be defined at $t = 0$, so that $\tilde{E}\,(t)$ is continuous in $[0, \varepsilon)$ for some $\varepsilon > 0$.

Under the condition (ii) or (iii), by the definition of $T_0\,(u_0, u_1, p)$ and (0.4) we get the critical value of u at $T_0\,(u_0, u_1, p)$, $u\,(T_0\,(u_0, u_1, p)) = -\left(\frac{p+1}{2} E\,(0)\right)^{\frac{1}{p+1}}$.

Using the continuity and negativity of u' in $[0, T_0\,(u_0, u_1, p)]$ we find

$$u'\,(t) = -\sqrt{E\,(0) + \frac{2}{p+1} u\,(t)^{p+1}} \qquad \forall t \in [0, T_0\,(u_0, u_1, p)]\,. \tag{3.4}$$

From (3.4), the identity (3.1) follows.

3.2 Some Properties Concerning $T_0\,(0, u_1, p)$

Because of some difficulties in the graphing of $T_0\,(u_0, u_1, p)$, we consider the property of $T_0\,(u_0, u_1, p)$ only for the case that $u_0 = 0 > u_1$ and p is even. After some computations one can easily find that

$$T_0\,(0, u_1, p) = (-u_1)^{\frac{1-p}{1+p}} \left(\frac{p+1}{2}\right)^{\frac{1}{1+p}} \frac{\sqrt{\pi}}{1+p} \frac{\Gamma\left(\frac{1}{1+p}\right)}{\Gamma\left(\frac{p+3}{2p+2}\right)}.$$

By Maple we get the graph of $T_0(0, u_1, p)$ below

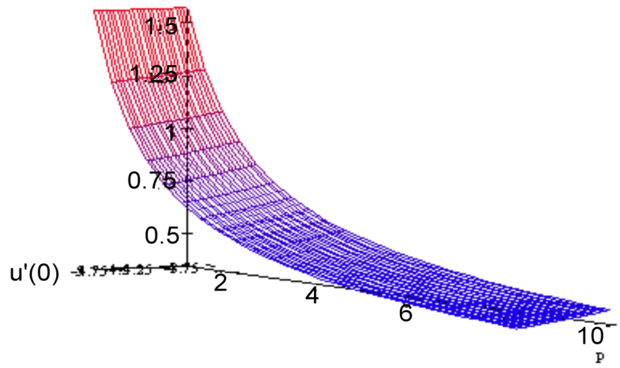

Fig. 5. The graph of $T_0(0, u_1, p), u_1 \in [-5, -4]$

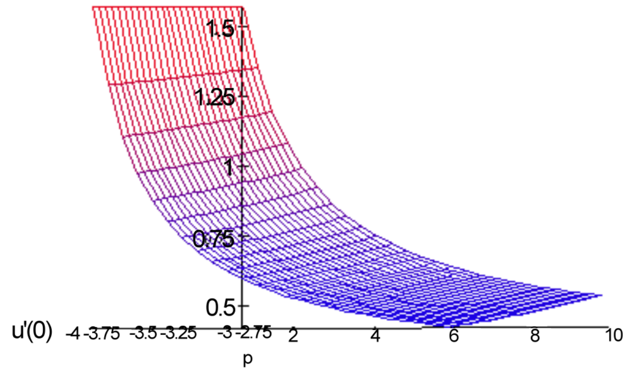

Fig. 6. The graph of $T_0(0, u_1, p), u_1 \in [-4, -3]$

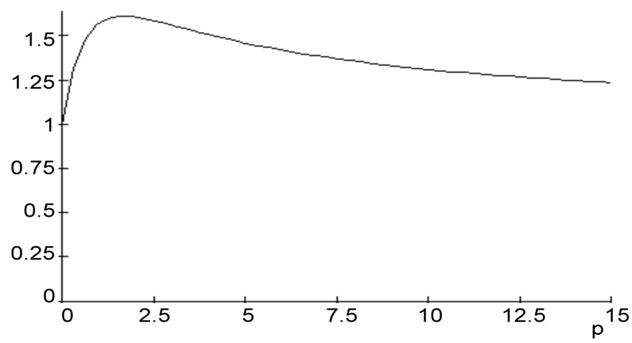

Fig. 7. Graph of $T_0(0, -1, p)$

From the pictures we see:

(i) for each fixed u_1, there exists a constant c depends on u_1 with $T_0\left(0, u_1, p\right) \geq c\left(u_1\right)$ for each $p > 1$ and

(ii) there exists a constant p_0 so that $T_0\left(0, -1, p\right)$ decreases in $p \geq p_0$ and increases in $p \in \left(1, p_0\right)$.

4. Existence of Zero and Critical Point

To the cases that $E\left(0\right) > 0, a'\left(0\right)^2 < 4a\left(0\right)E\left(0\right)$ and that $E\left(0\right) > 0 = u_0 > u_1$, we have the result.

Theorem 8. *Suppose that* u *is the solution of the problem* (0.1) *with* $E\left(0\right) > 0$*and one of the following holds*

(i) $a'\left(0\right)^2 < 4a\left(0\right)E\left(0\right)$

(ii) $a'\left(0\right)^2 = 4a\left(0\right)E\left(0\right)$ *and* $u_1 < 0$.

Then u *possesses a critical point* $T_0\left(u_0, u_1, p\right)$ *given by* (3.1), *provided condition* (ii) *holds or condition* (i) *together with* $a'\left(0\right) > 0$ *holds ; under* (i), *there exists* $z < \infty$ *such that*

$$a\left(z\right) = 0. \tag{4.1}$$

For $a'\left(0\right) \leq 0$ *and* $z_1\left(u_0, u_1, p\right)$ *given by*

$$z_1\left(u_0, u_1, p\right) = \frac{\sqrt{p^2 - 1}}{\sqrt{2}} \int_0^{\sqrt{\frac{4a(0)}{(p^2-1)E(0)}}} \frac{dr}{\sqrt{2 - \left(p - 1\right)k_3^2 r^{p+1}}}, \tag{4.2}$$

is the zero of a. *Further we have*

$$T \leq T_7^*\left(u_0, u_1, p\right) := \left(z_1 + T_5^*\right)\left(u_0, u_1, p\right). \tag{4.3}$$

where $k_3\left(u_0, u_1, p\right) = \left(\frac{p^2-1}{4}E\left(0\right)\right)^{\frac{p-1}{4}}$.

Furthermore we have

$$\lim_{t \to z_1(u_0, u_1, p)} a\left(t\right)\left(z_1\left(u_0, u_1, p\right) - t\right)^{-2} = E\left(0\right), \tag{4.4}$$

$$\lim_{t \to z_1(u_0, u_1, p)} \left(z_1\left(u_0, u_1, p\right) - t\right)^{-1} a'\left(t\right) = -2E\left(0\right), \tag{4.5}$$

(4.6) $$\lim_{t\to z_1(u_0,u_1,p)} a''(t) = 2E(0),$$

and a *blows up at* $T_7^*(u_0,u_1,p)$, *that is,* $\lim_{t\to T_7^*(u_0,u_1,p)} 1/a(t) = 0$.
For $a'(0) > 0$ *the zero* $z_2(u_0,u_1,p)$ *of* a *is given by*

(4.7) $$z_2(u_0,u_1,p) = \frac{\sqrt{p^2-1}}{\sqrt{2}}\left(\int_0^\beta \frac{dr}{\sqrt{2-(p-1)k_3^2 r^{p+1}}} + \int_\alpha^\beta \frac{dr}{\sqrt{2-(p-1)k_3^2 r^{p+1}}}\right),$$

where $\alpha = 2\sqrt{\frac{a(0)}{(p^2-1)E(0)}}$, $\beta = \sqrt[p+1]{\frac{2}{(p-1)k_3^2}}$ *and*

(4.8) $$T \le T_8^*(u_0,u_1,p) := (z_2 + T_6^*)(u_0,u_1,p).$$

Furthermore we have

(4.9) $$\lim_{t\to z_2(u_0,u_1,p)} a(t)(z_2(u_0,u_1,p)-t)^{-2} = E(0),$$

(4.10) $$\lim_{t\to z_2(u_0,u_1,p)} (z_2(u_0,u_1,p)-t)^{-1} a'(t) = -2E(0),$$

(4.11) $$\lim_{t\to z_2(u_0,u_1,p)} a''(t) = 2E(0),$$

and a *blows up at* $T_8^*(u_0,u_1,p)$, *that is,* $\lim_{t\to T_8^*(u_0,u_1,p)} 1/a(t) = 0$.
Further, under the condition (ii) *we have also that* $z_3(u_0,u_1,p)$*given by*

(4.12) $$z_3(u_0,u_1,p) = 2t_0(u_0,u_1,p),$$

is a zero of a *and*

(4.13) $$T \le T_9^*(u_0,u_1,p) = (z_3 + T_5^*)(u_0,u_1,p)$$

and a *blows up at* $T_9^*(u_0,u_1,p)$. *Furthermore we have*

(4.14) $$\lim_{t\to z_2(u_0,u_1,p)} a(t)(z_2(u_0,u_1,p)-t)^{-2} = E(0),$$

(4.15) $$\lim_{t\to z_2(u_0,u_1,p)} (z_2(u_0,u_1,p)-t)^{-1} a'(t) = -2E(0),$$

(4.16)
$$\lim_{t \to z_2(u_0,u_1,p)} a''(t) = 2E(0),$$

Proof. We prove this theorem in four steps. First we show the estimates (4.1), (4.2) and (4.7); secondly $(4.4), (4.5), (4.6)$; (4.9), (4.10) and (4.11); thirdly (4.3) and (4.8), at last $(4.12) - (4.16)$.

Step 1. At first we prove the existence of z zero of a. By theorem 7 we find that $J(t)$ is defined in interval $(0, T)$. Let us set

$$\tilde{J}(t) := k_3(u_0, u_1, p) J(t), \ q := \frac{p+3}{p-1},$$
$$A(t) := \tilde{J}(t)^2, \ I(t) := A(t)^{-\frac{q-1}{4}}.$$

Then $I(t) = k_3(u_0, u_1, p)^{-\frac{2}{p-1}} a(t)^{\frac{1}{2}}$, by (0.6) we have

(4.17)
$$\tilde{J}''(t) = \tilde{J}(t)^q \quad \text{in} \quad (0, T).$$

By (3.5), $\tilde{E}(t)$ can be defined at $t = 0$; under the conditions $E(0) > 0$ and $4a(0)E(0) > a'(0)^2$,

(4.18)
$$\tilde{E}(t) = \tilde{E}(0) = \tilde{J}'(t)^2 - \frac{2}{q+1}\tilde{J}(t)^{q+1} = -\frac{(p-1)^2}{2(p+1)} k_3(u_0, u_1, p)^2.$$

Employing theorem 4 we obtain the existence of z, a zero of a, therefore (4.1) is proved.

For $A'(0) = 2k_3(u_0, u_1, p)^2 J(0) J'(0) \geq 0$, that is, $a'(0) \leq 0$, by theorem 4 we get $\tilde{J}(z_1(u_0, u_1, p)) = 0$, where

(4.19)
$$\begin{aligned} z_1(u_0, u_1, p) &= \frac{2}{q-1} \int_0^{I(0)} \frac{dr}{\sqrt{\frac{2}{q+1} + \tilde{E}(0) r^{\frac{2q+2}{q-1}}}} \\ &= \frac{p-1}{2} \int_0^{\sqrt{\frac{4a(0)}{(p^2-1)E(0)}}} \frac{dr}{\sqrt{\frac{p-1}{p+1} - \frac{(p-1)^2}{2(p+1)} k_3(u_0, u_1, p)^2 r^{p+1}}}. \end{aligned}$$

The estimates (4.19) and (4.2) are equivalent.

For $A'(0) < 0$, that is, $a'(0) > 0$ by $(1.1.2)$ we get $\tilde{J}(z_2(u_0, u_1, p)) = 0$, where

$$z_2(u_0,u_1,p) \tag{4.20}$$
$$= \frac{\sqrt{p^2-1}}{2}\left(\int_0^{\sqrt[p+1]{\frac{2}{(p-1)k_3^2}}} + \int_{I(0)}^{\sqrt[p+1]{\frac{2}{(p-1)k_3^2}}}\right)\frac{dr}{\sqrt{1-\frac{p-1}{2}k_3^2 r^{p+1}}},$$

where $k_3 = k_3(u_0,u_1,p)$. The estimates (4.20) and (4.7) are equivalent.

Step 2. To (4.4) and (4.9) for $m = 1,2$, by $(2.1.1)$ we get

$$\begin{aligned} &\lim_{t\to z_m(u_0,u_1,p)}(z_m(u_0,u_1,p)-t)^{\frac{4}{q-1}}A(t) \\ &= 2^{\frac{2}{q-1}}(q+1)^{\frac{2}{q-1}}(q-1)^{-\frac{4}{q-1}}. \end{aligned} \tag{4.21}$$

Using (4.21) we obtain that

$$\begin{aligned} &\lim_{t\to z_m(u_0,u_1,p)}(z_m(u_0,u_1,p)-t)^{p-1}k_3^2 J(t)^2 \\ &= 2^{\frac{p-1}{2}}\left(\frac{2p+2}{p-1}\right)^{\frac{p-1}{2}}\left(\frac{4}{p-1}\right)^{1-p} \quad \text{for } m=1,2. \end{aligned} \tag{4.22}$$

The estimates (4.22) and (4.4), (4.9) are equivalent for $m = 1,2$ respectively.

To (4.5) and (4.10) for $m = 1,2$, applying $(2.1.2)$ we find

$$\begin{aligned} &\lim_{t\to z_m(u_0,u_1,p)}(z_m(u_0,u_1,p)-t)^{\frac{q+3}{q-1}}A'(t) \\ &= 2^{\frac{2q}{q-1}}(q+1)^{\frac{2}{q-1}}(q-1)^{-\frac{q+3}{q-1}}. \end{aligned} \tag{4.23}$$

From (4.23) it follows

$$\begin{aligned} &\lim_{t\to z_m(u_0,u_1,p)}\tfrac{p-1}{4a(t)}k_3(u_0,u_1,p)^2(z_m(u_0,u_1,p)-t)^p J(t)^2 a'(t) \\ &= 2^{\frac{p+1}{2}}\left(\tfrac{2p+2}{p-1}\right)^{\frac{p-1}{2}}\left(\tfrac{4}{p-1}\right)^{-p} \quad \text{for } m=1,2. \end{aligned} \tag{4.24}$$

Together with (4.22) and (4.24) we obtain that

(4.25) $$\lim_{t\to z_m(u_0,u_1,p)} (z_m(u_0,u_1,p) - t)\, a(t)^{-1} a'(t) = -2, \quad m = 1, 2.$$

Together (4.25) , (4.4) and (4.9) , imply $(4.2.5)$ and $(4.2.10)$.

The estimates (4.6) and (4.11) follow from (0.5) and the fact that

$$\lim_{t\to z_m^-(u_0,u_1,p)} u'(t)^2 = E(0), \; m = 1, 2.$$

Step 3. Suppose that $T > T_7^*(u_0,u_1,p)$, then by the fact that $u(z_1(u_0,u_1,p)) = 0$ and $u \in H2$ we find that

$$a'(z_1(u_0,u_1,p))^2 = 4a(z_1(u_0,u_1,p))\, u'(z_1(u_0,u_1,p))^2 = 0.$$

Using $u'(z_1(u_0,u_1,p)) > 0$ and theorem 5, u must blow up in a finite time since $E(z_1(u_0,u_1,p)) = E(0) > 0$ at $T_7^*(u_0,u_1,p)$.

For the case that $a'(0) > 0$, the arguments for the assertion that $T \le T_8^*(u_0,u_1,p)$ are similar to the above and the existence of critical point of u is obtained by the mean value theorem for ordinary C^1-function under the condition (i) .

Step 4. Under the condition (ii) we claim that there exists no strictly monotone negative solution, that is, if u is the solution of (0.1) , then u' posses a zero; if not, according to the negativeness of u, u' in the neighborhood of zero and (0.4) , one can see that

$$-u(t) \le \left(\frac{p+1}{2}u_1^2\right)^{1/(p+1)}$$

and

$$u'(t) = -\sqrt{u_1^2 + \frac{2}{p+1}u(t)^{p+1}} \qquad \forall t \ge 0,$$

therefore we find

$$\begin{aligned} t &= \int_0^{-u(t)} \frac{dr}{\sqrt{u_1^2 - \frac{2}{p+1}r^{p+1}}} \\ &\le (-u_1)^{\frac{1-p}{1+p}} \left(\frac{p+1}{2}\right)^{\frac{1}{1+p}} \int_0^1 \frac{dr}{\sqrt{1-s^{p+1}}} \\ &< 2^{-\frac{p+2}{p+1}} (-u_1)^{\frac{1-p}{1+p}} (p+1)^{\frac{1}{p+1}} \pi, \end{aligned}$$

but this is impossible for large t; thus u must posses a critical point at $t = T_0(u_0, u_1, p)$.

Now we show the existence of zero of u. Suppose that $u(t) < 0$ for each $t > 0$, then by the increasing nature of u' we know that

$$u'(t) = \sqrt{u_1^2 + \frac{2}{p+1} u(t)^{p+1}} \qquad \forall t \geq T_0(u_0, u_1, p)$$

and

$$\int_{-u(t)}^{-u(T_0(u_0,u_1,p))} \frac{dr}{\sqrt{u_1^2 - \frac{2}{p+1} r^{p+1}}} = t - T_0(u_0, u_1, p), \tag{4.26}$$

by a similar argument to the above, we get also a contradiction; therefore we get the existence of zero of u. Using (4.26) one can easily obtain the assertions (4.12) – (4.16).

Property Concerning Zero $z_1(u_0, u_1, p)$

Since the analysis concerning the zeros is very complex, we merely discuss $z_1(u_0, u_1, p)$ and $u_1 = 0$, and by (4.2) we have

$$z_1(u_0, 0, p) = (-u_0)^{-\frac{p-1}{2}} \sqrt{\frac{\pi}{2p+2}} \frac{\Gamma\left(\dfrac{1}{p+1}\right)}{\Gamma\left(\dfrac{p+3}{2p+2}\right)}.$$

Using Maple we get the graphs of $z_1(u_0, 0, p)$

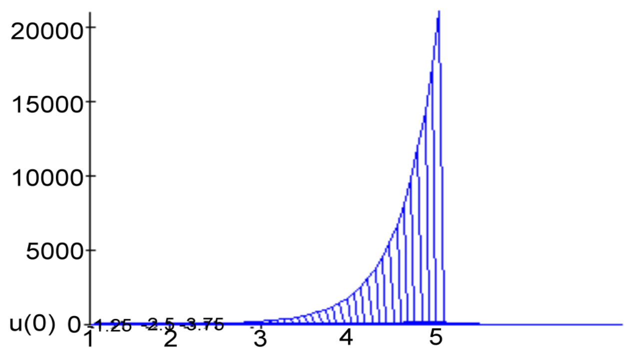

Fig. 8. Graph of $z_1(u_0, 0, p), u_0 < 0$.

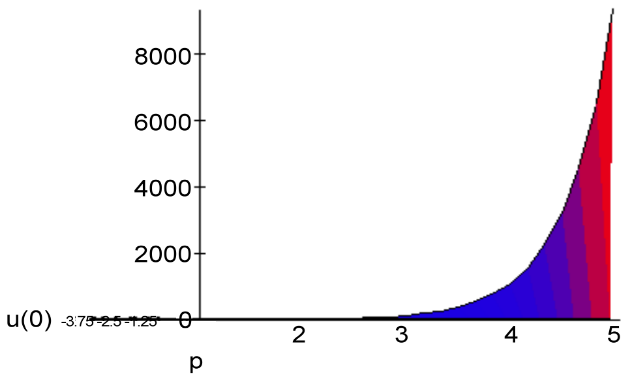

Fig. 9. Graph of $z_1\left(u_{0,}0, p\right), u_0 > 0$.

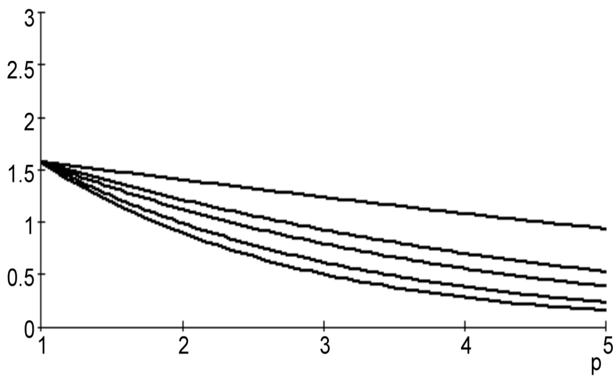

Fig. 10. Graph of z_1 for some $-u_0 > 1$.

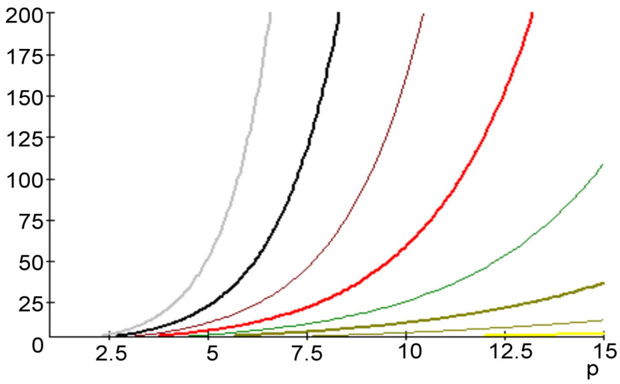

Fig. 11. Graph of z_1 for some $-u_0 \leq 1$.

From the above pictures one can easily find that:

- $z_1(u_0, 0, p)$ decreases in $-u_0$ for fixed p.
- for fixed u_0 with $-u_0 > 1$, $z_1(u_0, 0, p)$ decreases in p.
- for fixed u_0 with $-u_0 \leq 1$, $z_1(u_0, 0, p)$ increases in p.

5. Stability and Instability

We now consider the applications of the theorems above to the stability theory for the problem

$$(*) \qquad \begin{cases} u''(t) = u(t)^p, \\ u(0) = \varepsilon_1, u'(0) = \varepsilon_2. \end{cases}$$

We say the problem $(*)$ is stable under condition F, if any nontrivial global solution $u \in C^2(\mathbb{R}^+)$ of $(*)$ under the condition F satisfies

$$\|u\|_{C^2} \to 0 \quad \text{for} \quad |\varepsilon_1| + |\varepsilon_2| \to 0.$$

According to the theorems 4-9 we have the following result.

Cor 11. *The problem $(*)$ is stable under $E_u(0) = 0$, $\varepsilon_1\varepsilon_2 < 0$ and unstable under the following one of the followings*

(i) $\qquad E_u(0) < 0,$

(ii) $\qquad E_u(0) = 0 < \varepsilon_1\varepsilon_2,$

(iii) $\qquad E_u(0) > 0,\ \varepsilon_2^2 + \dfrac{1}{p+1}\varepsilon_1^{p+1} > 0,$

(iv) $\qquad E_u(0) > 0,\ \varepsilon_1 = 0, \varepsilon_2 > 0,$

(v) $\qquad E_u(0) > 0 > \varepsilon_2$ *and* p *is odd.*

Theorems 4 through 9 may be summarized in the following tables

$Life-span\ of\ a := T^*,\ T_i^* := T_i^*(u_0, u_1, p), i = 1, 2, \cdots 6,$
$z_j(u_0, u_1, p) := z_j,\ j = 1, 2;\ t_0 := t_0(u_0, u_1, p),$
$Energy = E(0),\ \hat{E}(0) := a'(0)^2 - 4a(0)E(0),$
$Blow-up\ rate$ for $a := \alpha_1,\ Blow-up\ constant\ for\ a := K_1,$
$Blow-up\ rate$ for $a' := \alpha_2,\ Blow-up\ constant\ for\ a' := K_2,$
$Blow-up\ rate$ for $a'' := \alpha_3,\ Blow-up\ constant\ for\ a'' := K_3.$

$E(0)$	$E(0) < 0$	$E(0) = 0$
T^*	(i) $a'(0) \geq 0, T^* \leq T_1^*$ (ii) $a'(0) < 0, T^* \leq T_2^*$	(i) $a'(0) > 0, T^* \leq T_3^*$. (ii) $a'(0) < 0, T^* = \infty$. (iii) $a'(0) = 0, T^* = \infty, u \equiv 0$.
α_1, K_1	$\frac{4}{p-1},\ K1(p)$	$\frac{4}{p-1},\ K1(p)$
α_2, K_2	$\frac{p+3}{p-1},\ K2(p)$	$\frac{p+3}{p-1},\ K2(p)$
α_3, K_3	$\frac{2p+2}{p-1},\ K3(p)$	$\frac{2p+2}{p-1},\ K3(p)$

$E(0) > 0$	$\hat{E}(0) < 0,\ a'(0) \leq 0$	$\hat{E}(0) < 0,\ a'(0) > 0$	$\hat{E}(0) = 0,\ u_1 < 0$, p is even
T^*	$T^* \leq z_1 + T_5^*$	$T^* \leq z_2 + T_6^*$	$T^* \leq 2t_0 + T_5^*$
Zero	$z = z_1$	$z = z_2$	$z = 2t_0$

$E(0) > 0, \hat{E}(0)$	$\hat{E}(0) > 0$	$\hat{E}(0) = 0,\ u_1 > 0$	$\hat{E}(0) = 0,\ u_1 < 0$, p is odd
T^*	$T^* \leq T_4^*$	$T^* \leq T_5^*$	$T^* \leq T_6^*$
α_1, K_1	$\frac{4}{p-1}, K1(p)$	$\frac{4}{p-1}, K1(p)$	$\frac{4}{p-1}, K1(p)$
α_2, K_2	$\frac{p+3}{p-1}, K2(p)$	$\frac{p+3}{p-1}, K2(p)$	$\frac{p+3}{p-1}, K2(p)$
α_3, K_3	$\frac{2p+2}{p-1}, K3(p)$	$\frac{2p+2}{p-1}, K3(p)$	$\frac{2p+2}{p-1}, K3(p)$

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Departments of Mathematical Sciences,
Chengchi University,
Taipei 107,
Taiwan, R.O.C.