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PERTURBATION BOUNDS FOR SUBSPACES ASSOCIATED WITH PERIODIC EIGENPROBLEMS

Ji-guang Sun

Abstract. The concept of periodic deflating subspaces of regular periodic matrix pairs $\{(A_j, B_j)\}_{j=1}^K$ is a generalization of deflating subspaces of a regular matrix pair (A, B). In this paper we derive perturbation bounds for each individual subspace of simple periodic deflating subspaces.

1. Introduction

Consider the multivariate eigenproblem

$$\begin{pmatrix} \alpha_1 B_1 & 0 & \cdots & 0 & -\beta_1 A_1 \\ -\beta_2 A_2 & \alpha_2 B_2 & & & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -\beta_K A_K & \alpha_K B_K \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_K \end{pmatrix} \tag{1.1}$$

$$\equiv C \begin{pmatrix} \alpha_1, \cdots, \alpha_K \\ \beta_1, \cdots, \beta_K \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_K \end{pmatrix} = 0,$$

where A_j and B_j are complex $n \times n$ matrices, α_j and β_j are complex variable, and x_j are n-dimensional nonzero vectors for $j = 1, \ldots, K$. In this paper we assume that the periodic matrix pairs $\{(A_j, B_j)\}_{j=1}^K$ are regular; that is,

$$\det \left[C \begin{pmatrix} \alpha_1, \cdots, \alpha_K \\ \beta_1, \cdots, \beta_K \end{pmatrix} \right] \not\equiv 0 \quad \text{for } \alpha_j \text{ and } \beta_j, \ 1 \le j \le K.$$

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The eigenproblem (1.1) with the period $K > 1$ arises in the area of periodic control systems, and there are some contributions in the literature on the theory, applications, and numerical solution of the eigenproblem (see, e.g., [1-4,6]). The object of this paper is to derive perturbation bounds for certain subspaces associated with the eigenproblem (1.1).

Throughout this paper we shall use the following notation. $\mathcal{C}^{m\times n}$ denotes the set of complex $m \times n$ matrices, and $\mathcal{C}^n = \mathcal{C}^{n\times 1}$. $\emptyset$ is the empty set. A^T stands for the transpose of a matrix, and A^H for the conjugate transpose of A. I is the identity matrix, I_n is the $n \times n$ identity matrix, and 0 is the null matrix. $\mathcal{R}(A)$ denotes the column space of a matrix A. The symbol $\| \ \|$ stands for any unitarily invariant norm, $\| \ \|_F$ is the Frobenius norm, and $\| \ \|_2$ is the Euclidean norm for vectors and the spectral norm for matrices. For $A = (a_1, \dots, a_n) = (\alpha_{ij}) \in \mathcal{C}^{m\times n}$ and a matrix B, $A \otimes B = (\alpha_{ij}B)$ is a Kronecker product, and $\text{vec}(A)$ is the vector defined by $\text{vec}(A) = (a_1^T, \dots, a_n^T)^T$.

We first cite some definitions related to regular periodic matrix pairs.

Definition 1.1 [3, 4]. Let $\{(A_j, B_j)\}_{j=1}^K$ be regular periodic matrix pairs. If the complex numbers $\alpha_1, \dots, \alpha_K$ and $\beta_1, \dots, \beta_K$ satisfy

$$\det \left[C \begin{pmatrix} \alpha_1, \cdots, \alpha_K \\ \beta_1, \cdots, \beta_K \end{pmatrix} \right] = 0$$

and

$$(\pi_\alpha, \pi_\beta) \equiv \left(\prod_{j=1}^K \alpha_j, \prod_{j=1}^K \beta_j \right) \neq (0, 0),$$

then (π_α, π_β) is called an eigenvalue of $\{(A_j, B_j)\}_{j=1}^K$.

From Definition 1.1 we see that any eigenvalue (π_α, π_β) of $\{(A_j, B_j)\}_{j=1}^K$ lies on the complex projective plane, or equivalently, any eigenvalue (π_α, π_β) lies on the Riemann sphere; that is, $(\pi_\alpha, \pi_\beta) \neq (0, 0)$ and $(\tau\pi_\alpha, \tau\pi_\beta)$ for any nonzero complex number τ represent the same eigenvalue. If an eigenvalue (π_α, π_β) satisfies $\pi_\beta \neq 0$, then π_α/π_β is a finite eigenvalue; otherwise, (π_α, π_β) is the infinite eigenvalue.

It is known [3,4] that there are exactly n eigenvalues (counting multiplicity) for $\{(A_j, B_j)\}_{j=1}^K$. The set of all eigenvalues of $\{(A_j, B_j)\}_{j=1}^K$ is denoted by $\lambda\left(\{(A_j, B_j)\}_{j=1}^K\right)$.

Let

$$(\pi_\alpha, \pi_\beta) \in \lambda\left(\{(A_j, B_j)\}_{j=1}^K\right).$$

If (π_α, π_β) is different from the others in $\lambda\left(\{(A_j, B_j)\}_{j=1}^K\right)$, then (π_α, π_β) is said to be a simple eigenvalue of $\{(A_j, B_j)\}_{j=1}^K$.

Definition 1.2 [3, 4]. Let $\{(A_j, B_j)\}_{j=1}^K$ be regular periodic $n \times n$ matrix pairs, and let $(\pi_\alpha, \pi_\beta) \equiv \left(\prod_{j=1}^K \alpha_j, \prod_{j=1}^K \beta_j\right)$ be an eigenvalue of $\{(A_j, B_j)\}_{j=1}^K$. If the nonzero vectors $x_1, \ldots, x_K \in \mathcal{C}^n$ satisfy

$$\beta_j A_j x_{j-1} = \alpha_j B_j x_j$$

for $j = 1, \ldots, K$, where $x_0 = x_K$, then $\{x_j\}_{j=1}^K$ are called periodic right eigenvectors of $\{(A_j, B_j)\}_{j=1}^K$ belonging to the eigenvalue (π_α, π_β).

Definition 1.3 [3, 4]. Let $\mathcal{X}_j$ and $\mathcal{Y}_j$ $(j = 1, \ldots, K)$ be subspaces of $\mathcal{C}^n$ with the same dimension. The pairs of subspaces $\{(\mathcal{X}_j, \mathcal{Y}_j)\}_{j=1}^K$ are called periodic deflating subspaces of $\{(A_j, B_j)\}_{j=1}^K$ if

$$A_j \mathcal{X}_{j-1} \subset \mathcal{Y}_j, \quad B_j \mathcal{X}_j \subset \mathcal{Y}_j$$

for $j = 1, \ldots, K$, where $\mathcal{X}_0 = \mathcal{X}_K$. Furthermore, the subspaces $\{\mathcal{X}_j\}_{j=1}^K$ are called periodic invariant subspaces (or periodic eigenspaces) of $\{(A_j, B_j)\}_{j=1}^K$.

Let $Z_1^{(j)}, Q_1^{(j)} \in \mathcal{C}^{n\times r}$ satisfy ${Z_1^{(j)}}^H Z_1^{(j)} = {Q_1^{(j)}}^H Q_1^{(j)} = I$, and let

$$\mathcal{X}_j = \mathcal{R}(Z_1^{(j)}), \quad \mathcal{Y}_j = \mathcal{R}(Q_1^{(j)}), \quad j = 1, \ldots, K.$$

It has been proved by [4] that the periodic subspaces $\{(\mathcal{X}_j, \mathcal{Y}_j)\}_{j=1}^K$ are periodic deflating subspaces of $\{(A_j, B_j)\}_{j=1}^K$ if and only if there are $n \times n$ unitary matrices $Z_j = \left(Z_1^{(j)}, Z_2^{(j)}\right)$ and $Q_j = \left(Q_1^{(j)}, Q_2^{(j)}\right)$ with $Z_0 = Z_K$ such that

$$Q_j^H A_j Z_{j-1} = \begin{pmatrix} A_{11}^{(j)} & A_{12}^{(j)} \\ 0 & A_{22}^{(j)} \end{pmatrix}, \quad Q_j^H B_j Z_j = \begin{pmatrix} B_{11}^{(j)} & B_{12}^{(j)} \\ 0 & B_{22}^{(j)} \end{pmatrix}, \tag{1.2}$$

for $j = 1, \ldots, K$, where $A_{11}^{(j)}, B_{11}^{(j)} \in \mathcal{C}^{r\times r}$, and the matrix pairs

$$\left\{\left(A_{11}^{(j)}, B_{11}^{(j)}\right)\right\}_{j=1}^K \quad \text{and} \quad \left\{\left(A_{22}^{(j)}, B_{22}^{(j)}\right)\right\}_{j=1}^K$$

are regular.

If

$$\lambda\left(\left\{\left(A_{11}^{(j)}, B_{11}^{(j)}\right)\right\}_{j=1}^K\right) \bigcap \lambda\left(\left\{\left(A_{22}^{(j)}, B_{22}^{(j)}\right)\right\}_{j=1}^K\right) = \emptyset, \tag{1.3}$$

then the periodic deflating subspaces $\{(\mathcal{X}_j, \mathcal{Y}_j)\}_{j=1}^K$ are called simple periodic deflating subspaces.

Let the relations (1.2) and (1.3) be satisfied. We now define the matrices T_{11}, T_{12}, T_{21}, T_{22} and T by

$$T_{11} = \begin{pmatrix} 0 & \cdots & \cdots & I_r \otimes A_{22}^{(1)} \\ I_r \otimes A_{22}^{(2)} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & I_r \otimes A_{22}^{(K)} & 0 \end{pmatrix}, \tag{1.4}$$

$$T_{12} = \begin{pmatrix} -A_{11}^{(1)^T} \otimes I_{n-r} & \cdots & \cdots & 0 \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & -A_{11}^{(K)^T} \otimes I_{n-r} \end{pmatrix}, \tag{1.5}$$

$$T_{21} = \begin{pmatrix} I_r \otimes B_{22}^{(1)} & \cdots & \cdots & 0 \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & I_r \otimes B_{22}^{(K)} \end{pmatrix}, \tag{1.6}$$

$$T_{22} = \begin{pmatrix} -B_{11}^{(1)^T} \otimes I_{n-r} & \cdots & \cdots & 0 \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & -B_{11}^{(K)^T} \otimes I_{n-r} \end{pmatrix}, \tag{1.7}$$

and

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}. \tag{1.8}$$

It is known [3] that the matrix T is nonsingular. Write

$$T^{-1} = \begin{pmatrix} C_{11}^{(1)} & \cdots & C_{1K}^{(1)} & C_{11}^{(2)} & \cdots & C_{1K}^{(2)} \\ \vdots & & \vdots & \vdots & & \vdots \\ C_{K1}^{(1)} & \cdots & C_{KK}^{(1)} & C_{K1}^{(2)} & \cdots & C_{KK}^{(2)} \\ D_{11}^{(1)} & \cdots & D_{1K}^{(1)} & D_{11}^{(2)} & \cdots & D_{1K}^{(2)} \\ \vdots & & \vdots & \vdots & & \vdots \\ D_{K1}^{(1)} & \cdots & D_{KK}^{(1)} & D_{K1}^{(2)} & \cdots & D_{KK}^{(2)} \end{pmatrix} = \begin{pmatrix} C_1 \\ \vdots \\ C_K \\ D_1 \\ \vdots \\ D_K \end{pmatrix}, \tag{1.9}$$

where $C_{jk}^{(l)}, D_{jk}^{(l)} \in \mathcal{C}^{r(n-r)\times r(n-r)}$ $(1 \le j,k \le K,\ 1 \le l \le 2)$. Note that the matrix T^{-1} can be expressed by

$$T^{-1} = \begin{pmatrix} -T_{22} & S_{12} \\ T_{21} & T_{11} \end{pmatrix} \begin{pmatrix} T_1^{-1} & 0 \\ 0 & T_2^{-1} \end{pmatrix},$$

where T_{11}, T_{21} and T_{22} are the matrices defined by (1.4), (1.6) and (1.7), respectively, and

$$S_{12} = \begin{pmatrix} A_{11}^{(2)^T} \otimes I_{n-r} & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & A_{11}^{(K)^T} \otimes I_{n-r} & 0 \\ 0 & \cdots & 0 & A_{11}^{(1)^T} \otimes I_{n-r} \end{pmatrix},$$

$$T_1 = \begin{pmatrix} -A_{11}^{(1)^T} \otimes B_{22}^{(1)} & 0 & \cdots & B_{11}^{(K)^T} \otimes A_{22}^{(1)} \\ B_{11}^{(1)^T} \otimes A_{22}^{(2)} & -A_{11}^{(2)^T} \otimes B_{22}^{(2)} & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & B_{11}^{(K-1)^T} \otimes A_{22}^{(K)} & -A_{11}^{(K)^T} \otimes B_{22}^{(K)} \end{pmatrix},$$

and

$$T_2 = \begin{pmatrix} A_{11}^{(2)^T} \otimes B_{22}^{(1)} & 0 & \cdots & -B_{11}^{(1)^T} \otimes A_{22}^{(1)} \\ -B_{11}^{(2)^T} \otimes A_{22}^{(2)} & A_{11}^{(3)^T} \otimes B_{22}^{(2)} & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & -B_{11}^{(K)^T} \otimes A_{22}^{(K)} & A_{11}^{(1)^T} \otimes B_{22}^{(K)} \end{pmatrix}.$$

The following four lemmas will be used in §2 – §4.

Lemma 1.1 [3, Corollary 2.2]. *Let $\{(A_j, B_j)\}_{j=1}^K$ be regular periodic matrix pairs, and let $Z_j = (Z_1^{(j)}, Z_2^{(j)})$ and $Q_j = (Q_1^{(j)}, Q_2^{(j)})$ be the $n \times n$ unitary matrices satisfying (1.2) and (1.3), that is, $\{(\mathcal{X}_j, \mathcal{Y}_j)\}_{j=1}^K$ with $\mathcal{X}_j = \mathcal{R}(Z_1^{(j)})$ and $\mathcal{Y}_j = \mathcal{R}(Q_1^{(j)})$ are simple periodic deflating subspaces of $\{(A_j, B_j)\}_{j=1}^K$. Then for $E_j, F_j \in \mathcal{C}^{n\times n}$, $j = 1, \ldots, K$, with sufficiently small $\|E_j\|_F$ and $\|F_j\|_F$, there are unique simple periodic deflating subspaces $\{(\tilde{\mathcal{X}}_j, \tilde{\mathcal{Y}}_j)\}_{j=1}^K$ of the matrix pairs $\{(A_j + E_j, B_j + F_j)\}_{j=1}^K$, where $\tilde{\mathcal{X}}_j = \mathcal{R}(\tilde{Z}_1^{(j)})$, $\tilde{\mathcal{Y}}_j = \mathcal{R}(\tilde{Q}_1^{(j)})$, and*

$$\tilde{Z}_1^{(j)} = Z_1^{(j)} + Z_2^{(j)}\Phi_j + \cdots, \quad \tilde{Q}_1^{(j)} = Q_1^{(j)} + Q_2^{(j)}\Psi_j + \cdots, \tag{1.10}$$

in which, $\mathrm{vec}(\Phi_j)$ *and* $\mathrm{vec}(\Psi_j)$ $(j = 1, \ldots, K)$ *have the expressions*

(1.11)

$$\mathrm{vec}(\Phi_j) = \sum_{k=1}^{K} C_{jk}^{(1)} \mathrm{vec}\left({Q_2^{(k)}}^H E_k Z_1^{(k-1)} \right) + \sum_{k=1}^{K} C_{jk}^{(2)} \mathrm{vec}\left({Q_2^{(k)}}^H F_k Z_1^{(k)} \right),$$

and

(1.12)

$$\mathrm{vec}(\Psi_j) = \sum_{k=1}^{K} D_{jk}^{(1)} \mathrm{vec}\left({Q_2^{(k)}}^H E_k Z_1^{(k-1)} \right) + \sum_{k=1}^{K} D_{jk}^{(2)} \mathrm{vec}\left({Q_2^{(k)}}^H F_k Z_1^{(k)} \right),$$

where $Z_1^{(0)} = Z_1^{(K)}$*, and* $C_{jk}^{(l)}$ *and* $D_{jk}^{(l)}$ $(1 \le j, k \le K,\ 1 \le l \le 2)$ *are the submatrices of* T^{-1} *given by (1.9).*

Let $\mathcal{X}_1 = \mathcal{R}(X_1)$ and $\tilde{\mathcal{X}}_1 = \mathcal{R}(\tilde{X}_1)$ be two $r-$dimensional subspaces of $\mathcal{C}^n$, where $X_1, \tilde{X}_1 \in \mathcal{C}^{n\times r}$, and $X_1^H X_1 = \tilde{X}_1^H \tilde{X}_1 = I$. Define $\Theta(X_1, \tilde{X}_1)$ by

$$\Theta(X_1, \tilde{X}_1) = \arccos\left(X_1^H \tilde{X}_1 \tilde{X}_1^H X_1 \right)^{1/2} \ge 0.$$

Then it is known [7, Chapter II] that

$$\rho(\mathcal{X}_1, \tilde{\mathcal{X}}_1) \equiv \| \sin\Theta(X_1, \tilde{X}_1) \|$$

is a generalized chordal metric on $r-$dimensional subspaces of $\mathcal{C}^n$. Particularly, we have the generalized chordal metric $\rho_F(\mathcal{X}_1, \tilde{\mathcal{X}}_1)$:

$$\rho_F(\mathcal{X}_1, \tilde{\mathcal{X}}_1) \equiv \| \sin\Theta(X_1, \tilde{X}_1) \|_F. \tag{1.13}$$

Lemma 1.2 [9, Theorem 1.3.2]**.** *Let* $X = (X_1, X_2)$ *be an* $n \times n$ *unitary matrix with* $X_1 \in \mathcal{C}^{n\times r}$*. Let*

$$\tilde{X}_1 = X \begin{pmatrix} I \\ Z \end{pmatrix}, \quad Z \in \mathcal{C}^{(n-r)\times r}, \tag{1.14}$$

and let $\mathcal{X}_1 = \mathcal{R}(X_1), \tilde{\mathcal{X}}_1 = \mathcal{R}(\tilde{X}_1)$*. Then*

$$\rho(\mathcal{X}_1, \tilde{\mathcal{X}}_1) = \|Z\| + O\left(\|Z\|^3\right) \quad \text{as } Z \to 0. \tag{1.15}$$

Let $X_1, \tilde{X}_1, \mathcal{X}_1, \tilde{\mathcal{X}}_1$ and Z be as in Lemma 1.2, and let $W_1 = \tilde{X}_1(\tilde{X}_1^H \tilde{X}_1)^{-1/2}$. Then [7, p.232]

$$\|Z\| = \| \tan\Theta(X_1, W_1) \|.$$

Combining it with

$$\rho(\mathcal{X}_1, \tilde{\mathcal{X}}_1) = \|\sin\Theta(X_1, W_1)\| \le \|\tan\Theta(X_1, W_1)\|$$

gives

$$\rho(\mathcal{X}_1, \tilde{\mathcal{X}}_1) \le \|Z\|. \tag{1.16}$$

From (1.15) and (1.16) we see that if ϵ is a sharper upper bound for $\|Z\|$, and if ϵ is very small, then ϵ is a sharper upper bound for $\rho(\mathcal{X}_1, \tilde{\mathcal{X}}_1)$, too.

Lemma 1.3 [8, Theorem 1.3.1]. *Let $z = (z_1^T, \ldots, z_p^T)^T$ with $z_j \in \mathcal{C}^{m_j}$ $\forall j$, $g \in \mathcal{C}^n$, and*

$$C = \begin{pmatrix} C_1 \\ \vdots \\ C_p \end{pmatrix} \quad \text{with} \quad C_j \in \mathcal{C}^{m_j \times n}, \;\; j = 1, \ldots, p,$$

where $m = m_1 + \cdots + m_p$. Let

$$\gamma = \|g\|_2, \quad \kappa_* = \sqrt{\kappa_1^2 + \cdots + \kappa_p^2} \quad \text{with} \quad \kappa_j = \|C_j\|_2, \; j = 1, \ldots, p,$$

and let $f, h : \mathcal{C}^m \to \mathcal{C}^n$ be two continuous mappings satisfying

$$\|f(z)\|_2 \le \epsilon\|z\|_2, \quad \|f(\tilde{z}) - f(z)\|_2 \le \epsilon\|\tilde{z} - z\|_2 \tag{1.17}$$

and

$$\|h(z)\|_2 \le \eta\|z\|_2^2, \quad \|h(\tilde{z}) - h(z)\|_2 \le 2\eta\max\{\|\tilde{z}\|_2, \|z\|_2\}\|\tilde{z} - z\|_2 \tag{1.18}$$

for some $\epsilon, \eta \ge 0$. If

$$\kappa_*\epsilon < 1 \quad \text{and} \quad \frac{4\kappa_*^2\gamma\eta}{(1 - \kappa_*\epsilon)^2} < 1,$$

or equivalently, if

$$\kappa_*(\epsilon + 2\sqrt{\gamma\eta}) < 1,$$

then there is a unique solution z of the nonlinear equation

$$z = C[g + f(z) + h(z)] \tag{1.19}$$

that satisfies

$$\|z_j\|_2 \le \frac{2\kappa_j\gamma}{1 - \kappa_*\epsilon + \sqrt{(1 - \kappa_*\epsilon)^2 - 4\kappa_*^2\gamma\eta}} \quad \forall j.$$

Referring to the proof of Lemma 1.3 [8, Theorem 1.3.1] we can prove the following result.

Lemma 1.4. *Let z, g, C, γ, f and h be as in Lemma 1.3, in which f and h satisfy (1.17) and (1.18), respectively. Define*

$$\delta = \|C\|^{-1}.$$

If

$$\frac{\epsilon}{\delta} < 1 \quad \text{and} \quad \frac{4\gamma\eta}{\delta^2(1-\epsilon/\delta)^2} < 1,$$

or equivalently, if

$$\frac{\epsilon + 2\sqrt{\gamma\eta}}{\delta} < 1,$$

then there is a unique solution z of the nonlinear equation (1.19) that satisfies

$$\|z\|_2 \le \frac{2\gamma}{\delta - \epsilon + \sqrt{(\delta-\epsilon)^2 - 4\gamma\eta}}.$$

It has been proved by [3] that each simple deflating subspace of regular periodic matrix pairs $\{(A_j, B_j)\}_{j=1}^K$ has an individual condition number, and by using the condition number we obtain a first order perturbation bound for each simple deflating subspace. However, the first order perturbation bounds only formally give perturbation results for simple deflating subspaces. In the case that perturbations in $\{(A_j, B_j)\}_{j=1}^K$ are not sufficiently small, the corresponding perturbed deflating subspaces may not exist. Consequently, it is necessary to derive perturbation bounds which guarantee the existence of corresponding perturbed deflating subspaces. This paper, as a supplement to the work [3], will derive such perturbation bounds (see §3).

Note that the technique and results of this paper are different from those of [1]. By [1, Subsection 2.1] one can obtain a perturbation bound for simple periodic deflating subspaces of a regular periodic matrix pairs $\{(A_j, B_j)\}_{j=1}^K$; the drawback of the bound is that it is governed by the ill-conditioning of the most sensitive deflating subspace. In this paper we obtain perturbation bounds for each individual subspace of simple periodic deflating subspaces $\{(\mathcal{X}_j, \mathcal{Y}_j)\}_{j=1}^K$.

2. Condition Numbers

In this section we introduce a definition of condition number for each individual subspace of simple periodic deflating subspaces $\{(\mathcal{X}_j, \mathcal{Y}_j)\}_{j=1}^K$, and derive expressions of the condition numbers. The condition numbers will appear in the perturbation bounds for periodic deflating subspaces derived in the next section. Note that the definition of condition number introduced in this section is a slight modification of that defined by [3, (3.15)].

Let $\{(\mathcal{X}_j, \mathcal{Y}_j\}_{j=1}^K$ be simple periodic deflating subspaces of regular periodic matrix pairs $\{(A_j, B_j)\}_{j=1}^K$, where $\mathcal{X}_j = \mathcal{R}(Z_1^{(j)})$ and $\mathcal{Y}_j = \mathcal{R}(Q_1^{(j)})$, $Z_1^{(j)}, Q_1^{(j)} \in \mathcal{C}^{n\times r}$, and ${Z_1^{(j)}}^H Z_1^{(j)} = {Q_1^{(j)}}^H Q_1^{(j)} = I$. Let $\{(E_j, F_j)\}_{j=1}^K$ be small perturbations in $\{(A_j, B_j)\}_{j=1}^K$, and let $\{(\tilde{\mathcal{X}}_j, \tilde{\mathcal{Y}}_j\}_{j=1}^K$ be the corresponding perturbations of $\{(\mathcal{X}_j, \mathcal{Y}_j\}_{j=1}^K$. By Lemma 1.1, for sufficiently small perturbations E_j and F_j $(j = 1, \ldots, K)$ the matrix pairs $\{(A_j + E_j, B_j + F_j)\}_{j=1}^K$ have unique simple periodic deflating subspaces $\{(\tilde{\mathcal{X}}_j, \tilde{\mathcal{Y}}_j\}_{j=1}^K$ with $\tilde{\mathcal{X}}_j = \mathcal{R}(\tilde{Z}_1^{(j)})$ and $\tilde{\mathcal{Y}}_j = \mathcal{R}(\tilde{Q}_1^{(j)})$, and $\tilde{Z}_1^{(j)}$, $\tilde{Q}_1^{(j)}$ have the expansions (1.10), in which $\mathrm{vec}(\Phi_j)$ and $\mathrm{vec}(\Psi_j)$ are given by (1.11) and (1.12), respectively.

We now apply the condition theory developed by Rice [5] to define condition numbers $\kappa(\mathcal{X}_j)$ and $\kappa(\mathcal{Y}_j)$ $(j = 1, \ldots, K)$ of the simple periodic deflating subspaces $\{(\mathcal{X}_j, \mathcal{Y}_j\}_{j=1}^K$ by

$$\kappa(\mathcal{X}_j) = \lim_{\delta\to 0} \sup_{\left(\sum_{j=1}^K (\|E_j\|_F^2 + \|F_j\|_F^2)\right)^{1/2} \le \delta} \frac{\rho_F(\mathcal{X}_j, \tilde{\mathcal{X}}_j)}{\delta}, \tag{2.1}$$

and

$$\kappa(\mathcal{Y}_j) = \lim_{\delta\to 0} \sup_{\left(\sum_{j=1}^K (\|E_j\|_F^2 + \|F_j\|_F^2)\right)^{1/2} \le \delta} \frac{\rho_F(\mathcal{Y}_j, \tilde{\mathcal{Y}}_j)}{\delta}. \tag{2.2}$$

Combining (2.1) with (1.10), (1.14) and (1.15) gives

$$\kappa(\mathcal{X}_j) = \lim_{\delta\to 0} \sup_{\left(\sum_{j=1}^K (\|E_j\|_F^2 + \|F_j\|_F^2)\right)^{1/2} \le \delta} \frac{\|\mathrm{vec}(\Phi_j)\|_2}{\delta}, \quad j = 1, \ldots, K. \tag{2.3}$$

Let

$$\begin{aligned} \widehat{E}_{21}^{(k)} &= {Q_2^{(k)}}^H E_k Z_1^{(k-1)}, \quad \widehat{F}_{21}^{(k)} = {Q_2^{(k)}}^H F_k Z_1^{(k)}, \\ \widehat{E}_k &= Q_k^H E_k Z_{k-1}, \quad \widehat{F}_k = Q_k^H F_k Z_k, \quad k = 1, \ldots, K, \end{aligned} \tag{2.4}$$

where $Z_1^{(0)} = Z_1^{(K)}, Z_0 = Z_K$. Then from (2.3), (2.4), (1.9) and (1.11) we get

$$\begin{aligned}
\kappa(\mathcal{X}_j) &= \lim_{\delta\to 0} \sup_{\left(\sum_{j=1}^K \left(\|E_j\|_F^2+\|F_j\|_F^2\right)\right)^{1/2}\le\delta} \frac{1}{\delta}\left\| C_j \begin{pmatrix} \mathrm{vec}(\widehat{E}_{21}^{(1)}) \\ \vdots \\ \mathrm{vec}(\widehat{E}_{21}^{(K)}) \\ \mathrm{vec}(\widehat{F}_{21}^{(1)}) \\ \vdots \\ \mathrm{vec}(\widehat{F}_{21}^{(K)}) \end{pmatrix}\right\|_2 \\
&= \lim_{\delta\to 0} \sup_{\left(\sum_{j=1}^K \|E_{21}^{(j)}\|_F^2+\|F_{21}^{(j)}\|_F^2\right)^{1/2}\le\delta} \frac{1}{\delta}\left\| C_j \begin{pmatrix} \mathrm{vec}(\widehat{E}_{21}^{(1)}) \\ \vdots \\ \mathrm{vec}(\widehat{E}_{21}^{(K)}) \\ \mathrm{vec}(\widehat{F}_{21}^{(1)}) \\ \vdots \\ \mathrm{vec}(\widehat{F}_{21}^{(K)}) \end{pmatrix}\right\|_2,
\end{aligned}$$

for $j = 1,\ldots,K$. Consequently, we have

$$\kappa(\mathcal{X}_j) = \|C_j\|_2, \quad j = 1,\ldots,K. \tag{2.5}$$

Similarly, we have

$$\kappa(\mathcal{Y}_j) = \|D_j\|_2, \quad j = 1,\ldots,K. \tag{2.6}$$

Thus, we have the following result.

Theorem 2.1. *Let $\{(\mathcal{X}_j, \mathcal{Y}_j)\}_{j=1}^K$ be simple periodic deflating subspaces of the matrix pairs $\{(A_j, B_j)\}_{j=1}^K$, and let $\kappa(\mathcal{X}_j)$ and $\kappa(\mathcal{Y}_j)$ be the condition numbers defined by (2.1) and* (2.2)*, respectively. Then $\kappa(\mathcal{X}_j)$ and $\kappa(\mathcal{Y}_j)$ have the expressions* (2.5) *and* (2.6)*, respectively, in which C_j and D_j are submatrices of T^{-1} given by* (1.9).

From the definitions (2.1) and (2.2) we get the first order perturbation bounds for simple periodic deflating subspaces $\{(\mathcal{X}_j, \mathcal{Y}_j)\}_{j=1}^K$:

(2.7)

$$\begin{aligned}
\rho_F(\mathcal{X}_j, \tilde{\mathcal{X}}_j) &\le \kappa(\mathcal{X}_j)\left[\sum_{j=1}^K \left(\|E_j\|_F^2 + \|F_j\|_F^2\right)\right]^{1/2} + O\left(\sum_{j=1}^K \left(\|E_j\|_F^2 + \|F_j\|_F^2\right)\right), \\
\rho_F(\mathcal{Y}_j, \tilde{\mathcal{Y}}_j) &\le \kappa(\mathcal{Y}_j)\left[\sum_{j=1}^K \left(\|E_j\|_F^2 + \|F_j\|_F^2\right)\right]^{1/2} + O\left(\sum_{j=1}^K \left(\|E_j\|_F^2 + \|F_j\|_F^2\right)\right),
\end{aligned}$$

for $j = 1, \ldots, K$, as $\sum_{j=1}^{K} \left(\|E_j\|_F^2 + \|F_j\|_F^2 \right) \to 0$.

3. Perturbation Bounds

3.1 Main Result

Theorem 3.1. *Let $\{(A_j, B_j)\}_{j=1}^K$, Z_j, Q_j and $\{(\mathcal{X}_j, \mathcal{Y}_j)\}_{j=1}^K$ be as in Lemma 1.1. For $E_j, F_j \in \mathcal{C}^{n\times n}$ $(j = 1, \ldots, K)$, let*

$$Q_j^H E_j Z_{j-1} = \begin{pmatrix} E_{11}^{(j)} & E_{12}^{(j)} \\ E_{21}^{(j)} & E_{22}^{(j)} \end{pmatrix}, \quad Q_j^H F_j Z_j = \begin{pmatrix} F_{11}^{(j)} & F_{12}^{(j)} \\ F_{21}^{(j)} & F_{22}^{(j)} \end{pmatrix}. \tag{3.1}$$

Moreover, let $\kappa(\mathcal{X}_j)$ and $\kappa(\mathcal{Y}_j)$ be the condition numbers expressed by (2.5) and (2.6), and let

$$\kappa_* = \left[\sum_{j=1}^{K} \left([\kappa(\mathcal{X}_j)]^2 + [\kappa(\mathcal{Y}_j)]^2 \right) \right]^{1/2}, \tag{3.2}$$

$$\gamma = \left[\sum_{j=1}^{K} \left(\|E_{21}^{(j)}\|_F^2 + \|F_{21}^{(j)}\|_F^2 \right) \right]^{1/2}, \tag{3.3}$$

$$\epsilon = \max_{1\le j\le K} \left\| \left(E_{11}^{(j)},\ F_{11}^{(j)} \right) \right\|_2 + \max_{1\le j\le K} \left\| \begin{pmatrix} E_{22}^{(j)} \\ F_{22}^{(j-1)} \end{pmatrix} \right\|_2, \tag{3.4}$$

and

$$\eta = \max\left\{ \max_{1\le j\le K} \left(\left\|A_{12}^{(j)}\right\|_2 + \left\|E_{12}^{(j)}\right\|_2 \right),\ \max_{1\le j\le K} \left(\left\|B_{12}^{(j)}\right\|_2 + \left\|F_{12}^{(j)}\right\|_2 \right) \right\}. \tag{3.5}$$

If

$$\kappa_*(\epsilon + 2\sqrt{\gamma\eta}) < 1, \tag{3.6}$$

then there are unique simple periodic deflating subspaces $\{(\tilde{\mathcal{X}}_j, \tilde{\mathcal{Y}}_j)\}_{j=1}^K$ of the matrix pairs $\{(A_j + E_j, B_j + F_j)\}_{j=1}^K$, and

$$\begin{aligned} \rho_F(\mathcal{X}_j, \tilde{\mathcal{X}}_j) &\le \frac{2\kappa(\mathcal{X}_j)\gamma}{1 - \kappa_*\epsilon + \sqrt{(1-\kappa_*\epsilon)^2 - 4\kappa_*^2\gamma\eta}}, \\ \rho_F(\mathcal{Y}_j, \tilde{\mathcal{Y}}_j) &\le \frac{2\kappa(\mathcal{Y}_j)\gamma}{1 - \kappa_*\epsilon + \sqrt{(1-\kappa_*\epsilon)^2 - 4\kappa_*^2\gamma\eta}}, \end{aligned} \tag{3.7}$$

for $j = 1, \ldots, K$, *where* $\rho_F(\cdot, \cdot)$ *is the generalized chordal metric defined by* (1.13).

Proof. It can be verified that the set of matrices

$$X^{(1)}, \ldots, X^{(K)}, Y^{(1)}, \ldots, Y^{(K)} \in \mathcal{C}^{(n-r)\times r}$$

is a solution to the system of equations

$$\begin{aligned}
&A_{22}^{(j)} X^{(j-1)} - Y^{(j)} A_{11}^{(j)} \\
&\quad = -E_{21}^{(j)} + \left(Y^{(j)} E_{11}^{(j)} - E_{22}^{(j)} X^{(j-1)} \right) + Y^{(j)} \left(A_{12}^{(j)} + E_{12}^{(j)} \right) X^{(j-1)}, \\
&B_{22}^{(j)} X^{(j-1)} - Y^{(j)} B_{11}^{(j)} \\
&\quad = -F_{21}^{(j)} + \left(Y^{(j)} F_{11}^{(j)} - F_{22}^{(j)} X^{(j)} \right) + Y^{(j)} \left(B_{12}^{(j)} + F_{12}^{(j)} \right) X^{(j)}, \\
&\quad j = 1, \ldots, K
\end{aligned} \tag{3.8}$$

if and only if the matrices $X^{(j)}$ and $Y^{(j)}$ $(j = 1, \ldots, K)$ satisfy

$$\begin{aligned}
&\begin{pmatrix} I & 0 \\ -Y^{(j)} & I \end{pmatrix} \begin{pmatrix} A_{11}^{(j)} + E_{11}^{(j)} & A_{12}^{(j)} + E_{12}^{(j)} \\ E_{21}^{(j)} & A_{22}^{(j)} + E_{22}^{(j)} \end{pmatrix} \begin{pmatrix} I & 0 \\ X^{(j-1)} & I \end{pmatrix} = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}, \\
&\begin{pmatrix} I & 0 \\ -Y^{(j)} & I \end{pmatrix} \begin{pmatrix} B_{11}^{(j)} + F_{11}^{(j)} & B_{12}^{(j)} + F_{12}^{(j)} \\ F_{21}^{(j)} & B_{22}^{(j)} + F_{22}^{(j)} \end{pmatrix} \begin{pmatrix} I & 0 \\ X^{(j)} & I \end{pmatrix} = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}, \\
&\quad j = 1, \ldots, K.
\end{aligned} \tag{3.9}$$

The relations of (3.9) imply that the subspaces $\{\tilde{\mathcal{X}}_j, \tilde{\mathcal{Y}}_j\}_{j=1}^K$ defined by

$$\tilde{\mathcal{X}}_j = \mathcal{R}\left(Z_j \begin{pmatrix} I \\ X^{(j)} \end{pmatrix} \right), \quad \tilde{\mathcal{Y}}_j = \mathcal{R}\left(Q_j \begin{pmatrix} I \\ -Y^{(j)} \end{pmatrix} \right), \quad j = 1, \ldots, K$$

are $r-$dimensional periodic deflating subspaces of $\{(A_j + E_j, B_j + F_j)\}_{j=1}^K$. Consequently, by Lemma1.2 and the relation (1.16), the problem of proving the inequalities of (3.7) is reduced to find a set of solutions $X_*^{(j)}, Y_*^{(j)}$ $(j = 1, \ldots, K)$ to (3.8), and bound the sizes of $\left\| X_*^{(j)} \right\|_F$ and $\left\| Y_*^{(j)} \right\|_F$ for $j = 1, \ldots, K$.

Let T be the matrix defined by (1.4)–(1.8), and let

$$L_{11} = \begin{pmatrix} 0 & \cdots & \cdots & -I_r \otimes E_{22}^{(1)} \\ -I_r \otimes E_{22}^{(2)} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & -I_r \otimes E_{22}^{(K)} & 0 \end{pmatrix}, \tag{3.10}$$

$$L_{12} = \begin{pmatrix} E_{11}^{(1)^T} \otimes I_{n-r} & \cdots & \cdots & 0 \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & E_{11}^{(K)^T} \otimes I_{n-r} \end{pmatrix}, \tag{3.11}$$

$$L_{21} = \begin{pmatrix} -I_r \otimes F_{22}^{(1)} & \cdots & \cdots & 0 \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & -I_r \otimes F_{22}^{(K)} \end{pmatrix}, \tag{3.12}$$

$$L_{22} = \begin{pmatrix} F_{11}^{(1)^T} \otimes I_{n-r} & \cdots & \cdots & 0 \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & F_{11}^{(K)^T} \otimes I_{n-r} \end{pmatrix}, \tag{3.13}$$

and

$$L = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}. \tag{3.14}$$

Moreover, let

$$z = \begin{pmatrix} \operatorname{vec}\left(X^{(1)}\right) \\ \vdots \\ \operatorname{vec}\left(X^{(K)}\right) \\ \operatorname{vec}\left(Y^{(1)}\right) \\ \vdots \\ \operatorname{vec}\left(Y^{(K)}\right) \end{pmatrix}, \quad g = -\begin{pmatrix} \operatorname{vec}\left(E_{21}^{(1)}\right) \\ \vdots \\ \operatorname{vec}\left(E_{21}^{(K)}\right) \\ \operatorname{vec}\left(F_{21}^{(1)}\right) \\ \vdots \\ \operatorname{vec}\left(F_{21}^{(K)}\right) \end{pmatrix}, \tag{3.15}$$

and

$$f(z) = Lz, \quad h(z) = \begin{pmatrix} \mathrm{vec}\left(Y^{(1)}(A_{12}^{(1)} + E_{12}^{(1)})X^{(K)}\right) \\ \mathrm{vec}\left(Y^{(2)}(A_{12}^{(2)} + E_{12}^{(2)})X^{(1)}\right) \\ \vdots \\ \mathrm{vec}\left(Y^{(K)}(A_{12}^{(K)} + E_{12}^{(K)})X^{(K-1)}\right) \\ \mathrm{vec}\left(Y^{(1)}(B_{12}^{(1)} + F_{12}^{(1)})X^{(1)}\right) \\ \mathrm{vec}\left(Y^{(2)}(B_{12}^{(2)} + F_{12}^{(2)})X^{(2)}\right) \\ \vdots \\ \mathrm{vec}\left(Y^{(K)}(B_{12}^{(K)} + F_{12}^{(K)})X^{(K)}\right) \end{pmatrix}. \tag{3.16}$$

Then the system of equations (3.8) can be written in an equivalent form

$$Tz = g + f(z) + h(z),$$

or equivalently,

$$z = T^{-1}[g + f(z) + h(z)]. \tag{3.17}$$

Observe that the functions f and h satisfy the conditions (1.17) and (1.18), where ϵ and η are the scalars defined by (3.4)and (3.5), respectively. In fact, from the first relation of (3.16) we get

$$\|f(z)\|_2 \le \|L\|_2\|z\|_2, \quad \|f(\tilde{z}) - f(z)\|_2 \le \|L\|_2\|\tilde{z} - z\|_2,$$

where the matrix L is defined by (3.10)–(3.14), and

$$\begin{aligned} \|L\|_2 &\le \left\| \begin{pmatrix} L_{11} \\ L_{21} \end{pmatrix} \right\|_2 + \left\| \begin{pmatrix} L_{12} \\ L_{22} \end{pmatrix} \right\|_2 \\ &\le \max_{1\le j\le K} \left\| \begin{pmatrix} I_r \otimes E_{22}^{(j)} \\ I_r \otimes F_{22}^{(j-1)} \end{pmatrix} \right\|_2 + \max_{1\le j\le K} \left\| \begin{pmatrix} E_{11}^{(j)^T} \otimes I_{n-r} \\ F_{11}^{(j)^T} \otimes I_{n-r} \end{pmatrix} \right\|_2 \\ &= \epsilon. \end{aligned}$$

Moreover, from the second relation of (3.16) we get

$$\begin{aligned} \|h(z)\|_2 &= \sqrt{\sum_{j=1}^{K} \left\| Y^{(j)} \left(A_{12}^{(j)} + E_{12}^{(j)} \right) X^{(j-1)} \right\|_F^2 + \sum_{j=1}^{K} \left\| Y^{(j)} \left(B_{12}^{(j)} + F_{12}^{(j)} \right) X^{(j)} \right\|_F^2} \\ &\le \eta \sqrt{\sum_{j=1}^{K} \|Y^{(j)}\|_F^2 \|X^{(j-1)}\|_F^2 + \sum_{j=1}^{K} \|Y^{(j)}\|_F^2 \|X^{(j)}\|_F^2} \\ &\le \eta \|z\|_2^2, \end{aligned}$$

and

$$\begin{aligned}\|h(\tilde{z})-h(z)\|_2^2 \leq 2 \Bigg(& \sum_{j=1}^{K} \left\|\tilde{Y}^{(j)}-Y^{(j)}\right\|_F^2 \left\|A_{12}^{(j)}+E_{12}^{(j)}\right\|_2^2 \left\|\tilde{X}^{(j-1)}\right\|_F^2 \\ & + \sum_{j=1}^{K} \left\|Y^{(j)}\right\|_F^2 \left\|A_{12}^{(j)}+E_{12}^{(j)}\right\|_2^2 \left\|\tilde{X}^{(j-1)}-X^{(j-1)}\right\|_F^2 \\ & + \sum_{j=1}^{K} \left\|\tilde{Y}^{(j)}-Y^{(j)}\right\|_F^2 \left\|B_{12}^{(j)}+F_{12}^{(j)}\right\|_2^2 \left\|\tilde{X}^{(j)}\right\|_F^2 \\ & + \sum_{j=1}^{K} \left\|Y^{(j)}\right\|_F^2 \left\|B_{12}^{(j)}+F_{12}^{(j)}\right\|_2^2 \left\|\tilde{X}^{(j)}-X^{(j)}\right\|_F^2 \Bigg) \\ \leq 2\eta^2 \Bigg(& \sum_{j=1}^{K} \left\|\tilde{Y}^{(j)}-Y^{(j)}\right\|_F^2 \left\|\tilde{X}^{(j-1)}\right\|_F^2 + \sum_{j=1}^{K} \left\|Y^{(j)}\right\|_F^2 \left\|\tilde{X}^{(j-1)}-X^{(j-1)}\right\|_F^2 \\ & + \sum_{j=1}^{K} \left\|\tilde{Y}^{(j)}-Y^{(j)}\right\|_F^2 \left\|\tilde{X}^{(j)}\right\|_F^2 + \sum_{j=1}^{K} \left\|Y^{(j)}\right\|_F^2 \left\|\tilde{X}^{(j)}-X^{(j)}\right\|_F^2 \Bigg) \\ \leq & \left[2\eta \max\{\|\tilde{z}\|_2, \|z\|_2\} \|\tilde{z}-z\|_2\right]^2. \end{aligned}$$

Hence, by Lemma 1.3 and (3.2)–(3.5), if $\kappa_*, \gamma, \epsilon, \eta$ satisfy (3.6), then the equation (3.17) has a unique solution

$$z_* = \left(\operatorname{vec}\left(X_*^{(1)}\right)^T, \ldots, \operatorname{vec}\left(X_*^{(K)}\right)^T, \operatorname{vec}\left(Y_*^{(1)}\right)^T, \ldots, \operatorname{vec}\left(Y_*^{(K)}\right)^T \right)^T$$

satisfying

$$\begin{aligned} \left\|X_*^{(j)}\right\|_F &= \left\|\operatorname{vec}\left(X_*^{(j)}\right)\right\|_2 \leq \frac{2\|C_j\|_2\gamma}{1-\kappa_*\epsilon+\sqrt{(1-\kappa_*\epsilon)^2-4\kappa_*^2\gamma\eta}}, \\ \left\|Y_*^{(j)}\right\|_F &= \left\|\operatorname{vec}\left(Y_*^{(j)}\right)\right\|_2 \leq \frac{2\|D_j\|_2\gamma}{1-\kappa_*\epsilon+\sqrt{(1-\kappa_*\epsilon)^2-4\kappa_*^2\gamma\eta}}, \end{aligned} \tag{3.18}$$

for $j = 1, \ldots, K$, where C_j and D_j are the submatrices of T^{-1} given by (1.9). Combining (3.18) with (1.16) and (2.5)–(2.6) shows the estimates of (3.7). ■

From (3.7) we get the first order perturbation bounds for simple periodic deflating subspaces $\{(\mathcal{X}_j, \mathcal{Y}_j)\}_{j=1}^K$:

$$
(3.19)\qquad
\begin{aligned}
\rho_F(\mathcal{X}_j, \tilde{\mathcal{X}}_j) &\lesssim \kappa(\mathcal{X}_j)\gamma \le \kappa(\mathcal{X}_j)\left[\sum_{j=1}^K \left(\|E_j\|_F^2 + \|F_j\|_F^2\right)\right]^{1/2},\\
\rho_F(\mathcal{Y}_j, \tilde{\mathcal{Y}}_j) &\lesssim \kappa(\mathcal{Y}_j)\gamma \le \kappa(\mathcal{Y}_j)\left[\sum_{j=1}^K \left(\|E_j\|_F^2 + \|F_j\|_F^2\right)\right]^{1/2},
\end{aligned}
$$

for $j = 1, \ldots, K$. The estimates of (3.19) coincide with those of (2.7).

3.1 Residual Bounds

Let $\{(A_j, B_j)\}_{j=1}^K$, Z_j, Q_j and $\{(\mathcal{X}_j, \mathcal{Y}_j)\}_{j=1}^K$ be as in Lemma 1.1, and let $\{(A_j, B_j)\}_{j=1}^K$ be perturbed to $\{(\tilde{A}_j, \tilde{B}_j)\}_{j=1}^K$. Let $\tilde{Z}_j = \left(\tilde{Z}_1^{(j)}, \tilde{Z}_2^{(j)}\right)$ and $\tilde{Q}_j = \left(\tilde{Q}_1^{(j)}, \tilde{Q}_2^{(j)}\right)$ be $n \times n$ unitary matrices with $\tilde{Z}_0 = \tilde{Z}_K$ such that

$$
(3.20)\qquad \tilde{Q}_j^H \tilde{A}_j \tilde{Z}_{j-1} = \begin{pmatrix} \tilde{A}_{11}^{(j)} & \tilde{A}_{12}^{(j)} \\ 0 & \tilde{A}_{22}^{(j)} \end{pmatrix}, \quad \tilde{Q}_j^H \tilde{B}_j \tilde{Z}_{j-1} = \begin{pmatrix} \tilde{B}_{11}^{(j)} & \tilde{B}_{12}^{(j)} \\ 0 & \tilde{B}_{22}^{(j)} \end{pmatrix},
$$

for $j = 1, \ldots, K$, where $\tilde{A}_{11}^{(j)}, \tilde{B}_{11}^{(j)} \in \mathcal{C}^{r \times r}$, and assume that

$$
(3.21)\qquad \lambda\left(\left\{\left(A_{11}^{(j)}, B_{11}^{(j)}\right)\right\}_{j=1}^K\right) \bigcap \lambda\left(\left\{\left(\tilde{A}_{22}^{(j)}, \tilde{B}_{22}^{(j)}\right)\right\}_{j=1}^K\right) = \emptyset.
$$

Let T_{12} and T_{22} be the matrices defined by (1.5) and (1.7), and define $\tilde{T}_{11}, \tilde{T}_{21}$ and $\tilde{T}$ by

$$
(3.22)\qquad \tilde{T}_{11} = \begin{pmatrix} 0 & \cdots & \cdots & I_r \otimes \tilde{A}_{22}^{(1)} \\ I_r \otimes \tilde{A}_{22}^{(2)} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & I_r \otimes \tilde{A}_{22}^{(K)} & 0 \end{pmatrix},
$$

$$
(3.23)\qquad \tilde{T}_{21} = \begin{pmatrix} I_r \otimes \tilde{B}_{22}^{(1)} & \cdots & \cdots & 0 \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & I_r \otimes \tilde{B}_{22}^{(K)} \end{pmatrix},
$$

and

$$
(3.24)\qquad \tilde{T} = \begin{pmatrix} \tilde{T}_{11} & T_{12} \\ \tilde{T}_{21} & T_{22} \end{pmatrix}.
$$

The assumption (3.21) implies that the matrix $\tilde{T}$ is nonsingular. (The proof is similar to that of the nonsingularity of the matrix T defined by (1.4)–(1.8) [3, 164–166].) Write

$$\tilde{T}^{-1} = \begin{pmatrix} \tilde{C}_1 \\ \vdots \\ \tilde{C}_K \\ \tilde{D}_1 \\ \vdots \\ \tilde{D}_K \end{pmatrix}, \quad \tilde{C}_j, \tilde{D}_j \in \mathcal{C}^{2r(n-r)K\times r(n-r)}, \quad j = 1, \ldots, K. \tag{3.25}$$

Let $\tilde{\mathcal{X}}_j = \mathcal{R}(\tilde{Z}_1^{(j)})$, $\tilde{\mathcal{Y}}_j = \mathcal{R}(\tilde{Q}_1^{(j)})$, and let

$$R_A^{(j)} = \tilde{A}_j Z_1^{(j-1)} - Q_1^{(j)} A_{11}^{(j)}, \quad R_B^{(j)} = \tilde{B}_j Z_1^{(j)} - Q_1^{(j)} A_{11}^{(j)}, \tag{3.26}$$

be the residuals of $\{(\tilde{A}_j, \tilde{B}_j)\}_{j=1}^K$ with respect to $Z_1^{(j-1)}$, $Z_1^{(j)}$, $Q_1^{(j)}$, $A_{11}^{(j)}$ and $B_{11}^{(j)}$, $j = 1, \ldots, K$. The following result gives residual bounds for $\rho_F(\mathcal{X}_j, \tilde{\mathcal{X}}_j)$ and $\rho_F(\mathcal{Y}_j, \tilde{\mathcal{Y}}_j)$, $j = 1, \ldots, K$.

Theorem 3.2. *For the above mentioned* $\{(A_j, B_j)\}_{j=1}^K$, $\{(\tilde{A}_j, \tilde{B}_j)\}_{j=1}^K$, $\mathcal{X}_j$, $\mathcal{Y}_j$, *and* $\tilde{\mathcal{X}}_j, \tilde{\mathcal{Y}}_j$, *we have*

$$\begin{aligned} \rho_F(\mathcal{X}_j, \tilde{\mathcal{X}}_j) &\le \|\tilde{C}_j\|_2 \left[\sum_{j=1}^K \left(\left\| P^{\perp}_{\tilde{Q}_1^{(j)}} R_A^{(j)} \right\|_F^2 + \left\| P^{\perp}_{\tilde{Q}_1^{(j)}} R_B^{(j)} \right\|_F^2 \right)\right]^{1/2}, \\ \rho_F(\mathcal{Y}_j, \tilde{\mathcal{Y}}_j) &\le \|\tilde{D}_j\|_2 \left[\sum_{j=1}^K \left(\left\| P^{\perp}_{\tilde{Q}_1^{(j)}} R_A^{(j)} \right\|_F^2 + \left\| P^{\perp}_{\tilde{Q}_1^{(j)}} R_B^{(j)} \right\|_F^2 \right)\right]^{1/2}, \end{aligned} \tag{3.27}$$

where $\tilde{C}_j, \tilde{D}_j$ *are the submatrices of* $\tilde{T}^{-1}$ *given by (3.25),* $R_A^{(j)}$, $R_B^{(j)}$ *are the residuals defined by (3.26),* $j = 1, \ldots, K$, *and* $P^{\perp}_{\tilde{Q}_1^{(j)}} = I - P_{\tilde{Q}_1^{(j)}}$, *in which* $P_{\tilde{Q}_1^{(j)}}$ *is the orthogonal projection onto* $\mathcal{R}\left(\tilde{Q}_1^{(j)}\right) \left(= \tilde{\mathcal{Y}}_j\right)$.

Proof. From (3.26) and (3.20) we get

$$\begin{aligned} R_A^{(j)} &= \tilde{Q}_1^{(j)} \tilde{A}_{11}^{(j)} \tilde{Z}_1^{(j-1)^H} Z_1^{(j-1)} + \tilde{Q}_1^{(j)} \tilde{A}_{12}^{(j)} \tilde{Z}_2^{(j-1)^H} Z_1^{(j-1)} + \tilde{Q}_2^{(j)} \tilde{A}_{22}^{(j)} \tilde{Z}_2^{(j-1)^H} Z_1^{(j-1)} \\ &\quad - Q_1^{(j)} A_{11}^{(j)}, \\ R_B^{(j)} &= \tilde{Q}_1^{(j)} \tilde{B}_{11}^{(j)} \tilde{Z}_1^{(j)^H} Z_1^{(j)} + \tilde{Q}_1^{(j)} \tilde{B}_{12}^{(j)} \tilde{Z}_2^{(j)^H} Z_1^{(j)} + \tilde{Q}_2^{(j)} \tilde{B}_{22}^{(j)} \tilde{Z}_2^{(j)^H} Z_1^{(j)} - Q_1^{(j)} B_{11}^{(j)}, \end{aligned}$$

and

$$(3.28)\qquad \begin{aligned} \tilde{Q}_2^{(j)^H} R_A^{(j)} &= \tilde{A}_{22}^{(j)} \tilde{Z}_2^{(j-1)^H} Z_1^{(j-1)} - \tilde{Q}_2^{(j)^H} Q_1^{(j)} A_{11}^{(j)}, \\ \tilde{Q}_2^{(j)^H} R_B^{(j)} &= \tilde{B}_{22}^{(j)} \tilde{Z}_2^{(j)^H} Z_1^{(j)} - \tilde{Q}_2^{(j)^H} Q_1^{(j)} B_{11}^{(j)}, \end{aligned}$$

for $j = 1, \ldots, K$.

Let

$$\begin{aligned} s_{\mathcal{X}}^{(j)} &= \mathrm{vec}\left(\tilde{Z}_2^{(j)^H} Z_1^{(j)}\right), \quad & s_{\mathcal{Y}}^{(j)} &= \mathrm{vec}\left(\tilde{Q}_2^{(j)^H} Q_1^{(j)}\right), \\ q_A^{(j)} &= \mathrm{vec}\left(\tilde{Q}_2^{(j)^H} R_A^{(j)}\right), \quad & q_B^{(j)} &= \mathrm{vec}\left(\tilde{Q}_2^{(j)^H} R_B^{(j)}\right), \end{aligned}$$

for $j = 1, \ldots, K$, and let

$$s = \begin{pmatrix} s_{\mathcal{X}}^{(1)} \\ \vdots \\ s_{\mathcal{X}}^{(K)} \\ s_{\mathcal{Y}}^{(1)} \\ \vdots \\ s_{\mathcal{Y}}^{(K)} \end{pmatrix}, \qquad q = \begin{pmatrix} q_A^{(1)} \\ \vdots \\ q_A^{(K)} \\ q_B^{(1)} \\ \vdots \\ q_B^{(K)} \end{pmatrix}.$$

Then by (3.22)–(3.24), the relation (3.28) can be written in an equivalent form:

$$\tilde{T}s = q, \quad \text{or} \quad s = \tilde{T}^{-1}q,$$

which implies

$$(3.29)\qquad s_{\mathcal{X}}^{(j)} = \tilde{C}_j q, \quad s_{\mathcal{Y}}^{(j)} = \tilde{D}_j q, \quad j = 1, \ldots, K.$$

Observe that

$$\left\| s_{\mathcal{X}}^{(j)} \right\|_2 = \left\| \tilde{Z}_2^{(j)^H} Z_1^{(j)} \right\|_F = \left[\mathrm{tr}\left(I - Z_1^{(j)^H} \tilde{Z}_1^{(j)} \tilde{Z}_1^{(j)^H} Z_1^{(j)} \right) \right]^{1/2} = \rho_F(\mathcal{X}_j, \tilde{\mathcal{X}}_j),$$

for $j = 1, \ldots, K$. Similarly,

$$\left\| s_{\mathcal{Y}}^{(j)} \right\|_2 = \left\| \tilde{Q}_2^{(j)^H} Q_1^{(j)} \right\|_F = \rho_F(\mathcal{Y}_j, \tilde{\mathcal{Y}}_j), \quad j = 1, \ldots, K.$$

Hence, from (3.29) we get

$$\rho_F(\mathcal{X}_j, \tilde{\mathcal{X}}_j) \le \|\tilde{C}_j\|_2 \left[\sum_{j=1}^{K} \left(\left\| \tilde{Q}_2^{(j)^H} R_A^{(j)} \right\|_F^2 + \left\| \tilde{Q}_2^{(j)^H} R_B^{(j)} \right\|_F^2 \right) \right]^{1/2},$$

$$\rho_F(\mathcal{Y}_j, \tilde{\mathcal{Y}}_j) \le \|\tilde{D}_j\|_2 \left[\sum_{j=1}^{K} \left(\left\| \tilde{Q}_2^{(j)^H} R_A^{(j)} \right\|_F^2 + \left\| \tilde{Q}_2^{(j)^H} R_B^{(j)} \right\|_F^2 \right) \right]^{1/2},$$

which imply the estimates of (3.27). ■

Note that one can use the formulas

$$\left\|P^{\perp}_{\tilde{Q}_1^{(j)}} R_A^{(j)}\right\|_F^2 = \left\|R_A^{(j)}\right\|_F^2 - \left\|P_{\tilde{Q}_1^{(j)}} R_A^{(j)}\right\|_F^2, \quad \left\|P^{\perp}_{\tilde{Q}_1^{(j)}} R_B^{(j)}\right\|_F^2 = \left\|R_B^{(j)}\right\|_F^2 - \left\|P_{\tilde{Q}_1^{(j)}} R_B^{(j)}\right\|_F^2$$

to modify the form of the estimates (3.27).

4. Final Remarks

4.1. Applying Lemma 1.4 to the nonlinear equation (3.17) shows the following result

Theorem 4.1. *Let* $\{(A_j, B_j)\}_{j=1}^K$, Z_j, Q_j, E_j, F_j $(j = 1, \ldots, K)$, $\{(\mathcal{X}_j, \mathcal{Y}_j)\}_{j=1}^K$, γ, ϵ *and* η *be as in Theorem 3.1. Let* T *be the matrix defined by (1.4)–(1.8), and define*

$$\delta = \left\|T^{-1}\right\|_2^{-1}. \tag{4.1}$$

If

$$\frac{\epsilon + 2\sqrt{\gamma\eta}}{\delta} < 1,$$

then there are unique simple periodic deflating subspaces $\left\{\left(\tilde{\mathcal{X}}_j, \tilde{\mathcal{Y}}_j\right)\right\}_{j=1}^K$ *of the periodic matrix pairs* $\{(A_j + E_j, B_j + F_j)\}_{j=1}^K$, *and*

$$\left[\sum_{j=1}^K \left(\rho_F^2\left(\mathcal{X}_j, \tilde{\mathcal{X}}_j\right) + \rho_F^2\left(\mathcal{Y}_j, \tilde{\mathcal{Y}}_j\right)\right)\right]^{1/2} \le \frac{2\gamma}{\delta - \epsilon + \sqrt{(\delta - \epsilon)^2 - 4\gamma\eta}}. \tag{4.2}$$

Theorem 4.1 is a generalization of the perturbation result [7, Chapter VI, Theorem 2.14] for deflating subspaces of a regular matrix pair. The drawback of the perturbation bound given by (4.2) is that it is governed by the ill-conditioning of the most sensitive deflating subspace.

4.2. From (4.2) we get

$$\left[\sum_{j=1}^K \left(\rho_F^2\left(\mathcal{X}_j, \tilde{\mathcal{X}}_j\right) + \rho_F^2\left(\mathcal{Y}_j, \tilde{\mathcal{Y}}_j\right)\right)\right]^{1/2} \lesssim \frac{\gamma}{\delta}$$

when $\sum_{j=1}^{K} \left(\|E_j\|_F^2 + \|F_j\|_F^2\right)$ is sufficiently small. Consequently, the quantity

$$\kappa\left(\{(\mathcal{X}_j, \mathcal{Y}_j)\}_{j=1}^K\right) \equiv \frac{1}{\delta} \;\left(= \|T^{-1}\|_2\right) \tag{4.3}$$

can be regarded as a condition estimator of the simple periodic deflating subspaces $\{(\mathcal{X}_j, \mathcal{Y}_j)\}_{j=1}^K$. By (1.9), (2.5), (2.6) and (4.1) we have the relations

$$\kappa(\mathcal{X}_j) < \kappa\left(\{(\mathcal{X}_j, \mathcal{Y}_j)\}_{j=1}^K\right) \quad \text{and} \quad \kappa(\mathcal{Y}_j) < \kappa\left(\{(\mathcal{X}_j, \mathcal{Y}_j)\}_{j=1}^K\right)$$

for $j = 1, \ldots, K$, and there is the possibility that in some cases

$$\kappa(\mathcal{X}_j) \ll \kappa\left(\{(\mathcal{X}_j, \mathcal{Y}_j)\}_{j=1}^K\right) \quad \text{and/or} \quad \kappa(\mathcal{Y}_j) \ll \kappa\left(\{(\mathcal{X}_j, \mathcal{Y}_j)\}_{j=1}^K\right)$$

for some j $(1 \le j \le K)$. Consequently, in some cases the quantity $\kappa\left(\{(\mathcal{X}_j, \mathcal{Y}_j)\}_{j=1}^K\right)$ may be a severe overestimate of the sensitivity of some subspaces $\mathcal{X}_j$ and/or $\mathcal{Y}_j$ $(1 \le j \le K)$. This fact is illustrated by the following numerical example.

Consider the regular periodic matrix pairs $\{(A_j, B_j)\}_{j=1}^3$ with

$$A_1 = \begin{pmatrix} 1 & 0 & -1 & 4 & 1 \\ 0 & 1 & 60 & 0 & -2 \\ 0 & 0 & 6 & 1 & 0 \\ 0 & 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 2 & 0 & 0 & -1 & 0 \\ 0 & 5 & 1 & 0 & 0 \\ 0 & 0 & 0.1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 1 & -50 & 0 & 3 & 0 \\ 0 & 4 & 70 & 2 & 0 \\ 0 & 0 & 1 & 10 & 0 \\ 0 & 0 & 0 & 8 & -1 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 2 & 1 & -2 & 0 & 1 \\ 0 & 4 & 0 & -2 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 2 & 100 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 80 \\ 0 & 0 & 0 & 10 & 1 \\ 0 & 0 & 0 & 0 & 7 \end{pmatrix}, \quad B_3 = \begin{pmatrix} 5 & 1 & 2 & 0 & -1 \\ 0 & 1 & -1 & 2 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We have

$$\lambda\left(\{(A_j, B_j)\}_{j=1}^3\right) = \{(2,\, 20),\ (4,\, 20),\ (6,\, 0.1),\ (240,\, 3),\ (105,\, 1)\},$$

or equivalently,

$$\lambda\left(\{(A_j, B_j)\}_{j=1}^3\right) = \{0.1,\ 0.2,\ 60,\ 80,\ 105\}.$$

It is easy to see that the pairs of subspaces $\{(\mathcal{X}_j, \mathcal{Y}_j)\}_{j=1}^3$ with

$$\mathcal{X}_1 = \mathcal{X}_2 = \mathcal{X}_3 = \mathcal{Y}_1 = \mathcal{Y}_2 = \mathcal{Y}_3 = \mathcal{R}\left(\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}^T\right)$$

are the simple periodic deflating subspaces of $\{(A_j, B_j)\}_{j=1}^3$ corresponding to the eigenvalues 0.1 and 0.2. Computing $\kappa(\mathcal{X}_j)$ and $\kappa(\mathcal{Y}_j)$ $(j = 1, 2, 3)$ by (2.5), (2.6) and (1.4)–(1.9), and computing $\kappa\left(\{(\mathcal{X}_j, \mathcal{Y}_j)\}_{j=1}^3\right)$ by (4.3) and (1.4)–(1.8), we get

$$\kappa(\mathcal{X}_1) \approx 326, \qquad \kappa(\mathcal{X}_2) \approx 243, \qquad \kappa(\mathcal{X}_3) \approx 1.2,$$
$$\kappa(\mathcal{Y}_1) \approx 6.8 \qquad \kappa(\mathcal{Y}_2) \approx 61, \qquad \kappa(\mathcal{Y}_3) \approx 3,$$

and

$$\kappa\left(\{(\mathcal{X}_j, \mathcal{Y}_j)\}_{j=1}^3\right) \approx 350.$$

Obviously, the quantity $\kappa\left(\{(\mathcal{X}_j, \mathcal{Y}_j)\}_{j=1}^3\right)$ is much larger than the condition numbers $\kappa(\mathcal{X}_3)$, $\kappa(\mathcal{Y}_1)$ and $\kappa(\mathcal{Y}_3)$.

Note that the computations were performed using MATLAB, version 6.5. The relative machine precision is 2.22×10^{-16}.

4.3. The difficulty about the computation of the condition numbers $\kappa(\mathcal{X}_j)$ and $\kappa(\mathcal{Y}_j)$ $(j = 1, \ldots, K)$ lies on the fact that each one of the condition numbers involves a computation of the spectral norm of an $r(n-r) \times 2r(n-r)K$ matrix C_j or D_j (see (2.5) and (2.6)), where C_j and D_j are submatrices of T^{-1} (see (1.9)), and T is a $2r(n-r)K \times 2r(n-r)K$ matrix given by (1.4)–(1.8). Therefore, the problem of how to compute the condition numbers $\kappa(\mathcal{X}_j)$ and $\kappa(\mathcal{Y}_j)$ $(j = 1, \ldots, K)$ efficiently is worth studying.

4.4. The eigenpronlem (1.1) with $K = 1$ is the generalized eigenvalue problem $\beta_1 A_1 x_1 = \alpha_1 B_1 x_1$. Condition numbers and perturbation bounds for each individual simple deflating subspace of the matrix pair (A_1, B_1) are given by [8, Section 3] (or see [9, Theorems 4.2.6 and 4.3.1]).

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Ji-guang Sun
Department of Computing Science,
Umeå University,
S-901 87 Umea,
Sweden
E-mail: jisun@cs.umu.se