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STRONG CONVERGENCE THEOREMS FOR COMMUTATIVE NONEXPANSIVE SEMIGROUPS IN GENERAL BANACH SPACES

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Abstract. In this paper, we introduce an iteration scheme of Mann's type for general commutative nonexpansive semigroups and then obtain a strong convergence theorem in compact sets of general Banach spaces by using the theory of means of abstract semigroups. Using this result, we prove some strong convergence theorems in cases of discrete and one-parameter semigroups.

1. INTRODUCTION

In 1975, Baillon [3] originally studied the first nonlinear ergodic theorem in the framework of Hilbert spaces: Let C be a closed and convex subset of a Hilbert space H and let T be a nonexpansive mapping of C into itself. If the set $F(T)$ of fixed points of T is nonempty, then for each $x \in C$, the Cesáro means

$$S_n(x) = \frac{1}{n}\sum_{k=0}^{n-1} T^k x$$

converge weakly to some $y \in F(T)$. In this case, putting $y = Px$ for each $x \in C$, P is a nonexpansive retraction of C onto $F(T)$ such that $PT = TP = P$ and Px is contained in the closure of convex hull of $\{T^n x : n = 1, 2, \dots\}$ for each $x \in C$. We call such a retraction "*an ergodic retraction*". Takahashi [18,20] proved the existence of such a retraction for an amenable semigroup of nonexpansive mappings on a Hilbert space. Rodé [13] also found a sequence of means on a semigroup, generalizing the Cesàro means, and extended Baillon's theorem. These results were extended to a uniformly convex Banach space whose norm is Fréchet differentiable in the case of a commutative semigroup of nonexpansive mappings by Hirano, Kido and Takahashi [5]. In 1999, Lau, Shioji and Takahashi [8] extended Takahashi's result and Rodé's result to an amenable semigroup of nonexpansive mappings in the Banach space.

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On the other hand, in 1997, Shimizu and Takahashi [14] introduced the first iterative scheme for finding common fixed points of families of nonexpansive mappings and then many authors have studied such iterative schemes for families of various mappings (cf. [15,21]). For example, Atsushiba and Takahashi [1] studied an iteration scheme of Mann's type [6,7,11,12] for two commutative nonexpansive mappings S and T in a Banach space: $x_0 \in C$ and

$$x_{n+1} = \frac{\alpha_n}{n^2} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} S^i T^j x_n + (1 - \alpha_n) x_n$$

for each $n \geq 1$, where $\{\alpha_n\}$ is a sequence in $[0, 1]$. Recently, Suzuki [16] and Suzuki and Takahashi [17] proved strong convergence theorems of the iteration scheme of Mann's type for two commutative nonexpansive mappings and one-parameter nonexpansive semigroups in compact sets of general Banach spaces, respectively.

In this paper, we introduce an iteration scheme of Mann's type for general commutative nonexpansive semigroups and then obtain a strong convergence theorem in compact sets of general Banach spaces by using the theory of means of abstract semigroups; for instance [4,18,22]. Using this result, we prove some strong convergence theorems in cases of discrete and one-parameter semigroups.

2. Preliminaries

Throughout this paper, we denote by E a real Banach space with the topological dual E^*. Let S be a semigroup. We denote by $l^\infty(S)$ the Banach space of all bounded real-valued functions on S with supremum norm. For each $s \in S$, we define two operators $l(s)$ and $r(s)$ on $l^\infty(S)$ by $(l(s)f)(t) = f(st)$ and $(r(s)f)(t) = f(ts)$ for each $t \in S$ and $f \in l^\infty(S)$, respectively. Let X be a subspace of $l^\infty(S)$ containing 1. An element μ of the topological dual X^* of X is said to be a *mean* on X if $\|\mu\| = \mu(1) = 1$. For $s \in S$, we can define a *point evaluation* δ_s by $\delta_s(f) = f(s)$ for each $f \in X$. A convex combination of point evaluations is called a *finite mean* on X. As is well known, μ is a mean on X if and only if

$$\inf_{s \in S} f(s) \leq \mu(f) \leq \sup_{s \in S} f(s)$$

for each $f \in X$. Suppose that $l(s)X \in X$ and $r(s)X \in X$ for each $s \in S$. Then, a mean μ on X is said to be *left invariant* (resp. *right invariant*) if $\mu(l(s)f) = \mu(f)$ (resp. $\mu(r(s)f) = \mu(f)$) for each $s \in S$ and $f \in X$. A mean μ on X is said to be *invariant* if μ is both left and right invariant. X is said to be *amenable* if there exists an invariant mean on X. For fixed point theorems for the semigroups, see [9]. We know from [4] that if S is commutative, then X is amenable. Let $\{\mu_\alpha\}$ be a net of means on X. Then $\{\mu_\alpha\}$ is said to be *asymptotically invariant* if for each $s \in S$, both $l(s)^*\mu_\alpha - \mu_\alpha$ and $r(s)^*\mu_\alpha - \mu_\alpha$ converge to 0 in the weak-star topology, where $l(s)^*$ and $r(s)^*$ are the adjoint operators of $l(s)$ and $r(s)$, respectively. Such nets were first studied by Day in [4]. We know that X is amenable if and only if there exists an asymptotically invariant net of finite means on X (cf. [4,22]).

Let S be a commutative semigroup with identity. Then S is a directed system when the binary relation $\leq$ on S is defined by $s \leq t$ if and only if there exists $u \in S$ such that $t = u + s$. We know that if S is a commutative semigroup and μ is an invariant mean on X, then

$$\liminf_{s \in S} f(s) \leq \mu(f) \leq \limsup_{s \in S} f(s)$$

for each $f \in X$; see [19,22] for more details.

Let T be a mapping of C into itself. Then T is said to be *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for each $x, y \in C$. Let S be a commutative semigroup with identity 0 and let $Nonex(C)$ be the set of all nonexpansive mappings of C into itself. Then $\mathcal{S} = \{T(s) : s \in S\}$ is called *a nonexpansive semigroup* on C if $T(s) \in Nonex(C)$ for each $s \in S$, $T(0) = I$ and $T(s+t) = T(s)T(t)$ for each $s, t \in S$. We denote by $F(\mathcal{S})$ the set of all common fixed points of $\{T(s) : s \in S\}$.

We denote by $l^\infty(S, E)$ the Banach space of all bounded mappings of S into E with supremum norm, and by $l_c^\infty(S, E)$ the subspace of all elements $f \in l^\infty(S, E)$ such that $f(S) = \{f(s) : s \in S\}$ is a relatively weakly compact subset of E. Let X be a subspace of $l^\infty(S)$ containing 1 such that $l(s)X \subset X$ for each $s \in S$, let μ be a mean on X and let $\mathcal{S} = \{T(s) : s \in S\}$ be a nonexpansive semigroup on C such that $T(\cdot)x \in l_c^\infty(S, E)$ for some $x \in C$. If for each $x^* \in E^*$ the function $s \mapsto \langle T(s)x, x^* \rangle$ is contained in X, then it follows from the bipolar theorem that there exists a unique point x_0 of E such that $\mu\langle T(\cdot)x, x^* \rangle = \langle x_0, x^* \rangle$ for each $x^* \in E^*$; see [18] and [5]. We denote such a point x_0 by $T_\mu x$. Note that $T_\mu x$ is contained in the closure of convex hull of $\{T(s)x : s \in S\}$; see [18] for more details.

To obtain the main result, we need the following lemma [16].

Lemma 1. *Let $\{z_n\}$ and $\{w_n\}$ be bounded sequences in a Banach space E and let $\{\alpha_n\}$ be a sequence in $[0, 1]$ such that*

$$0 < \liminf_{n \to \infty} \alpha_n \leq \limsup_{n \to \infty} \alpha_n < 1.$$

Suppose that $z_{n+1} = \alpha_n w_n + (1 - \alpha_n) z_n$ for each $n \geq 0$ and

$$\limsup_{n \to \infty} (\|w_n - w_{n+k}\| - \|z_n - z_{n+k}\|) \leq 0$$

for each $k \geq 0$. Then $\liminf_{n \to \infty} \|w_n - z_n\| = 0$.

3. Main Result

For proving our main theorem, we need the following proposition.

Proposition 1. *Let E be a Banach space, let C be a compact convex subset of E, let S be a commutative semigroup with identity, let $\mathcal{S} = \{T(s) : s \in S\}$ be a nonexpansive semigroup on C, let X be a subspace of $l^\infty(S)$ containing* 1 *such that $l(s)X \subset X$ for each $s \in S$ and the functions $s \mapsto \langle T(s)x, x^* \rangle$ and $s \mapsto \|T(s)x - y\|$ are contained in X for each $x, y \in C$ and $x^* \in E^*$ and let $\{\mu_n\}$ be an asymptically invariant sequence of means on X. If $z \in C$ and $\liminf_{n\to\infty} \|T_{\mu_n} z - z\| = 0$, then z is a common fixed point of $\mathcal{S}$.*

Proof. From $\liminf_{n\to\infty} \|T_{\mu_n} z - z\| = 0$, there exists a subsequence $\{T_{\mu_{n_k}} z\}$ of $\{T_{\mu_n} z\}$ such that $\{T_{\mu_{n_k}} z\}$ converges strongly to z. Since the set of means on X is compact in the weak-star topology, there exists a subnet $\{\mu_{n_{k_\alpha}} : \alpha \in \Lambda\}$ of $\{\mu_{n_k}\}$ such that $\{\mu_{n_{k_\alpha}}\}$ converges to μ in the weak-star topology. Then, it follows that μ is an invariant mean on X. In fact, since $\{\mu_n\}$ is asymptotically invariant, for each $\epsilon > 0$, $f \in X$ and $s \in S$, there exists $\alpha_0 \in \Lambda$ such that

$$|\mu_{n_{k_\alpha}}(f) - \mu_{n_{k_\alpha}}(l(s)f)| \leq \frac{\epsilon}{3}$$

for each $\alpha \geq \alpha_0$. Since $\{\mu_{n_{k_\alpha}}\}$ converges to μ in the weak-star topology, we can choose $\beta \geq \alpha_0$ such that

$$|\mu_{n_{k_\beta}}(f) - \mu(f)| \leq \frac{\epsilon}{3}$$

and

$$|\mu_{n_{k_\beta}}(l(s)f) - \mu(l(s)f)| \leq \frac{\epsilon}{3}.$$

Hence, we have

$$\begin{aligned}
|\mu(f) - \mu(l(s)f)| &\leq |\mu(f) - \mu_{n_{k_\beta}}(f)| + |\mu_{n_{k_\beta}}(f) - \mu_{n_{k_\beta}}(l(s)f)| \\
&\quad + |\mu_{n_{k_\beta}}(l(s)f) - \mu(l(s)f)| \\
&\leq \tfrac{\epsilon}{3} + \tfrac{\epsilon}{3} + \tfrac{\epsilon}{3} = \epsilon.
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we have $\mu(f) = \mu(l(s)f)$ for each $f \in X$ and $s \in S$. This implies that μ is an invariant mean on X. It also follows that $\{T_{\mu_{n_{k_\alpha}}} z\}$ converges weakly to $\{T_\mu z\}$. In fact, for each $x^* \in E^*$, we have

$$\langle T_{\mu_{n_{k_\alpha}}} z, x^* \rangle = \mu_{n_{k_\alpha}} \langle T(\cdot)z, x^* \rangle \to \mu \langle T(\cdot)z, x^* \rangle = \langle T_\mu z, x^* \rangle.$$

On the other hand, since $\{T_{\mu_{n_k}} z\}$ converges strongly to z, we have $\langle z, x^* \rangle = \langle T_\mu z, x^* \rangle$. This implies $z = T_\mu z$.

Let $d = \mu\|T(\cdot)z - z\|$. Assume $d > 0$. Since μ is an invariant mean on X, we have

$$d = \mu\|T(\cdot)z - z\| \le \limsup_{s \in S} \|T(s)z - z\|.$$

Since C is compact, there exists a cluster point u_1 of $\{T(s)z : s \in S\}$ in the norm topology such that $d \le \|u_1 - z\|$. Then, we have

$$\begin{aligned} d &\le \|u_1 - z\| \\ &= \|u_1 - T_\mu z\| \\ &= \sup\{\langle u_1 - T_\mu z, x^*\rangle : x^* \in E^*, \|x^*\| = 1\} \\ &= \sup\{\mu\langle u_1 - T(\cdot)z, x^*\rangle : x^* \in E^*, \|x^*\| = 1\} \\ &\le \sup\{\mu\|u_1 - T(\cdot)z\|\|x^*\| : x^* \in E^*, \|x^*\| = 1\} \\ &= \mu\|u_1 - T(\cdot)z\|. \end{aligned}$$

On the other hand, we have $\|T(t)z - z\| \le d$ for each $t \in S$. In fact, since μ is an invariant mean on X, for each $t \in S$, we have

$$\begin{aligned} \|T(t)z - z\| &= \|T(t)z - T_\mu z\| \\ &\le \mu\|T(t)z - T(\cdot)z\|. \end{aligned}$$

Putting $g(s) = \|T(t)z - T(s)z\|$ for each $s \in S$, we have $(l(t)g)(s) = \|T(t)z - T(t+s)z\|$ and hence

$$\begin{aligned} \mu\|T(t)z - T(\cdot)z\| &= \mu(g) = \mu(l(t)g) \\ &= \mu(l(t)\|T(t)z - T(\cdot)z\|) \\ &= \mu\|T(t)z - T(t+\cdot)z\| \\ &\le \mu\|z - T(\cdot)z\| \\ &= d. \end{aligned}$$

Let $\epsilon > 0$ and $s \in S$. Since u_1 is a cluster point of $\{T(s)z : s \in S\}$, there exists $p \ge s$ such that $\|T(p)z - u_1\| \le \epsilon/2$. Then, for each $t \ge p$ with $t = p + w \in p + S$, we have

$$\begin{aligned} \|T(t)z - u_1\| &\le \|T(t)z - T(p)z\| + \|T(p)z - u_1\| \\ &\le \|T(w)z - z\| + \|T(p)z - u_1\| \\ &\le d + \epsilon/2. \end{aligned}$$

Since μ is an invariant mean on X, we have

$$\begin{aligned} & \mu(\|T(p+\cdot)z - z\| + \|T(p+\cdot)z - u_1\|) \\ = \; & \mu\|T(p+\cdot)z - z\| + \mu\|T(p+\cdot)z - u_1\| \\ = \; & \mu\|T(\cdot)z - z\| + \mu\|T(\cdot)z - u_1\| \\ \geq \; & 2d. \end{aligned}$$

Therefore, there exists $q \geq p$ such that

$$\|T(q)z - z\| + \|T(q)z - u_1\| \geq 2d - \epsilon/2.$$

So, we have

$$\begin{aligned} \|T(q)z - u_1\| & \geq 2d - \epsilon/2 - \|T(q)z - z\| \\ & \geq 2d - \epsilon/2 - d \\ & \geq d - \epsilon/2 \\ & > d - \epsilon \end{aligned}$$

and

$$\begin{aligned} \|T(q)z - z\| & \geq 2d - \epsilon/2 - \|T(q)z - u_1\| \\ & \geq 2d - \epsilon/2 - (d + \epsilon/2) \\ & = d - \epsilon. \end{aligned}$$

Denote by $S(t)$ the closure of the set $\{T(s)z : t \leq s\}$ and set

$$A = \bigcap_{t \in S} S(t).$$

For $\epsilon > 0$ and $u \in C$, we also put

$$A(u, \epsilon) = \{x \in C : \|x - u\| \geq d - \epsilon\}.$$

It follows from the above argument that the family of closed subsets consisting of $\{S(t) : t \in S\}$ and $\{A(u, \epsilon) : u \in \{z, u_1\}$ and $\epsilon > 0\}$ has the finite intersection property. Since C is compact, there exists a point u_2 of A such that $d \leq \|u_2 - u_1\|$ and $d \leq \|u_2 - z\|$. We also have

$$d \leq \|u_2 - z\| = \|u_2 - T_\mu z\| \leq \mu\|u_2 - T(\cdot)z\|.$$

Let $\epsilon > 0$ and $s \in S$. Since u_1 and u_2 are a cluster point of $\{T(s)z : s \in S\}$, there exist $p_1, p_2 \geq s$ such that $\|T(p_1)z - u_1\| \leq \epsilon/3$ and $\|T(p_2)z - u_2\| \leq \epsilon/3$.

Then, for each $t \geq p_1 + p_2$, we get $w_1, w_2 \in S$ such that $t = w_1 + p_1$ and $t = w_2 + p_2$. So, we have, for each $i = 1, 2$,

$$\begin{aligned} \|T(t)z - u_i\| &\leq \|T(t)z - T(p_i)z\| + \|T(p_i)z - u_i\| \\ &\leq \|T(w_i)z - z\| + \|T(p_i)z - u_i\| \\ &\leq d + \epsilon/3. \end{aligned}$$

Choose $p \in S$ with $p \geq p_1 + p_2$. Since μ is an invariant mean on X, we have

$$\begin{aligned} & \mu(\|T(p+\cdot)z - z\| + \|T(p+\cdot)z - u_1\| + \|T(p+\cdot)z - u_2\|) \\ = \ & \mu\|T(p+\cdot)z - z\| + \mu\|T(p+\cdot)z - u_1\| + \mu\|T(p+\cdot)z - u_2\| \\ = \ & \mu\|T(\cdot)z - z\| + \mu\|T(\cdot)z - u_1\| + \mu\|T(\cdot)z - u_2\| \\ \geq \ & 3d. \end{aligned}$$

Therefore, there exists $q \geq p$ such that

$$\|T(q)z - z\| + \|T(q)z - u_1\| + \|T(q)z - u_2\| \geq 3d - \epsilon/3.$$

So, we have $\|T(q)z - z\| \geq d - \epsilon$ and $\|T(q)z - u_i\| \geq d - \epsilon$ $(i = 1, 2)$. Then, it follows that the family of closed subsets consisting of $\{S(t) : t \in S\}$ and $\{A(u, \epsilon) : u \in \{z, u_1, u_2\}$ and $\epsilon > 0\}$ has the finite intersection property. Since C is compact, there exists a point u_3 of A such that $d \leq \|u_i - u_j\|$ and $d \leq \|u_i - z\|$ for each $i, j = 1, 2, 3$ with $i \neq j$.

In the similar way, we can define inductively a sequence $\{u_n\}$ in A such that for each i, j with $i \neq j$, $d \leq \|u_i - u_j\|$ and for each $i \geq 1$, $d \leq \|u_i - z\|$. Since C is compact, this is a contradiction. Hence, we have $d = 0$.

Let $t \in S$. Then, we have, for each $s \in S$,

$$\|T(t)z - z\| \leq \|T(t)z - T(t+s)z\| + \|T(t+s)z - z\|.$$

Since μ is an invariant mean on X, we have

$$\begin{aligned} \|T(t)z - z\| &\leq \mu\|T(t)z - T(t+\cdot)z\| + \mu\|T(t+\cdot)z - z\| \\ &\leq \mu\|z - T(\cdot)z\| + \mu\|T(\cdot)z - z\| \\ &= d + d = 0. \end{aligned}$$

Then, z is a common fixed point of $\mathcal{S}$. This completes the proof. ■

Now, using Lemma 1 and Proposition 1, we can prove a strong convergence theorem of Mann's type for general commutative nonexpansive semigroups on a compact convex subset of a Banach space.

Theorem 1. *Let E be a Banach space, let C be a compact convex subset of E, let S be a commutative semigroup with identity, let $\mathcal{S} = \{T(s) : s \in S\}$ be a nonexpansive semigroup on C, let X be a subspace of $l^\infty(S)$ containing 1 such that $l(s)X \subset X$ for each $s \in S$ and the functions $s \mapsto \langle T(s)x, x^*\rangle$ and $s \mapsto \|T(s)x - y\|$ are contained in X for each $x, y \in C$ and $x^* \in E^*$ and let $\{\mu_n\}$ be an asymptotically invariant sequence of means on X such that $\lim_{n\to\infty} \|\mu_n - \mu_{n+1}\| = 0$. Let $\{\alpha_n\}$ be a sequence in $[0, 1]$ such that*

$$0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \alpha_n < 1.$$

Let $x_0 \in C$ and let $\{x_n\}$ be the sequence defined by

$$x_{n+1} = \alpha_n T_{\mu_n} x_n + (1 - \alpha_n)x_n, \quad n = 0, 1, 2, \ldots.$$

Then $\{x_n\}$ converges strongly to a common fixed point of $\mathcal{S}$.

Proof. For each $k, n \in \mathbb{N}$, we have

$$\begin{aligned}
&\|T_{\mu_n} x_n - T_{\mu_{n+k}} x_{n+k}\| - \|x_n - x_{n+k}\| \\
\le\ &\|T_{\mu_n} x_n - T_{\mu_n} x_{n+k}\| + \|T_{\mu_n} x_{n+k} - T_{\mu_{n+k}} x_{n+k}\| - \|x_n - x_{n+k}\| \\
\le\ &\|x_n - x_{n+k}\| + \|T_{\mu_n} x_{n+k} - T_{\mu_{n+k}} x_{n+k}\| - \|x_n - x_{n+k}\| \\
=\ &\|T_{\mu_n} x_{n+k} - T_{\mu_{n+k}} x_{n+k}\|.
\end{aligned}$$

Let $M = \sup_{x\in C} \|x\|$. Then, we have, for each $x^* \in E^*$ with $\|x^*\| = 1$,

$$\begin{aligned}
|\langle T_{\mu_n} x_{n+k} - T_{\mu_{n+k}} x_{n+k}, x^*\rangle| &= |\langle T_{\mu_n} x_{n+k}, x^*\rangle - \langle T_{\mu_{n+k}} x_{n+k}, x^*\rangle| \\
&= |\mu_n \langle T(\cdot)x_{n+k}, x^*\rangle - \mu_{n+k}\langle T(\cdot)x_{n+k}, x^*\rangle| \\
&= |(\mu_n - \mu_{n+k})\langle T(\cdot)x_{n+k}, x^*\rangle| \\
&\le \|\mu_n - \mu_{n+k}\| \sup_{s\in S} |\langle T(s)x_{n+k}, x^*\rangle| \\
&\le \|\mu_n - \mu_{n+k}\| \sup_{s\in S} \|T(s)x_{n+k}\|\|x^*\| \\
&\le M\|\mu_n - \mu_{n+k}\|
\end{aligned}$$

and hence

$$\|T_{\mu_n} x_{n+k} - T_{\mu_{n+k}} x_{n+k}\| \le M\|\mu_n - \mu_{n+k}\|.$$

From $\lim_{n\to\infty} \|\mu_n - \mu_{n+1}\| = 0$, we have

$$\begin{aligned}
&\limsup_{n\to\infty}(\|T_{\mu_n} x_n - T_{\mu_{n+k}} x_{n+k}\| - \|x_n - x_{n+k}\|) \\
\le\ &\lim_{n\to\infty} \|T_{\mu_n} x_{n+k} - T_{\mu_{n+k}} x_{n+k}\| \\
=\ &0.
\end{aligned}$$

Hence, it follows from Lemma 1 that $\liminf_{n\to\infty} \|T_{\mu_n}x_n - x_n\| = 0$. Thus, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\lim_{k\to\infty} \|T_{\mu_{n_k}}x_{n_k} - x_{n_k}\| = 0$. Further, since C is compact, there exist a subsequence $\{x_{n_{k_i}}\}$ of $\{x_{n_k}\}$ and a point z of C such that $\{x_{n_{k_i}}\}$ converges strongly to z. Without loss of generality, we assume that there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and a point $z \in C$ such that $\{x_{n_k}\}$ converges strongly to z and $\lim_{k\to\infty} \|T_{\mu_{n_k}}x_{n_k} - x_{n_k}\| = 0$. Then since

$$\begin{aligned} \|T_{\mu_{n_k}}z - z\| &\le \|T_{\mu_{n_k}}z - T_{\mu_{n_k}}x_{n_k}\| + \|T_{\mu_{n_k}}x_{n_k} - x_{n_k}\| + \|x_{n_k} - z\| \\ &\le 2\|x_{n_k} - z\| + \|T_{\mu_{n_k}}x_{n_k} - x_{n_k}\|, \end{aligned}$$

we have $\lim_{k\to\infty} \|T_{\mu_{n_k}}z - z\| = 0$ and hence $\liminf_{n\to\infty} \|T_{\mu_n}z - z\| = 0$. Therefore, it follows from Proposition 1 that z is a common fixed point of $\mathcal{S}$. Since

$$\begin{aligned} \|x_{n+1} - z\| &\le \alpha_n\|T_{\mu_n}x_n - z\| + (1-\alpha_n)\|x_n - z\| \\ &\le \alpha_n\|x_n - z\| + (1-\alpha_n)\|x_n - z\| \\ &= \|x_n - z\|, \end{aligned}$$

the $\lim_{n\to\infty} \|x_n - z\|$ exists. Thus, we have

$$\lim_{n\to\infty} \|x_n - z\| = \lim_{k\to\infty} \|x_{n_k} - z\| = 0.$$

This completes the proof. ■

4. Some Strong Convergence Theorems

In this section, applying the generalized strong convergence theorem of Mann's type for nonexpansive semigroups in Section 3, we obtain some strong convergence theorems in a Banach space. We denote by $\mathbb{N}$, $\mathbb{N}_+$ and $\mathbb{R}_+$ the set of all nonnegative integers, the set of all positive integers and the set of all nonnegative real numbers, respectively.

Theorem 2 ([16]). *Let E be a Banach space, let C be a compact convex subset of E, let S and T be nonexpansive mappings of C into itself with $ST = TS$. Let $x_0 \in C$ and let $\{x_n\}$ be the sequence defined by*

$$x_{n+1} = \frac{\alpha_n}{n^2}\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} S^iT^jx_n + (1-\alpha_n)x_n$$

for each $n \in \mathbb{N}$, where $\{\alpha_n\}$ is a sequence in $[0,1]$ such that

$$0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \alpha_n < 1.$$

Then $\{x_n\}$ converges strongly to a common fixed point of S and T.

Proof. Let $T(k) = S^i T^j$ for each $k = (i, j)$ in $\mathbb{N}^2$. Since S^i and T^j are nonexpansive for each $i, j \in \mathbb{N}$ and $ST = TS$, $\{T(k) : k \in \mathbb{N}^2\}$ is a nonexpansive semigroup on C. For each $n \in \mathbb{N}_+$, let us define

$$\mu_n(x) = \frac{1}{n^2} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} x(i, j)$$

for each $x \in l^\infty(\mathbb{N}^2)$. Then, $\{\mu_n\}$ is an asymptotically invariant sequence of means; for more details, see [22]. Next, for each $x \in C$, $x^* \in E^*$, $k \in \mathbb{N}^2$ and $n \in \mathbb{N}_+$, we have

$$\begin{aligned}\mu_n \langle T(k)x, x^* \rangle &= \frac{1}{n^2} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \langle S^i T^j x, x^* \rangle \\ &= \left\langle \frac{1}{n^2} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} S^i T^j x, x^* \right\rangle.\end{aligned}$$

Then, we have

$$T_{\mu_n} x = \frac{1}{n^2} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} S^i T^j x$$

for each $n \in \mathbb{N}_+$.

On the other hand, for each $x \in l^\infty(\mathbb{N}^2)$ with $\|x\| = 1$, we have

$$\begin{aligned}&|\mu_{n+1}(x) - \mu_n(x)| \\ &= \left| \frac{1}{(n+1)^2} \sum_{i=0}^{n} \sum_{j=0}^{n} x(i, j) - \frac{1}{n^2} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} x(i, j) \right| \\ &\leq \left| \frac{1}{(n+1)^2} \sum_{i=0}^{n} \sum_{j=0}^{n} x(i, j) - \frac{1}{(n+1)^2} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} x(i, j) \right| \\ &\quad + \left| \frac{1}{(n+1)^2} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} x(i, j) - \frac{1}{n^2} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} x(i, j) \right| \\ &\leq \frac{1}{(n+1)^2} \left| \sum_{\substack{i=n \\ \text{or } j=n}} x(i, j) \right| + \left(\frac{1}{n^2} - \frac{1}{(n+1)^2} \right) \left| \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} x(i, j) \right| \\ &\leq \frac{1}{(n+1)^2} \sum_{\substack{i=n \\ \text{or } j=n}} |x(i, j)| + \left(\frac{1}{n^2} - \frac{1}{(n+1)^2} \right) \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} |x(i, j)|\end{aligned}$$

$$\leq \frac{2n}{(n+1)^2} + \left(\frac{1}{n^2} - \frac{1}{(n+1)^2}\right) n^2$$

and hence

$$\|\mu_{n+1} - \mu_n\| \leq \frac{2n}{(n+1)^2} + \left(\frac{1}{n^2} - \frac{1}{(n+1)^2}\right) n^2$$

for each $n \in \mathbb{N}_+$. Thus, we have $\lim_{n\to\infty} \|\mu_{n+1} - \mu_n\| = 0$.

Therefore, it follows from Theorem 1 that $\{x_n\}$ converges strongly to a common fixed point of S and T. This completes the proof. ■

Let $Q = \{q_{n,m}\}_{n,m\in}$ be a matrix satisfying the following conditions:

(a) $\sup_{n\geq 0} \sum_{m=0}^{\infty} |q_{n,m}| < \infty$;

(b) $\sum_{m=0}^{\infty} q_{n,m} = 1$ for each $n \in \mathbb{N}$;

(c) $\lim_{n\to\infty} \sum_{m=0}^{\infty} |q_{n,m+1} - q_{n,m}| = 0$;

(d) $\lim_{n\to\infty} \sum_{m=0}^{\infty} |q_{n+1,m} - q_{n,m}| = 0$.

Then, Q is called a *weighted mean matrix*. Note that a weighted mean matrix Q is a special case of strongly regular matrix; see [10]. If Q is a strongly regular matrix, then for each $m \in \mathbb{N}$, we have $|q_{n,m}| \to 0$ as $n \to \infty$; see [5].

Theorem 3. *Let E be a Banach space, let C be a compact convex subset of E, let T be a nonexpansive mapping of C into itself and let $Q = \{q_{n,m}\}$ be a weighted mean matrix. Let $x_0 \in C$ and let the sequence $\{x_n\}$ be define by*

$$x_{n+1} = \alpha_n \sum_{m=0}^{\infty} q_{n,m} T^m x_n + (1 - \alpha_n) x_n$$

for each $n \in \mathbb{N}$, where $\{\alpha_n\}$ is a sequence in $[0, 1]$ such that

$$0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1.$$

Then $\{x_n\}$ converges strongly to a fixed point of T.

Proof. For each $n \in \mathbb{N}$, let us define

$$\mu_n(x) = \sum_{m=0}^{\infty} q_{n,m} x_m$$

for each $x = (x_0, x_1, x_2, \dots) \in l^\infty(\mathbb{N})$. Then, $\{\mu_n\}$ is an asymptotically invariant sequence of means; for more details, see [22]. Next, for each $x \in C$, $x^* \in E^*$ and $n \in \mathbb{N}$, we have

$$\begin{aligned}\mu_n\langle T^m x, y\rangle &= \sum_{m=0}^{\infty} q_{n,m}\langle T^m x, y\rangle \\ &= \left\langle \sum_{m=0}^{\infty} q_{n,m} T^m x, y \right\rangle.\end{aligned}$$

Then, we have $T_{\mu_n} x = \sum_{m=0}^{\infty} q_{n,m} T^m x$.

On the other hand, for each $x \in l^{\infty}(\mathbb{N})$ with $\|x\| = 1$, we have

$$\begin{aligned}|\mu_n(x) - \mu_{n+1}(x)| &= \left| \sum_{m=0}^{\infty} q_{n,m} x_m - \sum_{m=0}^{\infty} q_{n+1,m} x_m \right| \\ &= \left| \sum_{m=0}^{\infty} (q_{n,m} - q_{n+1,m}) x_m \right| \\ &\le \sum_{m=0}^{\infty} |q_{n,m} - q_{n+1,m}||x_m| \\ &\le \sum_{m=0}^{\infty} |q_{n,m} - q_{n+1,m}|\end{aligned}$$

and hence we have $\|\mu_n - \mu_{n+1}\| \le \sum_{m=0}^{\infty} |q_{n,m} - q_{n+1,m}|$ for each $n \in \mathbb{N}$. Thus, we have $\lim_{n\to\infty} \|\mu_{n+1} - \mu_n\| = 0$.

Therefore, it follows from Theorem 1 that $\{x_n\}$ converges strongly to a fixed point of T. This completes the proof. ∎

Theorem 4 ([17]). *Let E be a Banach space, let C be a compact convex subset of E and let $\mathcal{S} = \{T(t) : t \in \mathbb{R}_+\}$ be a strongly continuous nonexpansive semigroup on C. Let $x_0 \in C$ and let $\{x_n\}$ be the sequence define by*

$$x_{n+1} = \frac{\alpha_n}{t_n} \int_0^{t_n} T(s) x_n ds + (1 - \alpha_n) x_n$$

for each $n \in \mathbb{N}$, where $\{\alpha_n\}$ is a sequence in $[0, 1]$ such that

$$0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \alpha_n < 1$$

and $\{t_n\}$ is an increasing sequence in $(0, \infty]$ such that $\lim_{n\to\infty} t_n = \infty$ and $\lim_{n\to\infty} t_n/t_{n+1} = 1$. Then $\{x_n\}$ converges strongly to a common fixed point of $\mathcal{S}$.

Proof. For $n \in \mathbb{N}$, let us define

$$\mu_n(f) = \frac{1}{t_n} \int_0^{t_n} f(t) dt$$

for each $f \in C(\mathbb{R}_+)$, where $C(\mathbb{R}_+)$ denote the space of all real-valued bounded continuous functions on $\mathbb{R}_+$ with supremum norm. Then, $\{\mu_n\}$ is an asymptotically invariant sequence of means; for more details, see [22]. Further, for each $x \in C$ and $x^* \in E^*$, we have

$$\begin{aligned}\mu_n\langle T(\cdot)x, x^*\rangle &= \frac{1}{t_n}\int_0^{t_n}\langle T(s)x, x^*\rangle ds\\ &= \left\langle \frac{1}{t_n}\int_0^{t_n} T(s)x ds, x^*\right\rangle.\end{aligned}$$

Then, we have

$$T_{\mu_n}x = \frac{1}{t_n}\int_0^{t_n} T(s)x ds.$$

On the other hand, for each $f \in C(\mathbb{R}_+)$ with $\|f\| = 1$, we have

$$\begin{aligned}|\mu_{n+1}(f) - \mu_n(f)| &= \left|\frac{1}{t_{n+1}}\int_0^{t_{n+1}} f(s)ds - \frac{1}{t_n}\int_0^{t_n} f(s)ds\right|\\ &\leq \left|\frac{1}{t_{n+1}}\int_0^{t_{n+1}} f(s)ds - \frac{1}{t_{n+1}}\int_0^{t_n} f(s)ds\right|\\ &\quad + \left|\frac{1}{t_{n+1}}\int_0^{t_n} f(s)ds - \frac{1}{t_n}\int_0^{t_n} f(s)ds\right|\\ &= \frac{1}{t_{n+1}}\left|\int_{t_n}^{t_{n+1}} f(s)ds\right| + (\frac{1}{t_n} - \frac{1}{t_{n+1}})\left|\int_0^{t_n} f(s)ds\right|\\ &\leq \frac{1}{t_{n+1}}\int_{t_n}^{t_{n+1}} |f(s)|ds + (\frac{1}{t_n} - \frac{1}{t_{n+1}})\int_0^{t_n} |f(s)|ds\\ &\leq \frac{t_{n+1} - t_n}{t_{n+1}} + (\frac{1}{t_n} - \frac{1}{t_{n+1}})t_n\\ &= 2 - 2\frac{t_n}{t_{n+1}}\end{aligned}$$

and hence $\|\mu_{n+1} - \mu_n\| \leq 2 - 2t_n/t_{n+1}$ for each $n \in \mathbb{N}$. Thus, we have $\lim_{n\to\infty}\|\mu_{n+1} - \mu_n\| = 0$.

Therefore, it follows from Theorem 1 that $\{x_n\}$ converges strongly to a common fixed point of $\mathcal{S}$. This completes the proof. ■

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