TAIWANESE JOURNAL OF MATHEMATICS
Vol. 8, No. 4, pp. 761-770, December 2004
This paper is available online at http://www.math.nthu.edu.tw/tjm/

ONE-SIDED UNIT-REGULAR IDEALS OF REGULAR RINGS

Huanyin Chen

Abstract. In this paper, we investigate one-sided unit-regular ideals of regular rings. Let I be a purely infinite, simple and essential ideal of a regular ring R. It is shown that R is one-sided unit-regular if and only if so is R/I. Also we prove that every square matrix over one-sided unit-regular ideals of regular rings admits a diagonal matrix with idempotent entries.

Let R be an associative ring with identity. We say that R is a regular ring provided that for every $x \in R$ there exists $y \in R$ such that $x = xyx$(cf. [10]). We say that R is an one-sided unit-regular ring provided that for every $x \in R$ there exists right or left invertible $u \in R$ such that $x = xux$(see [9]). In [6, Corollary 7], the author proved that one-sided unit-regularity is Morita invariant. In addition, the author proved that a regular ring is one-sided unit-regular if and only if for all finitely generated projective right R-modules A, B and C, if $A \oplus B \cong A \oplus C$, then $B \lesssim C$ or $C \lesssim B$ (see [6, Theorem 8]). Also the author proved that every element in one-sided unit-regular rings is a product of an idempotent and a right or left invertible element of R (see [5, Theorem 4]).

In this paper, we investigate one-sided unit-regular ideals of regular rings. Let I be a purely infinite, simple and essential ideal of a regular ring R. Then I is one-sided unit-regular. Furthermore, we show that R is one-sided unit-regular if and only if so is R/I. Also we prove that every square matrix over one-sided unit-regular ideals of regular rings admits a diagonal matrix with idempotent entries.

Throughout this paper, We assume that all rings are associative with identity and all modules are right unital modules. We say that an element $u \in R$ is weak-invertible if there exist $a, b \in R$ such that $au = 1$ or $ub = 1$. Let $R_{<}^{-1}$ denote the set of all weak-invertible elements of R. If A and B are R-modules, the notation $B \lesssim A$ means that B is isomorphic to a submodule of A.

Lemma 1. *Let R be a ring with $u \in R$. Then the following are equivalent*:

Received November 5, 2002; accepted July 31, 2003.
Communicated by Pjek-Hwee Lee.
2000 *Mathematics Subject Classification*: 16E50, 16U99.
Key words and phrases: One-sided Unit-regularity, Purely infinite simple ideal, Diagonalization.

(1) u *is weak-invertible.*

(2) *There exists* $v \in R$ *such that* $uv = 1$ *or* $vu = 1$.

Proof. $(2) \Rightarrow (1)$ is trivial.

$(1) \Rightarrow (2)$. Since u is weak-invertible, there exist $a, b \in R$ such that $ua = 1$ or $bu = 1$. Set $v = a + b - bua$. Then $v = b$ (if $bu = 1$) or $v = a$ (if $ua = 1$). Therefore either $vu = 1$ or $uv = 1$. ∎

Let $u \in R$ be weak-invertible. Then we have some $v \in R$ such that $uv = 1$ or $vu = 1$. We denote v by $u_<^{-1}$. We note that $u_<^{-1}$ is not unique. In fact, if we have a fixed weak-inverse $u_<^{-1}$, then $uv = 1$ or $vu = 1$ if and only if there exist $a, b \in R$ such that $v = u_<^{-1} + a(1 - uu_<^{-1}) + (1 - u_<^{-1}u)b$. In the sequel, we will always choose some fixed weak-inverse.

Suppose that I is an ideal of a regular ring R. We say that I is one-sided unit-regular in case $aR + bR = R$ with $a \in 1 + I, b \in R$ implies that $a + by \in R_<^{-1}$ for a $y \in R$. Obviously, The one-sided unit-regularity for ideals of regular rings is a nontrivial generalization of the one-sided unit-regular rings. In [7, Theorem 2.9], the author showed that an ideal I of a regular ring R is one-sided unit-regular if and only if eRe is one-sided unit-regular for all idempotents $e \in I$. Also I is one-sided unit-regular if and only if for every $x \in 1 + I$, there exists $u \in R_<^{-1}$ such that $x = xux$ (cf. [7, Theorem 2.3]).

We say that $a \overset{\sim}{=} b$ via $1 + I$ provided that there exist $x, y, z \in 1 + I$ such that $a = zbx, b = xay, x = xyx = xzx$. We now extend [11, Theorem] and characterize one-sided unit-regularity for ideals of regular rings by pseudo-similarity.

Lemma 2. *Let* R *be a ring with* $a, b \in R$. *Then the following are equivalent*:

(1) $a \overset{\sim}{=} b$ *via* $1 + I$.

(2) *There exist some* $x, y \in 1 + I$ *such that* $a = xby, b = yax, x = xyx$ *and* $y = yxy$.

Proof. (2)⇒(1) is trivial.

(1)⇒(2). Since $a \overset{\sim}{=} b$ via $1 + I$, there are $x, y, z \in 1 + I$ such that $b = xay, zbx = a$ and $x = xyx = xzx$. By replacing y with yxy and z with zxz, we can assume $y = yxy$ and $z = zxz$. Clearly, $xazxy = xzbxzxy = xzbxy = xay = b, zxybx = zxyxayx = zxayx = zbx = a, zxy = zxyxzxy$ and $x = xzxyx$. Obviously, $1 + I$ is a submonoid of $(R, \cdot)$ and so $zxy \in 1 + I$ which completes the proof. ∎

Theorem 3. *Let* I *be an ideal of a regular ring* R. *Then the following are equivalent*:

(1) I *is one-sided unit-regular.*

(2) *Whenever* $a \eqsim b$ *via* $1+I$, *there exists weak-invertible* $u \in R$ *such that* $a = ubu_<^{-1}$.

Proof. (1)⇒(2). Suppose that $a \eqsim b$ via $1+I$. By Lemma 2, there exist $x, y \in 1+I$ such that $a = xby, b = yax, x = xyx$ and $y = yxy$. Since I is one-sided unit-regular, we have $u \in R_<^{-1}$ such that $y = yuy$. Set $w = y + (1-yu)u_<^{-1}(1-uy)$. Then $yuw = y$. Clearly, $1 - uw = (1-uy)(1-uu_<^{-1})$ and $1 - wu = (1-u_<^{-1}u)(1-yu)$. Set $k = (1-xy-uy)u(1-yx-yu), l = (1-yx-yu)w(1-xy-uy)$. Then $1-kl = (1-xy-uy)(1-uw)(1-xy-uy)$ and $1-lk = (1-yx-yu)(1-wu)(1-xy-uy)$; hence, $l = k_<^{-1}$. Furthermore, $kbk_<^{-1} = (1-xy-uy)u(1-yx-yu)b(1-yx-yu)w(1-xy-uy) = (1-xy-uy)(u-uyx-uyu)by = xyuby = xby = a$, as required.

(2)⇒(1). Given any $x \in 1+I$, there exists $y \in R$ such that $x = xyx$ and $y = yxy$. Clearly, we have $R = yxR \oplus (1-yx)R = xyR \oplus (1-xy)R$ and an isomorphism $\eta : xyR = xR \cong yxR$ given by $\eta(xr) = yxr$ for any $r \in R$. Clearly, $xy = x(yx)y, yx = y(xy)x, x = xyx, y = yxy$ and $x, y \in 1+I$. Hence $xy \eqsim yx$ via $1+I$, and then we have $u \in R_<^{-1}$ such that $yx = uxyu_<^{-1}$. Construct maps $\alpha : (1-xy)R \to (1-yx)R$ given by $(1-xy)r \to (1-yx)u(1-xy)r$ for any $r \in R$ and $\beta : (1-yx)R \to (1-xy)R$ given by $(1-yx)r \to (1-xy)u_<^{-1}(1-yx)r$ for any $r \in R$. Define $\phi : R = xR \oplus (1-xy)R \to yxR \oplus (1-yx)R$ given by $\phi(x_1 + x_2) = \eta(x_1) + \alpha(x_2)$ for any $x_1 \in xR, x_2 \in (1-xy)R$ and $\psi : R = yxR \oplus (1-yx)R \to xR \oplus (1-xy)R = R$ given by $\psi(y_1+y_2) = \eta^{-1}(y_1) + \beta(y_2)$ for any $y_1 \in yxR, y_2 \in (1-yx)R$.

If $uu_<^{-1} = 1$, then $(1-\phi\psi)(y_1+y_2) = (1-yx)(1-uu_<^{-1})y_2$ for any $y_1 \in yxR$ and $y_2 \in (1-yx)R$. So $\phi\psi = 1$.

If $u_<^{-1}u = 1$, then $(1-\psi\phi)(x_1+x_2) = (1-xy)(1-u_<^{-1}u)x_2$ for any $x_1 \in xR$ and $x_2 \in (1-xy)R$. So $\psi\phi = 1$. Thus we see that $\phi \in R_<^{-1}$. One easily checks that $x = x\phi x$. Therefore I is one-sided unit-regular by [7, Theorem 2.3]. ∎

Let $e, f \in R$ be idempotents. It is well known that $eR \cong fR$ if and only if there exist $a \in eRf, b \in fRe$ such that $e = ab$ and $f = ba$. If $a, b \in 1+I$, we say that $eR \cong fR$ via $1+I$.

Corollary 4. *Let* I *be an ideal of a regular ring* R. *Then the following are equivalent*:

(1) I *is one-sided unit-regular.*

(2) *For any idempotents* $e, f \in R$, $eR \cong fR$ *via* $1+I$ *implies that there exists* $u \in R_<^{-1}$ *such that* $e = ufu_<^{-1}$.

Proof. (1)⇒(2). Suppose that $eR \cong fR$ via $1+I$. Then there exist $a, b \in 1+I$ such that $e = ab$ and $f = ba$, where $a \in eRf, b \in fRe$. Clearly, $e = afb, f =$

$bea, a = aba$ and $b = bab$. That is, $e \overline{\sim} f$ via $1 + I$. According to Theorem 3, we have $u \in R_{<}^{-1}$ such that $e = ufu_{<}^{-1}$.

(2)⇒(1) is obtained by the proof of "$(2) \Rightarrow (1)$" in Theorem 3. ■

Corollary 5. *Let R be a regular ring. Then the following are equivalent*:

(1) *R is one-sided unit-regular.*

(2) *Whenever $a \overline{\sim} b$ with $a, b \in R$, there exists weak-invertible $u \in R$ such that $a = ubu_{<}^{-1}$.*

(3) *Whenever $eR \cong fR$ with idempotents $e, f \in R$, there exists weak-invertible $u \in R$ such that $e = ufu_{<}^{-1}$.*

Proof. We choose $I = R$. Then the result follows by Theorem 3 and Corollary 4. ■

In order to investigate the diagonal reduction of matrices over one-sided unit-regular ideals over regular rings, we extend [12, Lemma 1.1] as follows.

Lemma 6. *Let I be an ideal of a regular ring R and $x_1, x_2, \dots, x_m \in I$. Then there exists an idempotent $e \in I$ such that $x_i \in eRe$ for all $i = 1, 2, \dots, m$.*

Proof. Clearly there exist idempotents $u, v \in I$ such that $uR = \sum_{i=1}^{m} x_iR$ and $Rv = \sum_{i=1}^{m} Rx_i$. It is enough to show that there exists $e = e^2 \in I$ with $eu = u = ue$ and $ev = v = ve$. Next, $fR = uR + vR$ for some $f = f^2 \in I$. Clearly, $fu = u$ and $fv = v$. Set $g = f + u(1 - f)$. Obviously, $g^2 = g \in I$, $ug = u = gu$ and $gv = v$. It is enough to show that there exists $e = e^2 \in I$ with $eg = g = ge$ and $ev = v = ve$. Pick an idempotent $h \in I$ with $Rg + Rv = Rh$. Clearly, $gh = g$ and $vh = v$. Set $e = h + (1 - h)g$. Obviously, $e = e^2 \in I$, $eg = g = ge$ and $ev = v = ve$ because $gv = v$. ■

Theorem 7. *Let I be an ideal of a regular ring R. If I is one-sided unit-regular*, *then $M_n(I)$ is one-sided unit-regular as an ideal of $M_n(R)$.*

Proof. In view of [7, Theorem 2.9] it is enough to show that $WM_n(R)W$ is one-sided unit-regular for any idempotent $W = (w_{ij})_{i,j=1}^{n} \in M_n(I)$. By Lemma 6 there exists an idempotent $e \in I$ with $ew_{ij}e \in eRe$ for all i, j. Let E be the idempotent of $M_n(R)$ whose diagonal entries are equal to e while all the other ones are equal to 0. Obviously, $W \in EM_n(R)E = M_n(eRe)$. Next, by [7, Theorem 2.9], eRe is one-sided unit-regular and so [6, Corollary 7] yields that $M_n(eRe)$ is one-sided unit-regular. As $WM_n(R)W = WEM_n(R)EW = WM_n(eRe)W$, the result follows from [7, Theorem 2.9]. ■

Corollary 8. *Let I be an one-sided unit-regular ideal of a regular ring R. Then for any $A \in M_n(I)$, there exist idempotent matrix E and weak-invertible matrix U such that $A = EU$.*

Proof. Given $A \in M_n(I)$, then we have $B \in M_n(R)$ such that $A = ABA$ and $B = BAB$. Since $\big(A + (I_n - AB)\big)B + (I_n - AB)(I_n - B) = I_n$, it follows by Theorem 7 that $A + (I_n - AB) + (I_n - AB)(I_n - B)Y = U \in M_n(R)^{-1}_{<}$ such that $A = ABA = AB\big(A + (I_n - AB) + (I_n - AB)(I_n - B)Y\big) = EU$, where $E = AB = E^2 \in M_n(I)$. ∎

Denote by $FP(I)$ the set of finitely generated projective right R-module P such that $P = PI$.

Theorem 9. *Let I be an ideal of a regular ring R. Then the following are equivalent*:

(1) *I is one-sided unit-regular.*

(2) *For all $A \in FP(I)$, $A \oplus B \cong A \oplus C$ implies $B \lesssim C$ or $C \lesssim B$ for any right R-modules B and C.*

(3) *For any $A, B, C \in FP(I), A \oplus B \cong A \oplus C$ implies $B \lesssim C$ or $C \lesssim B$.*

Proof. $(1) \Rightarrow (2)$ Given $A \oplus B \cong A \oplus C$ with $A, B, C \in FP(I)$, we have idempotents $e_1, \cdots, e_n \in I$ such that $A \cong e_1R \oplus \cdots \oplus e_nR \cong diag(e_1, \cdots, e_n)R^n$. Clearly, $End_R(A) \cong diag(e_1, \cdots, e_n)M_n(R)diag(e_1, \cdots, e_n)$. By Theorem 7, $M_n(I)$ is one-sided unit-regular as an ideal of $M_n(R)$. According to [7, Theorem 2.9], $End_R(A)$ is one-sided unit-regular. It follows by [6, Proposition 2] that either $B \lesssim C$ or $C \lesssim B$.

$(2) \Rightarrow (3)$ is trivial.

$(3) \Rightarrow (1)$ Let $e \in I$ be an idempotent. Suppose that $A \oplus B \cong A \oplus C$ with $A, B, C \in FP(eRe)$. Then we have $A \bigotimes_{eRe} eR \oplus B \bigotimes_{eRe} eR \cong A \bigotimes_{eRe} eR \oplus C \bigotimes_{eRe} eR$. Clearly, $A \otimes_{eRe} eR, B \otimes_{eRe} eR, C \otimes_{eRe} eR \in FP(I)$. By our assumption either there exist an embedding of R-modules $f : B \otimes_{eRe} eR \to C \otimes_{eRe} eR$ or $f : C \otimes_{eRe} eR \to B \otimes_{eRe} eR$. Say, $f : B \otimes_{eRe} eR \to C \otimes_{eRe} eR$. Then

$$C \cong C \otimes_{eRe} eRe = (c \otimes_{eRe} eR)e \supseteq f(B \otimes_{eRe} eR)e = f(B \otimes_{eRe} eRe) \cong B$$

and so B can be embedded into C. According to [6, Theorem 8], eRe is one-sided unit-regular. Therefore I is one-sided unit-regular from [7, Theorem 2.9]. ∎

Set $cr(R^{-1}_{<}) = \{a \in R \mid$ If $ax + b = 1$ in R, then there exists $y \in R$ such that $a + by \in R^{-1}_{<}\}$. An element $e \in I$ is infinite if there exist orthogonal idempotents

$f, g \in I$ such that $e = f + g$ while $eR \cong fR$ and $g \neq 0$. A simple ideal I of a ring R is said to be purely infinite if every nonzero right ideal of I (as a ring without units) contains an infinite idempotent.

Lemma 10. *Let I be a purely infinite, simple and essential ideal of a regular ring R. Then $I + R_<^{-1} \subseteq cr(R_<^{-1})$.*

Proof. Suppose that $ax + b = 1$ with $a \in I + R_<^{-1}, x, b \in R$. Then we have $c \in R$ such that $a = aca$. Assume that there exists $u \in R_<^{-1}$ such that $a - u \in I$. If $uv = 1$ for a $v \in R$, then we have $a = aca \in acu + I$, so $av - ac \in I$. Clearly, $1 - av \in I$. Thus $1 - ac = (1 - av) + (av - ac) \in I$. Assume that $1 - ac \neq 0$ and $1 - ca \neq 0$. Since I is essential and simple, we have $(1 - ca)I(1 - ca) \neq 0$. As I is purely infinite and simple, we can find an infinite idempotent $r \in R$ such that $(1 - ac)R \cong rR \subseteq (1 - ca)R$; hence, $(1 - ac)R \lesssim (1 - ca)R$. By the regularity of R, there is an injection $\psi : (1 - ac)R \to (1 - ca)R$. Clearly, $R = caR \oplus (1 - ca)R = acR \oplus (1 - ac)R$ with $\phi : acR = aR \cong caR$ given by $\phi(are) = c(ar)$ for any $ar \in aR$. Define $u \in End_R(R)$ so that u restricts to ϕ and u restricts to ψ. Then $a = aua$ with left invertible $u \in R$. Hence $a \in R$ is one-sided unit-regular.

If $vu = 1$ for a $v \in R$, then $a = aca \in uca + I$, so $va - ca \in I$. Clearly, $1 - va \in I$; hence, $1 - ca = (1 - va) + (va - ca) \in I$. Analogously to the discussion above, we have either $(1 - ca)R \lesssim (1 - ac)R$ or $a \in R_<^{-1}$. Consequently, there is a $u \in R_<^{-1}$ such that $a = aua$. Therefore we always have $u \in R_<^{-1}$ such that $a = aua$. Set $ua = e$. Then $ex + ub = u$, so $e(x + ub) + (1 - e)ub = u$. Since R is regular, we have a $d \in R$ such that $(1 - e)ub = (1 - e)ubd(1 - e)ub$. Set $g = (1 - e)ubd(1 - e)$. Then $e = e^2, g = g^2$ and $eg = ge = 0$. Thus $e(x + ub) + gub = u$; hence $e(x + ub) = eu$ and $gub = gu$. Clearly,

$$
\begin{aligned}
& u\big(a + bd(1 - e)\big)\big(1 - eubd(1 - e)\big)u \\
&= \big(e(1 - eubd(1 - e)) + ubd(1 - e)\big)u \\
&= \big(e + (1 - e)ubd(1 - e)\big)u \\
&= (e + g)u \\
&= u.
\end{aligned}
$$

As $u \in R_<^{-1}$, $a + bd(1 - e) \in R_<^{-1}$. That is, $a \in cr(R_<^{-1})$. so $I + R_<^{-1} \subseteq cr(R_<^{-1})$.■

Let I be a purely infinite, simple and essential ideal of a regular ring R. By Lemma 10, $1 + I \subseteq cr(R_<^{-1})$; hence I is one-sided unit-regular. Using Theorem 9, we conclude that for all $A, B, C \in FP(I)$, $A \oplus B \cong A \oplus C$ implies $B \lesssim C$ or $C \lesssim B$.

Lemma 11. *Let I be an ideal of a regular ring R. Then R is one-sided unit-regular if and only if the following hold*:

(1) R/I *is one-sided unit-regular.*

(2) $(I + R_<^{-1})/I = (R/I)_<^{-1}$.

(3) $I + R_<^{-1} \subseteq cr(R_<^{-1})$.

Proof. Assume that R is one-sided unit-regular. It is easy to verify that R/I is one-sided unit-regular too. Clearly, $(I + R_<^{-1})/I \subseteq (R/I)_<^{-1}$. Let $\pi : R \to R/I$ be the quotient morphism. Given any $\pi(a) \in (R/I)_<^{-1}$, we have some $\pi(b) \in (R/I)_<^{-1}$ such that $\pi(a)\pi(b) = \pi(1)$ or $\pi(b)\pi(a) = \pi(1)$. Since R is one-sided unit-regular, it follows from $ab+(1-ab) = 1$ that $v = b+y(1-ab) \in R_<^{-1}$ for a $y \in R$. Assume that $uv = 1$ or $vu = 1$. Set $w = u + a(1 - vu) + (1 - uv)a$. We see that $wv = 1$ or $vw = 1$. That is, $w \in R_<^{-1}$. Since $\pi(a)\pi(b) = \pi(1)$ or $\pi(b)\pi(a) = \pi(1)$, we show that

$$\begin{aligned}\pi(v)\pi(a)\pi(v) &= \pi\big((b + y(1 - ab))a(b + y(1 - ab))\big)\\ &= \pi\big(ba(b + y(1 - ab))\big)\\ &= \pi\big(b + y(1 - ab)\big)\\ &= \pi(v).\end{aligned}$$

Clearly, $\pi(w) = \pi(u) + \pi(a)\big(\pi(1) - \pi(v)\pi(u)\big) + \big(\pi(1) - \pi(u)\pi(v)\big)\pi(a)$.

If $uv = 1$, then

$$\begin{aligned}\psi(v)\psi(w) &= \psi(v)\pi(u) + \psi(v)\pi(a)\big(\pi(1) - \pi(v)\pi(u)\big)\\ &= \psi(v)\pi(u) + \psi(v)\pi(a) - \psi(v)\pi(a)\pi(v)\pi(u)\\ &= \psi(v)\pi(a).\end{aligned}$$

So we have $\psi(w) = \psi(a)$.

If $vu = 1$, then

$$\begin{aligned}\psi(w)\psi(v) &= \pi(u)\psi(v) + \big(\pi(1) - \pi(u)\pi(v)\big)\psi(a)\psi(v)\\ &= \psi(u)\pi(v) + \psi(a)\pi(v) - \psi(u)\pi(v)\pi(a)\pi(v)\\ &= \psi(a)\pi(v).\end{aligned}$$

We also have $\psi(w) = \psi(a)$. Therefore $(I + R_<^{-1})/I = (R/I)_<^{-1}$. Because R is one-sided unit-regular, we easily get $I + R_<^{-1} \subseteq cr(R_<^{-1})$.

Conversely, assume now that the three conditions are satisfied. Suppose that $ax + b = 1$ in R. Then $\pi(a)\pi(x) + \pi(b) = \pi(1)$ in R/I. Since R/I is one-sided unit-regular, we have some $\pi(y) \in R/I$ such that $\pi(a)+\pi(b)\pi(y) \in (R/I)_<^{-1}$. Thus there exists $w \in R_<^{-1}$ such that $\pi(a)+\pi(b)\pi(y) = \pi(w)$. Hence $a+by-w \in I$, and then $a+by \in I+R_<^{-1}$. From $ax+b=1$, we have $(a+by)x+b(1-yx)=1$. Therefore $a+b\big(y+(1-yx)z\big) = a+by+b(1-yx)z \in R_<^{-1}$, as asserted. ■

In [4, Theorem 1.12], P. Ara et al. showed that if I is a purely infinite, simple and essential exchange ideal, then R is a QB-ring if and only if R/I is a QB-ring and $(R/I)_q^{-1} = (R/I)_r^{-1} \cup (R/I)_l^{-1}$. We now extend this result to one-sided unit-regular rings as follows.

Theorem 12. *Let I be a purely infinite, simple and essential ideal of a regular ring R. Then R is one-sided unit-regular if and only if so is R/I.*

Proof. One direction is clear. Conversely, assume now that R/I is one-sided unit-regular. It suffices to to prove that one-sided invertible elements lift modulo I. Assume that $\overline{xy} = \overline{1}$ in R/I. Since R is regular, we have a $z \in R$ such that $x = xzx$ and $z = zxz$. Clearly, $\overline{xz} = \overline{1}$; hence $1-xz \in I$. If $xz = 1$ or $zx = 1$, then $x \in R_<^{-1}$. So we assume that the idempotents $1-xz, 1-zx$ are both non-zero. As I is essential and simple, $(1-zx)I(1-zx) \neq 0$. On the other hand, I is purely infinite and simple, we can find an infinite idempotent $r \in R$ such that $(1-xz)R \cong rR \subseteq (1-zx)R$, whence $(1-xz)R \lesssim (1-zx)R$. By the regularity of R, we can find $s \in (1-xz)R(1-zx), t \in (1-zx)R(1-xz)R$ such that $1-xz = st$. Clearly, $xt = sz = 0$; hence, $(x+s)(t+z) = xz+st = 1$. That is, $x+s \in R$ is right invertible. Obviously, we have $s \in (1-xz)R(1-zx) \subseteq I$, and then $\overline{x} = \overline{x+s}$. That is, x can be lifted by a right invertible element modulo I. Therefore we complete the proof by Lemma 10 and Lemma 11. ■

Corollary 13. *Every purely infinite, simple regular ring is one-sided unit-regular.*

Proof. Since R is a purely infinitely, simple ideal of R, we get the result by Theorem 12. ■

Lemma 14. *Let I be an one-sided unit-regular ideal of a regular ring R. Then the following hold*:

(1) *For any $A, B \in M_n(I)$, $AM_n(R) = BM_n(R)$ implies that there exists $U \in M_n(R)_<^{-1}$ such that $A = BU$.*

(2) *For any $A, B \in M_n(I)$, $M_n(R)A = M_n(R)B$ implies that there exists $U \in M_n(R)_<^{-1}$ such that $A = UB$.*

Proof. (1) Suppose that $AM_n(R) = BM_n(R)$ with $A, B \in M_n(I)$. Then $A = BX$ and $B = AY$ for $X, Y \in M_n(R)$. Since R is regular, so is $M_n(R)$. Hence

A and B are both regular, so we may assume that $X, Y \in M_n(I)$. Furthermore, we have $B\big(X+(I_n-XY)\big) = BX = A$. Thus we may assume that $X \in I_n + M_n(I)$. Likewise, we may assume that $Y \in I_n + M_n(I)$. Since $XY + (I_n - XY) = I_n$, by Theorem 7, we have $Z \in M_n(R)$ such that $X + (I_n - XY)Z = U \in M_n(R)^{-1}_{<}$. Therefore $A = BX = B\big(X + (I_n - XY)Z\big) = BU$, as asserted.

(2) Clearly, I is one-sided unit-regular as an ideal of R if and only if I^{op} is one-sided unit-regular as an ideal of the opposite ring R^{op}. Thus we get the result by (1). ■

Theorem 15. *Let I be an one-sided unit-regular ideal of a regular ring R. Then for any matrix $A \in M_n(I)$, there exist weak-invertible $U, V \in M_n(R)$ such that $UAV = diag(e_1, \cdots, e_n)$ for idempotents $e_1, \cdots, e_n \in I$.*

Proof. Let $A \in M_n(I)$. Since R is regular, we have $E = E^2 \in M_n(I)$ such that $AM_n(R) = EM_n(R)$. Clearly, ER^n is a generated projective right R-module; hence, there are idempotents $e_1, \cdots, e_n \in I$ such that $ER^n \cong e_1R \oplus \cdots \oplus e_nR \cong diag(e_1, \cdots, e_n)R^n$ as right R-modules, so we have $ER^{n\times 1} \cong diag(e_1, \cdots, e_n)$ $R^{n\times 1}$, where $R^{n\times 1} = \{\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mid x_1, \cdots, x_n \in R\}$ is a right R-module and a left $M_n(R)$-module. Let $R^{1\times n} = \{(x_1, \cdots, x_n) \mid x_1, \cdots, x_n \in R\}$. Then $R^{1\times n}$ is a left R-module and a right $M_n(R)$-module; hence, $(ER^{n\times 1}) \bigotimes_R R^{1\times n} \cong diag(e_1, \cdots, e_n)R^{n\times 1} \bigotimes_R R^{1\times n}$. One easily checks that $R^{n\times 1} \bigotimes R^{1\times n} \cong M_n(R)$ as right $M_n(R)$-modules. So $\psi : AM_n(R) \cong diag(e_1, \cdots, e_n)M_n(R)$ with all $e_i \in R$. Clearly, $M_n(R)A = M_n(R)\psi(A)$ and $\psi(A)M_n(R) = diag(e_1, \cdots, e_n)M_n(R)$. It follows by Lemma 14 that $UA = \psi(A)$ and $\psi(A)V = diag(e_1, \cdots, e_n)$ for some $U, V \in M_n(R)^{-1}_{<}$. Therefore $UAV = diag(e_1, \cdots, e_n)$, as asserted. ■

Corollary 16. *Let I be a purely infinite, simple and essential ideal of a regular ring R. Then for any $A \in M_n(I)$, there exist weak-invertible $U, V \in M_n(R)$ such that $UAV = diag(e_1, \cdots, e_n)$ for idempotents $e_1, \cdots, e_n \in I$.*

Proof. In view of Lemma 10, I is one-sided unit-regular. So the proof is true from Theorem 15. ■

Corollary 17. *Let R be an one-sided unit-regular ring. Then for any $A \in M_n(R)$, there exist weak-invertible $U, V \in M_n(R)$ such that $UAV = diag(e_1, \cdots, e_n)$ for idempotents $e_1, \cdots, e_n \in R$.*

Proof. Letting $I = R$, we get the result by Theorem 15. ■

ACKNOWLEDGEMENTS

The author is grateful to the referee for his/her suggestions which led to the proofs of Lemma 6, Theorem 7, Theorem 9 and helped me to improve the manuscript. This work was supported by the National Natural Science Foundation of Zhejiang Province.

REFERENCES

1. P. Ara, K. R. Goodearl, K. C. O'Meara and E. Pardo, Diagonalization of matrices over regular rings, *Linear Algebra Appl.* **265** (1997), 147-163.
2. P. Ara, K. R. Goodearl and E. Pardo, K_0 of purely infinite simple regular rings, *K-Theory* **26** (2002), 69-100.
3. P. Ara, G. K. Pedersen and F. Perera, An infinite analogue of rings with stable range one, *J. Algebra* **230** (2000), 608-655.
4. P. Ara, G. K. Pedersen and F. Perera, *Extensions and pullbacks in QB-rings*, Preprint, 2002.
5. H. Chen, Elements in one-sided unit regular rings, *Comm. Algebra* **25** (1997), 2517-2529.
6. H. Chen, Comparability of modules over regular rings, *Comm. Algebra* **25** (1997), 3531-3544.
7. H. Chen and F. Li, On ideals of regular rings, *Acta Math. Sinica* **18** (2002), 347-356.
8. H. Chen and M. Chen, Generalized ideal-stable regular rings, *Comm. Algebra* **31** (2003), 4989-5001.
9. G. Ehrlich, Units and one-sided units in regular rings, *Trans. Amer. Math. Soc.* **216** (1976), 81-90.
10. K. R. Goodearl, *Von Neumann Regular Rings*, Pitman, London, San Francisco, Melbourne, 1979; second ed., Krieger, Malabar, Fl. 1991.
11. R. Guralnick and C. Lanski, Pseudosimilarity and cancellation of modules, *Linear Algebra and Appl.* **47** (1982), 111-115.
12. P. Menal and J. Moncasi, Lifting units in self-injective rings and an index theory for Rickart C^*-algebras, *Pacific J. Math.* **126** (1987), 295-329.

Huanyin Chen
Department of Mathematics,
Zhejiang Normal University,
Jinhua 321004,
People's Republic of China
E-mail: chyzxl@sparc2.hunnu.edu.cn