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EQUITABLE LIST COLORING OF GRAPHS

Wei-Fan Wang and Ko-Wei Lih

Abstract. A graph G is equitably k-choosable if, for any k-uniform list assignment L, G admits a proper coloring π such that $\pi(v) \in L(v)$ for all $v \in V(G)$ and each color appears on at most $\lceil |G|/k \rceil$ vertices. It was conjectured in [8] that every graph G with maximum degree Δ is equitably k-choosable whenever $k \geq \Delta + 1$. We prove the conjecture for the following cases: (i) $\Delta \leq 3$; (ii) $k \geq (\Delta - 1)^2$. Moreover, equitably 2-choosable graphs are completely characterized.

1. Introduction

We only consider simple graphs in this paper unless otherwise stated. For a graph G, we denote its vertex set, edge set, order, maximum degree, and minimum degree by $V(G)$, $E(G)$, $|G|$, $\Delta(G)$, and $\delta(G)$, respectively. For a vertex $v \in V(G)$, let $N_G(v)$ denote the set of neighbors of v in G and $d_G(v)$ the degree of v in G. For $S \subseteq V(G)$, we use $G[S]$ to denote the subgraph of G induced by S and simply write $G - S$ for $G[V(G) \setminus S]$. If $G[S]$ does not contain edges, then S is called an *independent set* of G. Let $\alpha(G)$ denote the maximal cardinality of an independent set of G.

A k-*coloring* of a graph G is a mapping π from the vertex set $V(G)$ to the set of colors $\{1, 2, \ldots, k\}$ such that $\pi(x) \neq \pi(y)$ for every edge $xy \in E(G)$. The graph G is k-*colorable* if it has a k-coloring. The *chromatic number* $\chi(G)$ of G is the smallest integer k such that G is k-colorable. A k-coloring π is called m-*bounded* if every color appears on at most m vertices. A coloring π is called *equitable* if the sizes of any two color classes differ by at most 1. Obviously, every equitable k-coloring of a graph G is $\lceil |G|/k \rceil$-bounded.

In 1973, Meyer [11] introduced the notion of equitable coloring of graphs and conjectured that the equitable chromatic number of a connected graph G, which

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is neither a complete graph nor an odd cycle, is at most $\Delta(G)$. This conjecture has been confirmed for trees [2], [11], bipartite graphs [9], and graphs satisfying $\Delta(G) \leq 3$ or $\Delta(G) \geq |G|/2$ (see [3].) An earlier result of Hajnal and Szemerédi [6] showed that every graph G is equitably k-colorable for all $k > \Delta(G)$. The reader is referred to [10] for a survey of research on equitable coloring of graphs.

The mapping L is said to be a *list assignment* for the graph G if it assigns a list $L(v)$ of possible colors to each vertex v of G. A list assignment L for G is *k-uniform* if $|L(v)| = k$ for all $v \in V(G)$. If G has a proper coloring π such that $\pi(v) \in L(v)$ for all vertices v, then we say that G is *L-colorable* or π is an *L-coloring* of G. We call G *k-choosable* if it is L-colorable for every k-uniform list assignment L; *equitably L-colorable* if it has a $\lceil |G|/k \rceil$-bounded L-coloring for a k-uniform list assignment L; *equitably list k-colorable* or *equitably k-choosable* if it is equitably L-colorable for every k-uniform list assignment L.

The concept of list-coloring was introduced by Vizing [13] and independently by Erdős, Rubin and Taylor [4]. Quite a number of interesting results have been obtained in recent years, e.g., [1,5,7,12,14]. Combining m-bounded coloring and list coloring of graphs, Kostochka, Pelsmajer, and West [8] investigated the equitable list coloring of graphs. They proposed the following conjectures.

Conjecture 1. *Every graph G is equitably k-choosable whenever $k > \Delta(G)$.*

Conjecture 2. *If G is a connected graph with maximum degree $\Delta \geq 3$ other than $K_{\Delta+1}$ and $K_{\Delta,\Delta}$, then G is equitably Δ-choosable.*

It was proved in [8] that a graph G of maximum degree Δ is equitably k-choosable if either $k \geq \max\{\Delta, |G|/2\}$ and $G \neq K_{k+1}, K_{k,k}$, or $k \geq 1 + \Delta/2$ and G is a forest, or $k \geq \Delta$ and G is a connected interval graph, or $k \geq \max\{\Delta, 5\}$ and G is a 2-degenerate graph. In this paper, we will prove that the conjecture 2 holds for graphs with maximum degree at most 3. Moreover, we prove that every graph G with $\Delta(G) \geq 3$ is equitably k-choosable for any $k \geq (\Delta(G) - 1)^2$.

2. Equitably 2-Choosable Graphs

Let G be a graph with a (not necessarily uniform) list assignment L. Suppose that π is an L-coloring of G. We use $B(\pi)$ to denote the maximum size of a color class in the coloring π. Let $B(G; L) = \min\{B(\pi) \mid \pi \text{ is an } L\text{-coloring of } G\}$. If L is k-uniform and $B(G; L) \leq \lceil |G|/k \rceil$, then G is equitably L-colorable.

A generalized Brooks' theorem by Erdős, Rubin and Taylor [4] asserts that a connected graph G that is neither a complete graph nor an odd cycle is $\Delta(G)$-choosable. Applying this result, we immediately get the following.

Lemma 3. *Let $k \geq 1$ be an integer. If a graph G is k-choosable and $\alpha(G) \leq \lceil |G|/k \rceil$, then G is equitably k-choosable. In particular, if $\alpha(G) \leq \lceil |G|/k \rceil$ and G is neither a complete graph nor an odd cycle, then G is equitably k-choosable whenever $k \geq \Delta(G)$.*

If we remove vertices of degree 1 recursively from a graph G, then the final graph has no vertices of degree 1 and is called the *core* of G. A graph is called a *$\theta_{2,2,p}$-graph* if it consists of two vertices x and y and three internally disjoint paths of lengths $2, 2$, and p joining x and y. Using these two concepts, Erdős, Rubin and Taylor [4] established the following characterization for the 2-choosability of a graph.

Lemma 4. *A connected graph G is 2-choosable if and only if the core of G is either a K_1, an even cycle, or a $\theta_{2,2,2r}$-graph, where $r \geq 1$.*

Theorem 5. *A connected graph G is equitably 2-choosable if and only if G is a bipartite graph satisfying the following two conditions.*

(i) *The core of G is either a K_1, an even cycle, or a $\theta_{2,2,2r}$-graph, where $r \geq 1$.*

(ii) *G has two parts X and Y such that $||X| - |Y|| \leq 1$.*

Proof. Suppose that G is equitably 2-choosable, hence 2-choosable. Thus G is a bipartite graph with two parts, say X and Y. Statement (i) follows from Lemma 4. Let L be a 2-uniform list assignment for G with $L(v) = \{1, 2\}$ for all $v \in V(G)$. Then G has a unique equitable L-coloring π such that $\pi(x) = 1$ for all $x \in X$ and $\pi(y) = 2$ for all $y \in Y$. Thus $|X| \leq \lceil |G|/2 \rceil = \lceil (|X| + |Y|)/2 \rceil$ and $|Y| \leq \lceil (|X| + |Y|)/2 \rceil$. It follows that $||X| - |Y|| \leq 1$, therefore (ii) holds.

Now suppose that G is a bipartite graph with two parts X and Y satisfying (i) and (ii). By (i) and Lemma 4, G is 2-choosable. For any 2-uniform list assignment L for G, we know that G has an L-coloring π. By (ii), $B(\pi) \leq \alpha(G) \leq \max\{|X|, |Y|\} \leq \lceil |G|/2 \rceil$. Hence π is equitable by Lemma 3. ■

3. Graphs with Maximum Degree 3

The following basic result was proved in [8], which will be frequently used in the subsequent sections.

Lemma 6. *Let G be graph with a k-uniform list assignment L. Let $S = \{x_1, x_2, \ldots, x_k\}$ be a set of k vertices in G such that $G - S$ has an equitable L-coloring. If*

$$|N_G(x_i) \setminus S| + (i - 1) \leq k - 1 \qquad (*)$$

for $1 \leq i \leq k$, then G has an equitable L-coloring.

We can generalize Lemma 6 to the following.

Lemma 7. *Let G be graph with a k-uniform list assignment L. Let $\emptyset \neq A \subseteq V(G)$ such that $G - A$ has an equitable L-coloring π. For every vertex $v \in A$, define a list assignment*

$$L_\pi(v) = L(v) \setminus \{\pi(x) \mid x \in N_G(v) \cap (V(G) \setminus A)\}.$$

If $G[A]$ has an L_π-coloring σ such that $B(\sigma) \leq \lfloor |A|/k \rfloor$, then G has an equitable L-coloring.

Proof. Clearly, by combining the colorings π and σ, we can set up an L-coloring ϕ of G. Furthermore, $B(G; L) \leq B(G - A; L) + B(G[A]; L_\pi) \leq \lceil |G - A|/k \rceil + \lfloor |A|/k \rfloor = \lceil (|G| - |A|)/k \rceil + \lfloor |A|/k \rfloor \leq \lceil |G|/k \rceil$. Thus ϕ is an equitable L-coloring of G. ∎

In the sequel, L_π is called an *induced list assignment* of the set A for the coloring π.

Lemma 8. *Let H be a graph with $V(H) = \{u_1, u_2, u_3, u_4\}$, and let L be a list assignment for H.*

If L satisfies one of the following conditions, then H has an L-coloring such that $B(H; L) = 1$.

(1) $|L(u_i)| \geq i$ *for* $i = 1, 2, 3, 4$;

(2) $|L(u_1)| \geq 1$, $|L(u_2)| \geq 2$, $|L(u_3)| = |L(u_4)| = 3$, *and* $L(u_3) \neq L(u_4)$;

(3) $|L(u_4)| = 4$, $|L(u_1)| \geq 1$, $|L(u_2)| = |L(u_3)| = 2$, *and* $L(u_2) \neq L(u_3)$.

Proof. The result is obvious if (1) holds. Suppose now that (2) holds. We first color u_1 with a color $a \in L(u_1)$, and u_2 with $b \in L(u_2) \setminus \{a\}$. Since $|L(u_3)| = |L(u_4)| = 3$ and $L(u_3) \neq L(u_4)$, it follows that $L(u_3) \setminus \{a, b\} \neq L(u_4) \setminus \{a, b\}$ and $|L(u_i) \setminus \{a, b\}| \geq 1$ for $i = 3, 4$. Thus there exist $c \in L(u_3) \setminus \{a, b\}$ and $d \in L(u_4) \setminus \{a, b\}$ such that $c \neq d$. We further color u_3 with c and u_4 with d. Since a, b, c, and d are distinct, we have $B(H; L) = 1$.

Finally suppose that (3) holds. First we color u_1 with some color a from $L(u_1)$. Since $|L(u_2)| = |L(u_3)| = 2$ and $L(u_2) \neq L(u_3)$, there exist $b \in L(u_2) \setminus \{a\}$ and $c \in L(u_3) \setminus \{a\}$ such that $b \neq c$. We color u_2 with b and u_3 with c. Afterwards, we color u_4 with some color from $L(u_4) \setminus \{a, b, c\}$. Therefore $B(H; L) = 1$, and the proof is complete. ∎

Let H^* denote the graph consisting of a 4-cycle $C = u_1u_2u_3u_4u_1$ and four pendant edges u_iv_i, $i = 1, 2, 3, 4$, such that all the vertices, u_i's and v_j's, are distinct.

Lemma 9. *Let L be a list assignment for H^* that satisfies $|L(u_i)| = 4$ and $|L(v_i)| \geq 2$ for $i = 1, 2, 3, 4$. Then H^* has an L-coloring such that $B(H^*; L) \leq 2$.*

Proof. We first give a partial L-coloring π for the vertices v_1, v_2, v_3 and v_4 such that every color is used at most twice. Such a coloring exists obviously as $|L(v_i)| \geq 2$ for all $i = 1, 2, 3, 4$. There are several possibilities as follows.

Case 1. $\{\pi(v_1), \pi(v_2), \pi(v_3), \pi(v_4)\} = \{1, 2\}$.

Define a list assignment $L'(u_i) = L(u_i) \setminus \{1, 2\}$ for $i = 1, 2, 3, 4$. It is easy to see that $|L'(u_i)| \geq 2$ and the 4-cycle $u_1u_2u_3u_4u_1$ is L'-colorable. We note that every color appears on the 4-cycle at most twice. Thus $B(H^*; L) \leq 2$.

Case 2. $|\{\pi(v_1), \pi(v_2), \pi(v_3), \pi(v_4)\}| = 4$.

We may suppose that $\pi(v_i) = i$ for $i = 1, 2, 3, 4$. Let $L'(u_i) = L(u_i) \setminus \{1, 2\}$ for $i = 1, 2$ and $L'(u_i) = L(u_i) \setminus \{3, 4\}$ for $i = 3, 4$. Since $|L'(u_i)| \geq 2$, the 4-cycle $u_1u_2u_3u_4u_1$ has an L'-coloring such that each of the colors $1, 2, 3, 4$ is used at most once on this cycle and other colors at most twice. Hence $B(H^*; L) \leq 2$.

Case 3. $|\{\pi(v_1), \pi(v_2), \pi(v_3), \pi(v_4)\}| = 3$.

Subcase 3.1. $\pi(v_1) = \pi(v_2) = 1$, $\pi(v_3) = 2$, and $\pi(v_4) = 3$.

If $3 \in L(u_3)$, we color u_3 with 3, u_2 with a color $a \in L(u_2) \setminus \{1, 2, 3\}$, u_1 with $b \in L(u_1) \setminus \{1, 3, a\}$, and u_4 with $c \in L(u_4) \setminus \{1, 3, b\}$. If $2 \in L(u_4)$, we have a similar proof. Hence suppose that $3 \notin L(u_3)$ and $2 \notin L(u_4)$. In this case, we color u_4 with $a \in L(u_4) \setminus \{1, 3\}$, u_3 with $b \in L(u_3) \setminus \{1, 2, a\}$, u_2 with $c \in L(u_2) \setminus \{1, b\}$, and u_1 with $d \in L(u_1) \setminus \{1, a, c\}$. It is easy to observe that every color is used at most twice, thus $B(H^*; L) \leq 2$.

Subcase 3.2. $\pi(v_1) = \pi(v_3) = 1$, $\pi(v_2) = 2$, and $\pi(v_4) = 3$.

If $2 \in L(u_1)$, we color u_1 with 2, u_4 with a color $a \in L(u_4) \setminus \{1, 2, 3\}$, u_3 with $b \in L(u_3) \setminus \{1, 2, a\}$, and u_2 with $c \in L(u_2) \setminus \{1, 2, b\}$. We can establish a similar coloring for cases $2 \in L(u_3)$ or $3 \in L(u_1) \cup L(u_3)$. Thus we assume that $2, 3 \notin L(u_1) \cup L(u_3)$. Color u_2 with a color $a \in L(u_2) \setminus \{1, 2\}$, u_4 with $b \in L(u_4) \setminus \{1, 3\}$, u_1 with $c \in L(u_1) \setminus \{1, a, b\}$, and u_3 with $d \in L(u_3) \setminus \{1, a, b\}$. It is not difficult to see that every color is used at most twice in the previous colorings. Therefore $B(H^*; L) \leq 2$. The proof of the lemma is complete. ∎

Lemma 10. *If G is a graph with $\Delta(G) \leq 2$, then G is equitably k-choosable for any $k \geq 3$.*

Proof. If $k \geq 5$ or G is a forest, the result follows from Theorems 2 or 4 of [8]. Thus suppose that $k \leq 4$ and G contains a cycle. We do induction on the order

$|G|$. If $|G| \le k$, the conclusion holds clearly because we may color all vertices with distinct colors. Let G be a graph with $\Delta(G) \le 2$ and $|G| \ge k+1$. Let $C = u_1u_2\cdots u_nu_1$ be a cycle of G, where $n \ge 3$. Suppose that L is a k-uniform list assignment for G. If $k = 3$, we let $x_3 = u_2$, $x_2 = u_1$, and $x_1 = u_3$. Suppose $k = 4$. If $n \ge 4$, we let $x_4 = u_2$, $x_3 = u_3$, $x_2 = u_1$, and $x_1 = u_4$. If $n = 3$, we let $x_4 = u_1$, $x_3 = u_2$, $x_2 = u_3$, and $x_1 \in V(G) \setminus V(C)$. It is easy to check that the set $S = \{x_1, x_2, \ldots, x_4\}$ satisfies $(*)$. By the induction assumption, $G - S$ is equitably L-colorable. By Lemma 6, G is equitably L-colorable. This completes the proof. ∎

Lemma 11. *Let G be a graph with $\Delta(G) = 3$. Then, for any $k \ge 5$, G is equitably k-choosable.*

Proof. We do induction on the order $|G|$. If $|G| \le k$, the result is straightforward. Let G be a graph with $\Delta(G) = 3$ and $|G| \ge k+1$. Suppose that L is a k-uniform list assignment for G. We are going to construct a set $S = \{x_1, x_2, \ldots, x_k\} \subset V(G)$ which satisfies $(*)$. Since $\Delta(G) = 3$, G contains a vertex u of degree 3 with neighbors v, w, and y. We need to treat the following cases.

Case 1. $G[\{u, v, w, y\}]$ *is a component of* G.

In this case, it suffices to let $x_k = u$, $x_{k-1} = v$, $x_{k-2} = w$, $x_{k-3} = y$, $x_{k-4}, \ldots, x_1 \in V(G) \setminus \{u, v, w, y\}$. In fact, when $k-3 \le i \le k$, we have $N_G(x_i) \setminus S = \emptyset$. Thus $|N_G(x_i) \setminus S| + (i-1) = i-1 \le k-1$. When $1 \le i \le k-4$, $|N_G(x_i) \setminus S| \le |N_G(x_i)| = d_G(x_i) \le \Delta(G) = 3$ and furthermore $|N_G(x_i) \setminus S| + (i-1) \le 3 + (i-1) \le 3 + (k-4-1) = k-2$. Hence the set $S = \{x_1, x_2, \ldots, x_k\}$ satisfies $(*)$.

Case 2. $G[\{u, v, w, y\}]$ *is not a component of* G.

Without loss of generality, suppose that v is adjacent to a vertex t that is different from u, w, and y. We define $x_k = u$, $x_{k-1} = v$, $x_{k-2} = t$, $x_{k-3} = w$, $x_{k-4} = y$, and $x_{k-5}, \ldots, x_1 \in V(G) \setminus \{u, v, w, y, t\}$. It is easy to see that, if $i \le k-3$, then $|N_G(x_i) \setminus S| + (i-1) \le 3 + (i-1) \le 3 + (k-3-1) = k-1$. Since $v \in S$ and t is adjacent to v in G, we derive that $|N_G(x_{k-2}) \setminus S| + (k-2-1) \le 2 + (k-3) = k-1$. Since $u, t \in S$ and v is adjacent to u and t, so $|N_G(x_{k-1}) \setminus S| + (k-1-1) \le 1 + (k-2) = k-1$. Similarly, since $N_G(x_k) \subseteq S$, we get $|N_G(x_{k-1}) \setminus S| + (k-1) = k-1$. The argument implies that the set $S = \{x_1, x_2, \ldots, x_k\}$ satisfies $(*)$.

Now let $H = G - S$. If $\Delta(H) \le 2$, H is equitably L-colorable by Lemma 10. If $\Delta(H) = 3$, the induction hypothesis asserts that H is equitably L-colorable. By Lemma 6, G is equitably L-colorable. The proof is complete. ∎

Lemma 12. *Every graph G with $\Delta(G) = 3$ is equitably 4-choosable.*

Proof. Suppose that the lemma is false. Let G be a counterexample graph with the fewest vertices. Let L be a 4-uniform list assignment such that G is not equitably L-colorable. Then G possesses the properties stated in the following claims.

Claim 1. *The minimum degree $\delta(G)$ is 3.*

Proof. Since an isolated vertex can be assigned any color from its list, we see that $\delta(G) \geq 1$. Assume that G contains a vertex u of degree 1. Let v denote the unique neighbor of u. If $d_G(v) = 1$, let $x_4 = u$, $x_3 = v$, $x_1, x_2 \in V(G) \setminus \{u, v\}$ with $x_1x_2 \in E(G)$. If $d_G(v) \geq 2$, let $x_4 = u$, $x_3 = v$, $x_2 \in N_G(v) \setminus \{u\}$, and $x_1 \in V(G) \setminus \{x_2, x_3, x_4\}$. It is easy to verify that $S = \{x_1, x_2, x_3, x_4\}$ satisfies ($*$). By the minimality of G or Lemma 10, $G - S$ is equitably L-colorable. Furthermore, it follows from Lemma 6 that G is equitably L-colorable, contradicting the assumption on G. Thus $\delta(G) \geq 2$.

Suppose that G contains a vertex u of degree 2 with two neighbors y and z. If $G[\{u, y, z\}]$ forms a component of G, we let $x_4 = u$, $x_3 = y$, $x_2 = z$, and $x_1 \in V(G) \setminus \{u, y, z\}$. Otherwise, we suppose that y is adjacent to a vertex t different from u and z. Let $x_4 = u$, $x_3 = y$, $x_2 = z$, and $x_1 = t$. It is easy to check that $S = \{x_1, x_2, x_3, x_4\}$ satisfies ($*$). A similar contradiction will follow. Therefore $\delta(G) = 3$.

Claim 2. *There are no 3-cycles in G.*

Proof. Suppose that G contains a 3-cycle $C = u_1u_2u_3u_1$. Let $A = \{u_1, u_2, u_3, u_4\}$, where u_4 is a neighbor of u_1 that differs from u_2 and u_3. Thus $G - A$ admits an equitable L-coloring π. The induced assignment L_π of A for π satisfies the following: $|L_\pi(u_1)| = 4$, $|L_\pi(u_i)| \geq 3$ for $i = 2, 3$, and $|L_\pi(u_4)| \geq 2$. By Lemma 8, we know that $G[A]$ has an L_π-coloring such that $B(G[A]; L_\pi) = 1$. By Lemma 7, it follows that G is equitably L-colorable. This contradiction proves Claim 2.

Claim 3. *There are no 4-cycles in G.*

Proof. Suppose that G contains a 4-cycle $C = u_1u_2u_3u_4u_1$. As G does not contain 3-cycles by Claim 2, $u_1u_3, u_2u_4 \notin E(G)$. Let $v_i \in N_G(u_i) \setminus V(C)$ for $i = 1, 2, 3, 4$. So $v_1 \neq v_2$, $v_2 \neq v_3$, $v_3 \neq v_4$, and $v_4 \neq v_1$. The proof is divided into two subcases.

Subcase 3.1. Assume that $v_1 = v_3$.

If $v_2 = v_4$, a similar proof can be established. Let $A = \{u_1, u_2, u_3, u_4, v_1\}$ and let π be an equitable L-coloring of $G - A$. Obviously, $|L_\pi(u_1)| = |L_\pi(u_3)| = 4$, and $|L_\pi(t)| \geq 3$ for each $t \in \{u_2, u_4, v_1\}$. If there exists a color $a \in (L_\pi(u_2) \cup L_\pi(u_4) \cup L_\pi(v_1)) \setminus L_\pi(u_j)$ for $j = 1$, or 3, say $a \in L_\pi(v_1) \setminus L_\pi(u_1)$, then we color

v_1 with the color a, u_2 with $b \in L_\pi(u_2) \setminus \{a\}$, u_4 with $c \in L_\pi(u_4) \setminus \{a, b\}$, u_3 with $d \in L_\pi(u_3) \setminus \{a, b, c\}$, and u_1 with a color from $L_\pi(u_1) \setminus \{b, c, d\}$. Thus an L_π-coloring of $G[A]$ is constructed with the property that $B(G[A]; L_\pi) \le 1$. By Lemma 7, G is equitably L-colorable. This is a contradiction. Hence suppose $L_\pi(u_2) \cup L_\pi(u_4) \cup L_\pi(v_1) \subseteq L_\pi(u_1) \cap L_\pi(u_3)$. If there exists a color $a \in L_\pi(u_1) \setminus L_\pi(u_3)$, clearly $a \notin L_\pi(u_2) \cup L_\pi(u_4) \cup L_\pi(v_1)$, an L_π-coloring of $G[A]$ can be constructed similarly to satisfy $B(G[A]; L_\pi) \le 1$. So suppose $L_\pi(u_1) = L_\pi(u_3) = \{1, 2, 3, 4\}$, and $L_\pi(t) \subseteq \{1, 2, 3, 4\}$ for all $t \in \{u_2, u_4, v_1\}$. It suffices to show that $G[A]$ has an L_π-coloring such that some color, say 1, is used twice on vertices of A and each of the remaining colors, 2, 3, 4, occurs exactly once on A. In fact, if the color 1 belongs to two of the sets $L_\pi(u_2)$, $L_\pi(u_4)$, and $L_\pi(v_1)$, say $1 \in L_\pi(u_2) \cap L_\pi(u_4)$, we color u_2 and u_4 with 1, v_1 with $a \in L_\pi(v_1) \setminus \{1\}$, u_1 with $b \in L_\pi(u_1) \setminus \{1, a\}$, and u_3 with a color from $L_\pi(u_3) \setminus \{1, a, b\}$. Otherwise, suppose $1 \notin L_\pi(u_2) \cup L_\pi(u_4)$. Color u_1 and u_3 with 1, v_1 with $a \in L_\pi(v_1) \setminus \{1\}$, u_2 with $b \in L_\pi(u_2) \setminus \{a\}$, and u_4 with a color from $L_\pi(u_4) \setminus \{a, b\}$. It is easy to check that the current L_π-coloring satisfies our requirements.

Subcase 3.2. Assume that $v_1 \ne v_3$ and $v_2 \ne v_4$.

Let $A = \{u_1, \ldots, u_4, v_1, \ldots, v_4\}$. Then $G[A]$ is a graph H^* as defined in Lemma 9. For any L-coloring π of $G - A$, the induced assignment L_π of A satisfies $|L_\pi(u_i)| = 4$ and $|L_\pi(v_i)| \ge 2$ for all $i = 1, 2, 3, 4$. By Lemma 9, $G[A]$ has an L_π-coloring such that $B(G[A]; L_\pi) \le 2$. It follows from Lemma 7 that G is equitably L-colorable, which is absurd. The proof of Claim 3 is complete.

Claim 4. *For each edge $xy \in E(G)$, $|L(x) \setminus L(y)| = 1$.*

Proof. Suppose that G has an edge xy such that $|L(x) \setminus L(y)| \ne 1$, i.e., $L(x) = L(y)$ or $|L(x) \setminus L(y)| \ge 2$. Let $u_1, u_2 \in N_G(x) \setminus \{y\}$ and $v_1, v_2 \in N_G(y) \setminus \{x\}$. Since G contains neither 3-cycles nor 4-cycles, x, y, u_1, u_2, v_1, and v_2 are distinct and $u_2v_2 \notin E(G)$. Let $A = \{u_1, x, y, v_1\}$ and $H = G - A + u_2v_2$. By the minimality of G, H has an equitable L-coloring π. Thus the induced assignment of A for π satisfies the following: $|L_\pi(u_1)| \ge 2$, $|L_\pi(v_1)| \ge 2$, $|L_\pi(x)| \ge 3$, and $|L_\pi(y)| \ge 3$. Noting that u_2 is adjacent to v_2 in H, we derive $\pi(u_2) \ne \pi(v_2)$. Together with the assumption that $|L(x) \setminus L(y)| \ne 1$, this implies that $L_\pi(x) \ne L_\pi(y)$. So $G[A]$ has an L_π-coloring with $B(G[A]; L_\pi) = 1$ by Lemma 8. We have arrived at a contradiction.

Claim 5. *There are no 5-cycles in G.*

Proof. Suppose that G contains a 5-cycle $C = u_1u_2 \cdots u_5u_1$. Since G does not contain 3-cycles by Claim 2, u_1 is adjacent to a vertex v outside $V(C)$. We use w_1 and w_2 to denote the neighbors of v that are different from u_1. Let $A = \{v, w_1, w_2, u_1, u_2, \ldots, u_5\}$. By Claims 2 and 3, $|A| = 8$ and there do not exist edges between the set $\{u_2, u_5\}$ and the set $\{w_1, w_2\}$. Let π denote an equitable L-coloring of $G - A$. We are going to construct an L_π-coloring of $G[A]$ such that $B(G[A]; L_\pi) \leq 2$. Consequently, a contradiction follows from Lemma 7 and the minimality of G.

Assume that $u_3w_2 \in E(G)$. Then $u_4w_2 \notin E(G)$ for, otherwise, G would contain a 3-cycle $u_3u_4w_2u_3$, contradicting Claim 2. Note that $|L_\pi(w_1)| \geq 2$, $|L_\pi(t)| \geq 3$ for each $t \in \{u_2, u_4, u_5, w_2\}$, and $L_\pi(s) = L(s)$ for each $s \in \{u_1, u_3, v\}$. Moreover, $L_\pi(u_1) \setminus L_\pi(u_2) \neq \emptyset$ because $L(u_1) \neq L(u_2)$ by Claim 4. We color u_1 with a color $a \in L_\pi(u_1) \setminus L_\pi(u_2)$, w_1 with $b \in L_\pi(w_1) \setminus \{a\}$, u_5 with $c \in L_\pi(u_5) \setminus \{a, b\}$, v with $d \in L_\pi(v) \setminus \{a, b, c\}$, u_4 with $a' \in L_\pi(u_4) \setminus \{b, c\}$, w_2 with $b' \in L_\pi(w_2) \setminus \{d, a'\}$, u_2 with $c' \in L_\pi(u_2) \setminus \{a', b'\}$, and u_3 with $d' \in L_\pi(u_3) \setminus \{a', b', c'\}$. We note that a, b, c, and d are distinct, and so are a', b', c', and d'. It follows that $B(G[A]; L_\pi) \leq 2$.

The above argument works if one of u_3w_1, u_4w_1, and u_4w_2 belongs to $E(G)$.

Assume now that $u_3w_1, u_3w_2, u_4w_1, u_4w_2 \notin E(G)$. Then $|L_\pi(w_i)| \geq 2$ for $i = 1, 2$, $L_\pi(t) = L(t)$ for $t \in \{u_1, v\}$, and $|L_\pi(u_i)| \geq 3$ for all $i = 2, 3, 4, 5$. Without loss of generality, we suppose that $|L_\pi(u_i)| = 3$ for $i \geq 2$. (If $|L_\pi(u_i)| = 4$, we may take a 3-set of $L_\pi(u_i)$.) If $L_\pi(u_2) \neq L_\pi(u_3)$, we first color w_1, w_2, u_5, and v with mutually distinct colors. Based on this coloring, we further color u_1, u_4, u_2, and u_3 with distinct colors by Lemma 8. If $L_\pi(u_4) \neq L_\pi(u_5)$, a similar coloring can be established. If $L_\pi(u_3) \neq L_\pi(u_4)$, we color w_1, w_2, u_1, and v with distinct colors. Afterwards, we color u_2, u_5, u_3, and u_4 with distinct colors. It is easy to see that $B(G[A]; L_\pi) \leq 2$ for the current colorings.

Now suppose $L_\pi(u_2) = L_\pi(u_3) = L_\pi(u_4) = L_\pi(u_5) = L^*$. If $L_\pi(w_1) \cap L_\pi(w_2) \neq \emptyset$, we color w_1 and w_2 with a color $a \in L_\pi(w_1) \cap L_\pi(w_2)$, u_2 and u_4 with $b \in L^* \setminus \{a\}$, and u_3 and u_5 with $c \in L^* \setminus \{a, b\}$. Because $L(u_1) \neq L(v)$ by Claim 4, it follows that $L(u_1) \setminus \{a, b, c\} \neq L(v) \setminus \{a, b, c\}$. We can color u_1 with $d \in L(u_1) \setminus \{a, b, c\}$ and v with $d' \in L(v) \setminus \{a, b, c\}$ such that $d \neq d'$. Thus $B(G[A]; L_\pi) \leq 2$.

So suppose $L_\pi(w_1) \cap L_\pi(w_2) = \emptyset$. Since $|(L_\pi(w_1) \cup L_\pi(w_2)) \setminus L^*| \geq |L_\pi(w_1)| + |L_\pi(w_2)| - |L^*| \geq 2 + 2 - 3 = 1$, there exists a color $a \in (L_\pi(w_1) \cup L_\pi(w_2)) \setminus L^*$. Assume that $a \in L_\pi(w_1) \setminus L^*$ and let $\beta \in L(u_1) \setminus L(v)$ by Claim 4. Color w_1 with a, u_5 with $b \in L^* \setminus \{\beta\}$, u_4 with $c \in L^* \setminus \{b\}$, and u_3 with $d \in L^* \setminus \{b, c\}$. Now let us define a list assignment L' for the set $A' = \{u_1, u_2, v, w_2\}$ with $L'(u_1) = L(u_1) \setminus \{b\}$, $L'(u_2) = L^* \setminus \{d\}$, $L'(v) = L(v) \setminus \{a\}$, and $L'(w_2) = L_\pi(w_2)$. It is easy to see that $|L'(u_1)| \geq 3$, $|L'(v)| \geq 3$, $|L'(u_2)| \geq 2$, and $|L'(w_2)| \geq 2$. Since $\beta \in L'(u_1)$, but $\beta \notin L'(v)$, we see $L'(u_1) \neq L'(v)$. Lemma 8 asserts that the induced subgraph $G[A']$ has an L'-coloring with $B(G[A']; L') = 1$. Hence $B(G[A]; L_\pi) \leq 2$. The proof of Claim 5 is complete.

Suppose that xy is an edge of G with $u, v \in N_G(x) \setminus \{y\}$ and $w, z \in N_G(y) \setminus$

$\{x\}$. By Claim 4, we assume that $L(x) = \{1, 2, 3, 4\}$ and $L(y) = \{1, 2, 3, 5\}$. We simply write P for $L(u) \cup L(v) \cup L(w) \cup L(z)$ and Q for $L(u) \cap L(v) \cap L(w) \cap L(z)$.

Observation 1. $4 \in L(u) \cup L(v)$ *and* $5 \in L(w) \cup L(z)$.

Suppose to the contrary that $4 \notin L(u)$. (A similar argument can be given in other cases.) Let $A = \{v, x, y, z\}$. So $G - A$ has an equitable L-coloring π, and A admits an induced assignment L_π such that $|L_\pi(v)| \geq 2$, $|L_\pi(z)| \geq 2$, $|L_\pi(x)| \geq 3$, and $|L_\pi(y)| \geq 3$. Since $4 \notin L(u)$, π gives u a color different from 4. Hence $4 \in L_\pi(x)$. On the other hand, it is obvious that $4 \notin L_\pi(y)$ as $4 \notin L(y)$. It follows that $L_\pi(x) \neq L_\pi(y)$ and hence $G[A]$ has an L_π-coloring with $B(G[A]; L_\pi) = 1$ by Lemma 8. However, G is equitably L-colorable by Lemma 7. A contradiction is obtained.

Observation 2. $1, 2, 3 \in P$.

Suppose that $1 \notin P$. (A similar proof can be established for other cases.) Let z' denote a neighbor of z and $z' \neq y$. Let $A = \{x, y, z, z'\}$. For each L-coloring π of $G - A$, A has an induced assignment L_π such that $|L_\pi(z')| \geq 2$, $|L_\pi(x)| \geq 2$, $|L_\pi(y)| \geq 3$, and $|L_\pi(z)| \geq 3$. Since $\pi(w) \neq 1$ and $1 \notin L(z)$, it follows that $1 \in L_\pi(y) \setminus L_\pi(z)$. Thus $L_\pi(y) \neq L_\pi(z)$. A contradiction follows again from Lemmas 7 and 8.

Observation 3. $|Q \cap \{1, 2, 3\}| \leq 1$.

Suppose that $|Q \cap \{1, 2, 3\}| \geq 2$, say, $1, 2 \in Q$. Since $4 \in L(u)$ by Observation 1 and $|L(u) \setminus L(x)| = 1$ by Claim 4, we see that $3 \notin L(u)$. Similarly, we can derive $3 \notin L(t)$ for each $t \in \{v, w, z\}$. This implies that $3 \notin P$, which contradicts Observation 2.

Observation 4. *There exist* $s^* \in \{u, v\}$ *and* $t^* \in \{w, z\}$ *such that* $|L(s^*) \cap L(t^*) \cap \{1, 2, 3\}| = 1$.

By Observation 1 and Claim 4, $L(r)$ contains exactly two of the colors 1, 2, and 3 for each $r \in \{u, v, w, z\}$. So we suppose $\{1, 2\} \subseteq L(u)$. If $\{1, 2\} \subseteq L(w) \cap L(z)$, then $\{1, 2\} \setminus L(v) \neq \emptyset$ by Observation 3. We thus take $s^* = v$ and $t^* = w$. Otherwise, suppose $\{1, 2\} \setminus L(z) \neq \emptyset$. It suffices to take $s^* = u$ and $t^* = z$. We always have $L(s^*) \cap L(t^*) \cap \{1, 2, 3\} = \{1\}$ or $\{2\}$. The proof of Observation 4 is complete.

By Observation 4, we suppose that $L(u) = \{1, 2, 4, a\}$ and $L(w) = \{2, 3, 5, b\}$, where $a \neq 3$ and $b \neq 1$. Let $u' \in N_G(u) \setminus \{x\}$ and $w' \in N_G(w) \setminus \{y\}$. Let $A = \{u, u', v, w, w', x, y, z\}$. Since G does not contain cycles of lengths at most 5, the vertices in A are distinct. Moreover, if $u'w' \notin E(G)$, then $G[A]$ is a tree. For any equitable L-coloring π of $G - A$, A has an induced assignment L_π such that $L_\pi(x) = L(x)$, $L_\pi(y) = L(y)$, $|L_\pi(u)| \geq 3$, $|L_\pi(w)| \geq 3$, and $|L_\pi(t)| \geq 2$ for all $t \in \{v, z, u', w'\}$. If u' is adjacent to w' in G, then both $|L_\pi(u')|$ and $|L_\pi(w')|$ are at least 3. Without loss of generality, suppose that $|L_\pi(u)| = |L_\pi(w)| = 3$. In the following, we are going to construct an L_π-coloring of $G[A]$ such that $B(G[A]; L_\pi) \leq 2$. Thus a contradiction will be derived.

Assume that $a \in L_\pi(u)$. At first, we color z, w', w, and y with four different colors, and use β to denote the color assigned to y. If $a \in L_\pi(v)$, we further color v with a, u' with $b \in L_\pi(u') \setminus \{a\}$, u with $c \in L_\pi(u) \setminus \{a, b\}$, and x with $d \in L(x) \setminus \{b, c, \beta\}$. If $a \notin L_\pi(v)$, we color u with a, u' with $b \in L_\pi(u') \setminus \{a\}$, v with $c \in L_\pi(v) \setminus \{b\}$, and x with $d \in L(x) \setminus \{b, c, \beta\}$. Because $a \notin L(x)$, the current colorings satisfy our requirements.

If $b \in L_\pi(w)$, an analogous proof can be given. Thus suppose that $L_\pi(u) = \{1, 2, 4\}$ and $L_\pi(w) = \{2, 3, 5\}$. If there exists a color $a \in L_\pi(v) \setminus \{1, 2, 4\}$, we first color z, w', w, and y with distinct colors. Let β denote the color of y. Afterwards, we color v with a, u' with $b \in L_\pi(u') \setminus \{a\}$, x with $c \in L_\pi(x) \setminus \{a, b, \beta\}$, and u with $d \in L_\pi(u) \setminus \{b, c\}$. If there exists $a \in L_\pi(u') \setminus \{1, 2, 4\}$ and $a \notin L_\pi(v)$, we color z, w', w, and y with distinct colors, then color u' with a, x with 4, v with $b \in L_\pi(v) \setminus \{4\}$, and u with a color from $L_\pi(u) \setminus \{4, b\}$. Since $4 \notin L(y)$, the coloring is available.

Finally, suppose $L_\pi(v) \cup L_\pi(u') \subseteq \{1, 2, 4\}$ and, similarly, $L_\pi(z) \cup L_\pi(w') \subseteq \{2, 3, 5\}$. First color x with 3 and y with 1. Then we color u, u', and v with $1, 2, 4$ and w, w', and z with $2, 3, 5$ such that all these vertices receive distinct colors. The proof of Lemma 12 is complete. ∎

Combining Lemmas 11 and 12, we can derive the following.

Theorem 13. *Conjecture 1 holds for a graph with maximum degree at most 3.*

4. Equitable $(\Delta - 1)^2$-Choosability

The *distance* between two vertices in a graph G is the length of a shortest path connecting them. For $v \in V(G)$, let $M_G(v)$ denote the set of vertices which have distance 2 to the vertex v.

Theorem 14. *Let G be a graph with $\Delta(G) \geq 3$. If $k \geq (\Delta(G) - 1)^2$, then G is equitably k-choosable.*

Proof. If $\Delta(G) = 3$, then $k \geq (3-1)^2 = 4$. By Theorem 13, G is equitably k-choosable. Suppose that the theorem holds for all graphs with maximum degree less than m, $m \geq 4$. We will prove the theorem for graphs with maximum degree m. Once m is fixed, we further use induction on the order $|G|$. If $|G| \leq k$, the conclusion is evident. Let G be a graph with $\Delta(G) = m$ and $|G| \geq k+1$. Suppose that L is a k-uniform list assignment for G, where $k \geq (m-1)^2$. We are going to define a set $S = \{x_1, x_2, \ldots, x_k\}$ satisfying $(*)$. Afterwards, we let $H = G - S$. If $\Delta(H) < \Delta(G)$, then $k \geq (\Delta(G) - 1)^2 > (\Delta(H) - 1)^2$. By the induction hypothesis on the maximum degree, H is equitably L-colorable. If $\Delta(H) = \Delta(G)$, H is equitably L-colorable by the induction hypothesis on the number of vertices. Therefore, G is equitably L-colorable by Lemma 6.

Suppose that u is a vertex of maximum degree in G. We see that $|N_G(u)| = d_G(u) = m$. Since $m \geq 4$, we have $k \geq (m-1)^2 > m+1$. Define $x_k = u$, and let $x_{k-1}, x_{k-2} \ldots, x_{k-m}$ be the m neighbors of u. Let

$$Y_i = M_G(u) \cap N_G(x_i),$$

for $i = k-m, k-m+1, \ldots, k-1$. Then let

$$Y = \bigcup_{i=k-m+3}^{k-1} Y_i,$$

and $p = |Y|$. Take $x_{k-m-1}, x_{k-m-2}, \ldots, x_{k-m-p} \in Y$, $x_{k-m-p-1} \in Y_{k-m+2}$, and $x_{k-m-p-2}, x_{k-m-p-3}, \ldots, x_1 \in V(G) \setminus \{x_{k-m-p-1}, x_{k-m-p}, \ldots, x_k\}$. Since $m \geq 4$ and

$$p = |Y| \leq \sum_{i=k-m+3}^{k-1} |Y_i| \leq (m-3)(m-1),$$

we derive

$$k - m - p - 1 \geq (m-1)^2 - m - (m-3)(m-1) - 1 = m - 3 \geq 1.$$

This implies that $Y \subseteq S$, and $x_{k-m-p-1} \in Y_{k-m+2}$. Hence S is well-defined. It remains to check that S satisfies $(*)$. First we note that $|N_G(x_i) \setminus S| \leq |N_G(x_i)| = d_G(x_i) \leq \Delta(G) = m$ for any $x_i \in S$. Thus, when $i \leq k-m$, we have $|N_G(x_i) \setminus S| + (i-1) \leq m + (k-m-1) = k-1$.

Assume that $i = k-m+1$. Since x_{k-m+1} is adjacent to x_k and $x_k \in S$, it follows that $|N_G(x_{k-m+1}) \setminus S| \leq m-1$ and thus $|N_G(x_{k-m+1}) \setminus S| + (k-m+1-1) \leq m-1+(k-m) = k-1$.

Assume that $i = k-m+2$. Since $x_{k-m+2}x_k, x_{k-m+2}x_{k-m-p-1} \in E(G)$ and $x_k, x_{k-m-p-1} \in S$, we have $|N_G(x_{k-m+2}) \setminus S| \leq m-2$. Thus $|N_G(x_{k-m+2}) \setminus S| + (k-m+2-1) \leq m-2+(k-m+1) = k-1$.

Assume that $k-m+3 \leq i \leq k$. It is easy to see that $N_G(x_i) \subseteq S$ by definition, so $|N_G(x_i) \setminus S| = 0$. Therefore $|N_G(x_i) \setminus S| + (i-1) = (i-1) \leq k-1$. ■

References

1. N. Alon and M. Tarsi, Colorings and orientations of graphs, *Combinatorica* **12** (1992), 125-134.

2. Bor-Liang Chen and Ko-Wei Lih, Equitable coloring of trees, *J. Combin. Theory Ser.* **B61** (1994), 83-87.

3. Bor-Liang Chen, Ko-Wei Lih and Pou-Lin Wu, Equitable coloring and the maximum degree, *Europ. J. Combin.* **15** (1994), 443-447.

4. P. Erdős, A. L. Rubin and H. Taylor, Choosability in graphs, *Congr. Numer.* **26** (1980), 125-157.

5. F. Galvin, The list chromatic index of a bipartite multigraph, *J. Combin. Theory Ser.* **B63** (1995), 153-158.

6. A. Hajnal and E. Szemerédi, Proof of a conjecture of Erdős. in: *Combinatorial Theory and Its Applications*, Vol. 2, Colloq. Math. Soc. János Bolyai 4, North-Holland, Amsterdam, 1970, pp. 601-623.

7. T. R. Jensen and B. Toft, *Graph Coloring Problems*, Wiley, New York, 1995.

8. A. V. Kostochka, M. J. Pelsmajer and D. B. West, A list analogue of equitable coloring, *J. Graph Theory*, **44** (2003), 166-177.

9. Ko-Wei Lih and Pou-Lin Wu, On equitable coloring of bipartite graphs, *Discrete Math.* **151** (1996), 155-160.

10. Ko-Wei Lih, The equitable coloring of graphs, in: D. Z. Du and P. Pardalos, eds. *Handbook of Combinatorial Optimization*, Vol. 3, Kluwer, Dordrecht, 1998, 543-566.

11. W. Meyer, Equitable coloring, *Amer. Math. Monthly* **80** (1973), 920-922.

12. C. Thomassen, Every planar graph is 5-choosable, *J. Combin. Theory Ser.* **B62** (1994), 180-181.

13. V. G. Vizing, Coloring the vertices of a graph in prescribed colors, *Diskret. Analiz.* **29** (1976), 3-10. (in Russian.)

14. M. Voigt, List colorings of planar graphs, *Discrete Math.* **120** (1993), 215-219.

Wei-Fan Wang
Department of Mathematics,
Zhejiang Normal University,
Jinhua, Zhejiang 321004,
P. R. China
E-mail: wangwf0062@sina.com.cn

Ko-Wei Lih
Institute of Mathematics,
Academia Sinica,
Nankang, Taipei 115,
Taiwan
E-mail: makwlih@sinica.edu.tw