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ON GENERALIZED k-DIAMETER OF k-REGULAR k-CONNECTED GRAPHS

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Abstract. In this paper, motivated by the study of the wide diameter and the Rabin number of graphs, we define the generalized k-diameter of k-connected graphs, and show that every k-regular k-connected graph on n vertices has the generalized k-diameter at most $n/2$ and this upper bound cannot be improved when $n = 4k - 6 + i(2k - 4)$.

1. Introduction

Let $G = G(V, E)$ be a simple connected graph on n vertices with $\kappa(G) = k$ and S, T be any pair of disjoint subsets of $V(G)$ such that $|S| = |T| = k$. Then there are k vertex disjoint paths connecting S and T by Menger's Theorem [1]. Let $P_k(S, T)$ be a family of k vertex disjoint paths joining S and T, i.e.

$$P_k(S,T) = \{P_1, P_2, \cdots, P_k\}, \quad |P_1| \le |P_2| \le \cdots \le |P_k|.$$

The generalized k-wide distance (or simply generalized k-distance), written as $gd_k(S,T)$, between S and T is the minimum $|P_k|$ among all $P_k(S,T)$, and the generalized k-wide diameter (or simply generalized k-diameter), denoted by $gd_k(G)$, of G is defined as the maximum generalized k-wide distance $gd_k(S,T)$ over all pairs S, T of disjoint subsets of $V(G)$ with $|S| = |T| = k = \kappa(G)$, i.e.

$$gd_k(S,T) = \min_{P_k(S,T)} |P_k|,$$

and

$$gd_k(G) = \max\{gd_k(S,T) : \ S, T \in V(G) \text{ and } |\mathrm{S}| = |\mathrm{T}| = \mathrm{k}, \mathrm{S} \cap \mathrm{T} = \phi\}.$$

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The definition of the generalized wide diameter of graph G is mainly motivated from the definitions of the wide diameter and the Rabin number of graphs, two parameters had been studied widely by various researchers (for example, see [2]∼[8]).

In this paper, we show that every k-regular k-connected graph on n vertices has generalized k-diameter at most $n/2$ and this upper bound is tight when $n = 4k-6+i(2k-4)$.

2. Main Results

Let

$$F(n,k) = \max\{gd_k(G) : G \text{ is } k\text{-regular } k\text{-connected graph with } n \text{ vertices } \}.$$

The similar function $f(n,k)$ about k-diameter $d_k(G)$ defined in [5] has been discussed in [5] and[3]. Clearly, $F(n,2) = n-3$, and $F(n,k) \leq n-2k+1$. The following proposition provides the value of $F(n,k)$ for large k.

Proposition 2.1. *If either kn is even and $k \geq n/4 + 10/4 \geq 5$ or $n = 4k-8 \geq 12$, then $F(n,k) = n-2k+1$.*

Proof. Note that for a cycle of length $n \geq 4$ we have $gd_2(C_n) = n-3$. Take graph G as $H_1 \cdot C_{n-2k+4} \cdot H_2$, where H_i $(i = 1,2)$ is a graph on $k-2$ vertices, i.e. G is a graph with vertex set $\{u_1, u_2, \cdots, u_{k-2}, v_1, v_2, \cdots, v_{n-2k+4}, w_1, w_2, \cdots, w_{k-2}\}$ such that subset $\{v_1, v_2, \cdots, v_{n-2k+4}\}$ spans C_{n-2k+4}, subgraph induced by $\{u_1, u_2, \cdots, u_{k-2}\}$ is isomorphic to H_1 and u_i is adjacent to $v_1, v_2, \cdots, v_{n-3k+7}$, subgraph induced by $\{w_1, w_2, \cdots, w_{k-2}\}$ is isomorphic to H_2 and w_i is adjacent to $v_{n-3k+8}, v_{n-3k+9}, \cdots, v_{n-2k+4}$, and u_i is adjacent to w_i, respectively, for $i = 1, 2, \cdots, k-2$. One can easily see that if H_1 is $k-1-(n-3k+7) = 4k-n-8$ connected and H_2 is 2-connected then G is k-connected and the generalized k-distance between vertex set $\{u_1, u_2, \cdots, u_{k-2}, v_1, v_2\}$ and vertex set $\{w_1, w_2, \cdots, w_{k-2}, v_{n-2k+3}, v_{n-2k+4}\}$ is equal to $n-2k+1$. Thus, in order to get k-regular k-connected graph G with $gd_k(G) = n-2k+1$, it is enough to take as H_1 a graph with no edges when $n = 4k-8$, and any l-regular l-connected graph with $k-2$ vertices when $l = 4k-n-8 \geq 2$, and take as H_2 a 2-regular 2-connected graph on $k-2$ vertices (note that, since kn is even, so is $l \cdot (k-2)$ and since $2(k-2)$ is even graphs H_1 and H_2 always exist). ■

The following theorem shows that even for small k, $F(n,k)$ is bounded by $n/2$.

Theorem 2.2. *If kn is even and $k \geq 3$ then $F(n,k) \leq n/2$.*

Proof. Let G be a k-regular k-connected graph on $n \geq 2k$ vertices and S, T be two disjoint k-subsets of $V(G)$ such that $gd_k(S,T) = gd_k(G)$ and

$$P_k(S,T) = \{P_1, P_2, \cdots, P_k\}, \quad |P_1| \leq |P_2| \leq \cdots \leq |P_k| = gd_k(G)$$

be such a family of k vertex disjoint paths between S and T that for every other family

$$P'_k(S,T) = \{P'_1, P'_2, \cdots, P'_k\}, \quad |P'_1| \leq |P'_2| \leq \cdots \leq |P'_k| = gd_k(G),$$

we have $\sum_{i=1}^{k} |P'_i| \geq \sum_{i=1}^{k} |P_i|$. Moreover, let A denotes the subset of all vertices of G which belong to none of the paths $P_1, P_2, \cdots, P_k$. G has n vertices, so

$$\sum_{i=1}^{k}(|P_i| + 1) + |A| = \sum_{i=1}^{k} |P_i| + k + |A| = n. \tag{1}$$

We estimate from below the number of edges in G. The number of edges which belong to paths from $P_k(S,T)$ is equal to $\sum_{i=1}^{k} |P_i|$. Furthermore, no two vertices which belong to path P_k are joined by an edge which does not belong to path P_k (otherwise P_k would be replaced by a shorter path, contradicting the choice of $P_k(S,T)$), so there exist precisely $(k-2)(|P_k| - 1) + 2(k-1) = (k-2)|P_k| + k$ edges incident to vertices from path P_k which are not contained in it. We shall show that there exist at least $|A|$ edges which are neither contained in one of the paths from $P_k(S,T)$ nor incident to vertices of P_k.

Let H be a component of a subgraph induced in G by set A, and let $|H|$ be the number of vertices of H. We shall prove that at least $|H|$ edges of G are incident to vertices from H and not incident to vertices from P_k. If H contains a cycle it contains at least $|H|$ edges so it is enough to consider the case when H is a tree.

Case 1. $k = 3$

Note that H is adjacent to at most $|H| + 2$ vertices of path $P_k = v_0 v_1 \cdots v_{|P_k|}$, say $v_{l+1}, v_{l+2}, \cdots, v_{l+|H|+2}$, where $v_0 \in S$ and $v_{|P_k|} \in T$. Indeed, otherwise one could find vertices v_i and v_j with $j - i \geq |H| + 2$, both adjacent to H, and replace P_k by a shorter path using vertices of H instead of $v_{i+1} v_{i+2} \cdots v_{j-1}$. Furthermore, at least one of the vertices $v_{l+2}, v_{l+3}, \cdots, v_{l+|H|+1}$ must have a neighbor outside H since otherwise graph G could be disconnected by deleting vertices v_{l+1} and $v_{l+|H|+2}$. We note that both vertices v_{l+1} and $v_{l+|H|+2}$ can be adjacent to only one vertex of H. Indeed, otherwise one could find vertices x and y with distance less than $|H| - 1$ in H adjacent to v_{l+1} and $v_{l+|H|+2}$, respectively , and replace P_k by a shorter path using vertices of the shortest path from x to y in H instead of $v_{l+2} v_{l+3} \cdots v_{l+|H|+1}$. Thus, P_k sends to H at most $|H| + 2 - 1 = |H| + 1$ edges, so at least

$$3|H| - (|H| - 1) - (|H| + 1) = |H|$$

edges incident to H are not incident to vertices from P_k.

Case 2. $k = 4$ and H is a path

Similarly as in the previous case, H must be adjacent to at most $|H| + 2$ vertices of path $P_k = v_0 v_1 \cdots v_{|P_k|}$, say $v_{l+1} v_{l+2} \cdots v_{l+|H|+2}$, where at least two of the vertices $v_{l+2}, v_{l+3}, \cdots, v_{l+|H|+1}$ have neighbors outside H. Furthermore, it is not hard to see that both vertices v_{l+1} and $v_{l+|H|+2}$ can be adjacent to only one vertex of the path H, namely to one of its ends. Hence, the number of edges between P_k and H is bounded above by $2 + 2|H| - 2$, so at least

$$4|H| - 2|H| - (|H| - 1) = |H| + 1$$

edges incident to H are not incident to vertices from P_k.

Case 3. $k = 4$ and H is not a path

Since now the diameter of H is less than $|H| - 1$, it is adjacent only to at most $|H| + 1$ vertices of path P_k, from which at least two have neighbors outside H. Thus, similarly as in the previous two cases, the number of edges incident to H but not to P_k is bounded below by

$$4|H| - 2(|H| + 1) + 2 - (|H| - 1) = |H| + 1.$$

Case 4. $k \geq 5$

Note that no vertex from H is adjacent to more than three vertices from P_k since otherwise path P_k could be replaced by a shorter one. Hence, G contains at least

$$k|H| - 3|H| - (|H| - 1) \geq |H| + 1.$$

edges incident to vertices from H not incident to vertices from P_k.

Thus we have shown that there are at least $|A|$ edges in G which are neither contained in some k paths nor incident to vertices from P_k, so

$$\sum_{i=1}^{k} |P_i| + (k-2)|P_k| + k + |A| \leq nk/2. \tag{2}$$

Now subtracting (1) from (2) and dividing by $k - 2$ gives $n/2$ as the upper bound for $|P_k|$. ∎

Remark. Note that from the proof it follows that, when $k > 5, gd_k(S,T) = n/2$ only if all vertices of G lies on some path from $P_k(S,T)$ and all edges of G either belong to a path from $P_k(S,T)$ or are incident to some vertices from P_k.

The above bound for $F(n,k)$ cannot be improved in general case. In fact, the equality $F(n,k) = \lfloor n/2 \rfloor$ holds for infinitely many pairs k and n.

Proposition 2.3. *Let* $n = 2k - 3 + i(k-2)$, *where* $3 \leq k \leq n,\ i = 1, 2, \cdots,$ *and* $i > 1$ *if* $k = 3$, *then* $F(2n, k) = n$.

Proof. We shall construct a k-regular k-connected graph $G(2n, k)$ with $2n = 4k - 6 + i(2k-4)$ vertices for which $gd_k(G(2n,k)) = n$. The set of vertices of $G(2n,k)$ contains vertices $v_j,\ j = 0, 1, \cdots, n$ and w_l^m, where $l = 1, 2 \cdots, k-2$ and $m = 0, 1, \cdots, i, i+1$. The set of edges of $G(2n,k)$ consists of the following pairs of vertices:

(a) $\{v_j, v_{j+1}\}$ for $j = 0, 1, \cdots, n-1$,

(b) $\{v_0, w_1^0\}$, $\{w_1^0, w_{k-2}^{i+1}\}$, and $\{v_n, w_{k-2}^{i+1}\}$,

(c) $\{v_0, w_l^0\}$ for $l = 2, \cdots, k-2$, and $\{v_0, w_1^1\}$,

(d) $\{v_n, w_l^{i+1}\}$ for $l = 1, 2, \cdots, k-3$ and $\{v_n, w_{k-2}^i\}$,

(e) $\{w_l^m, w_l^{m+1}\}$ for $l = 2, 3, \cdots, k-3,\ m = 0, 1, \cdots, i$,

(f) $\{w_1^m, w_1^{m+1}\}$ for $m = 1, 2, \cdots, i$,

(g) $\{w_{k-2}^m, w_{k-2}^{m+1}\}$ for $m = 0, 1, \cdots, i-1$,

(h) $\{w_l^m, v_{m(k-2)+s}\}$ for $l = 1, 2, \cdots, k-2, m = 0, 1, \cdots, i, i+1$ and $s = 1, 2, \cdots, k-2$.

Graph $G(14, 4)$ is given in Fig. 1 .

Let $S = \{w_l^0 |\ l = 1, 2, \cdots, k-2\} \cup \{w_1^1, v_0\}$ and $T = \{w_l^{i+1} |\ l = 1, 2, \cdots, k-2\} \cup \{w_{k-2}^i, v_n\}$. One can easily check that $G(2n,k)$ is k-regular k-connected and the only family of k vertex disjoint paths between S and T consists of paths $w_1^0 w_{k-2}^{i+1}$, $v_0 v_1 \cdots v_n$, $w_1^1 w_1^2 \cdots w_1^{i+1}$, $w_{k-2}^0 \cdots w_{k-2}^i$ and $k-4$ paths $w_l^0 w_l^1 \cdots w_l^{i+1}$, $l = 2, \cdots, k-$

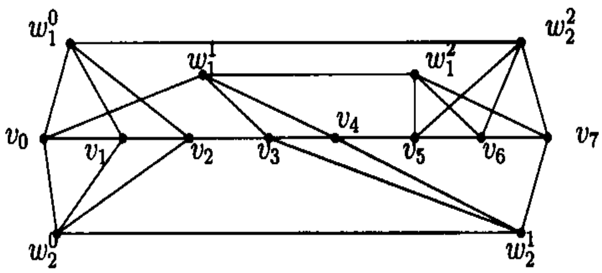

Fig. 1. G (14, 4)

One might expect that equality $F(n,k) = \lfloor n/2 \rfloor$ holds for every n and k such that nk is even and $3 \le k \le \lfloor n/2 \rfloor$. The next result shows that it is not true.

Proposition 2.4. *If* $n \ge 18$ *and* $n/3 + 3 \le k \le n/2$ *then* $F(2n,k) < n$.

Proof. Due to the observation we made after the proof of Theorem 2.2, the equality $F(2n,k) = n$ can hold only if for some disjoint vertex sets S, T, a family of paths $P_k(S,T)$ contains all vertices of the graph and each edge of the graph which dose not belong to paths from $P_k(S,T)$ is incident to P_k. Suppose that for the two disjoint vertex sets S, T with $|S| = |T| = k$ of a k-regular k-connected graph on $2n$ vertices we have $gd_k(S,T) = n$. Then, G contains $n-1$ vertices outside path $P_k = v_0v_1v_2\cdots v_{n-1}v_n,$ where $v_0 \in S, v_n \in T$. So, since $n/3+3 \le k \le n/2$, $P_{k-1} = w_0w_1w_2\cdots w_l$ for some $2 \le l \le n/3-4$. Vertex w_0 has $k-1 \ge n/3+2$ neighbors lying on P_k, and w_1 has $k-2 \ge n/3+1$ neighbors lying on P_k, and $k-1 \ge n/3+2$ neighbors lying on P_k for vertex w_l, so w_0 is adjacent to some vertex v_i with $i \ge n/3+1$ and w_l is adjacent to some vertex v_j with $j \le 2n/3-1$, then vertex w_1 is adjacent to some vertex v_m with $m < n/3+1$ or $m > 2n/3-1$. Thus, paths P_{k-1} and P_k could be replaced by $P'_{k-1} = v_0v_1v_2\cdots v_mw_1w_2\cdots w_l$ and $P'_k = w_0v_iv_{i+1}\cdots v_{n-1}v_n$ of lengths $|P'_{k-1}| = m+1+l-1 = m+l < n/3+1+n/3-4 = 2n/3-3 < n$ and $|P'_k| = 1+n-i = n-i+1 \le 2n/3 < n$ if $m < n/3+1$, or $P''_{k-1} = v_0v_1\cdots v_jw_l$ and $P''_k = w_0w_1v_mv_{m+1}\cdots v_n$ of lengths $|P''_{k-1}| = j+1 \le 2n/3 < n$ and $|P''_k| = 2+n-m < n/3+3 < n$ if $m > 2n/3-1$, so $gd_k(S,T) < n$, contradicts to $gd_k(S,T) = n$. ∎

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