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OPTIMALITY CONDITIONS FOR MINIMAX PROGRAMMING OF ANALYTIC FUNCTIONS

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Dedicated to Professor H. C. Wang on his honor retirement from NTHU.

Abstract. In this paper, we investigate a minimax complex programming problem. Several sufficient Optimality conditions are established under the framework of generalized convexity for analytic functions. Employing the sufficient optimality conditions, we have proved the weak, strong and strict converse duality theorems for the complex minimax programming problem.

1. Introduction and Preliminaries

It is known that a necessary optimality condition for a differentiable function f is satisfying $f'(x) = 0$. While the sufficient optimality condition, we may regard it as the converse of the necessary condition by adding some extra assumptions. For instance, if x_0 minimizes a differentiable function, then $f'(x_0) = 0$. Conversely, if $f'(x) = 0$ has a solution x_0, then x_0 may or may not be an optimal (min/ or max) for $f(x)$. If $f \in C^2$ and $f''(x_0) > 0$ ($f''(x_0) < 0$), then x_0 is a minimum (resp. maximum) for $f(x)$.

Based on the above reason, in an optimization problem as well as a mathematical programming problem, many authors made every effort to study the extra conditions for which a feasible solution might be optimal, that is, to establish the sufficient optimality conditions.

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In general, convexity for a function is very useful nature in optimization analysis to establishing the existence of optimal solution. If a real-valued nonlinear functional f is convex at $x_0 \in X$, a normed linear space, then for any $x \in X$ and $\lambda \in [0, 1]$, we have

$$f(\lambda x + (1-\lambda)x_0) \leq \lambda f(x) + (1-\lambda)f(x_0).$$

If $0 < \lambda < 1$, the above inequality yields

$$f(x) - f(x_0) \geq \frac{1}{\lambda}[f(\lambda(x - x_0) + x) - f(x_0)].$$

Further if f is also differentiable at x_0, then as $\lambda \to 0$, we obtain

$$f(x) - f(x_0) \geq f'(x_0)(x - x_0). \tag{1.1}$$

This $f'(x_0)$ is a bounded linear functional on X, that is, $f'(x_0) \in X^*$, the dual space of X. The equality in (1.1) can be expressed as

$$f(x) - f(x_0) = f'(x_0)(x - x_0) + \rho\theta \tag{1.2}$$

where $\theta = \theta(x, x_0)$ is small enough whenever x near x_0, and $\rho \in \mathbb{R}$. From (1.1) and (1.2), the convexity is then extended to generalized convexity; for example, invex, pseudoconvex, quasiconvex etc. If $f'(x_0)(x - x_0)$ in (1.1) (or (1.2)) is replaced by a functional $F : X \times X \times X^* \to \mathbb{R}$, the convexity is extended to the F-invex, that is

$$f(x) - f(x_0) \geq F(x, x_0 : f'(x_0)),$$

combining (1.1) and (1.2), we call that function f is (F, ρ, θ)-convex (resp. strictly (F, ρ, θ)-convex), if

$$\begin{gathered} f(x) - f(x_0) \geq F(x, x_0 : f'(x_0)) + \rho\theta(x, x_0) \\ (\text{resp. } f(x) - f(x_0) > F(x, x_0 : f'(x_0)) + \rho\theta(x, x_0)). \end{gathered} \tag{1.3}$$

For more detailed application, one can refer Chen and Lai [1] as well as Lee and Lai [13].

In this paper, we will apply such concept of generalized convexity to deal with the minimax programming problem in complex spaces described as the form:

$$(P_c) \qquad \min_{\xi \in X} \max_{\eta \in Y} \operatorname{Re} \varphi(\xi, \eta)$$
$$\text{subject to } -g(\xi) \in S \subset \mathbb{C}^p$$

where $X = \{\xi = (z, \bar{z}) \in \mathbb{C}^{2n} | z \in \mathbb{C}^n\}$; $Y = \{\eta = (w, \bar{w}) \in \mathbb{C}^{2m} | w \in \mathbb{C}^m\}$ is a compact subset in $\mathbb{C}^{2m}$; S is a polyhedral cone in $\mathbb{C}^p$; and for each $\eta \in Y$, the maps

$$\varphi(\bullet, \eta) : \mathbb{C}^{2n} \to \mathbb{C} \text{ and } g(\bullet) : \mathbb{C}^{2n} \to \mathbb{C}^p$$

are assumed to be analytic over the manifold X.

In real variable case, problem (P_c) was considered as the form

$$(P_r) \qquad \underset{x\in X}{\text{Min}} \quad \underset{y\in Y}{\text{Max}} \quad f(x,y)$$
$$\text{subject to } -g(x) \leq 0 \text{ for } x \in X \subset \mathbb{R}^n$$

where Y is a compact subset of $\mathbb{R}^m$, $f:\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}$ and $g:\mathbb{R}^n\to\mathbb{R}^m$ are continuously differentiable maps. In [16], Schmittendor established the necessary and sufficient optimality conditions for the minimax programming problem (P_r) under the assumptions of convexity in f and g. Later, Tanimoto [17] proved the duality theorems for (P_r). Henceafter, many authors investigated the minimax programming problem in fractional or nonfractional nonlinear programming (cf. Lai et al. [5, 9, 11] and their references).

On the other aspect, programming problems could be considered in complex spaces, so called complex programming problems. It has many applications in electrical network with alternating currents/voltages by using complex variable $z \in \mathbb{C}^n$ to stand for elements of network. In variant fields of electric engineering, like the complex programming problems which are employed in *blind deconvolution*, *blind equalization*, *minimal entropy*, *optimal receiver* etc (cf. Lai and Liu [7, 8]). Concerning the complex programming, Levinson [14] was firstly studied in linear case, then Swarup and Sharma [16] studied the linear fractional programming in complex space. Henceforth, many authors investigated nonlinear fractional as well as non-fractional nonlinear programming in complex spaces (see [2-4], [6-8] and [15]).

It is remarkable that the manifold X is a closed convex cone over the real field $\mathbb{R}$ (not over complex field $\mathbb{C}$). In order to have the convexity of the real part for a nonlinear complex function, the function $\varphi(\xi,\eta)$ in (P_c) is taking the variables in the form

$$\xi = (z,\bar{z}) \in \mathbb{C}^{2n} \text{ and } \eta = (w,\bar{w}) \in \mathbb{C}^{2m}$$

for our requirement of complex minimax problem since any nonlinear analytic function $f(z), z \in \mathbb{C}^n$ can not have convex real part (cf. Ferrero [3]).

In this paper, we will establish the sufficient optimality conditions for problem (P_c) under the framework of generalized quasi/pseudo-convexity which extends the results of Datta and Bhatia [2]. Employing the existent theorems for optimal solution, we also treat with the duality problem to the minimax problem (P_c), and prove the weak, strong as well as the strict converse duality theorems.

Rather than explore optimality conditions, let us now start our work by some preparation. We describe some notations and definitions for complex programming in the next section.

2. Definitions and Notations

We say that a subset $S \subset \mathbb{C}^p$ is a **polyhedral cone** if there is an integer $k \in \mathbb{N}$ and a matrix $A \in \mathbb{C}^{p\times k}$ such that

$$S = A\mathbb{R}^k_+ = \{Ax | x \in \mathbb{R}^k_+\},$$

that is, S is generated by a finite number of vectors. Correspondingly, $S \subset \mathbb{C}^p$ is a **polyhedral cone** if it is the intersection of a finite number of closed half-spaces having the origin on the boundary or there is an integer $k \in \mathbb{N}$ and k-points $u_1, u_2, \cdots, u_k$ in $\mathbb{C}^p$ such that

$$S = \bigcap_{j=1}^{k} H(u_j) = \{z \in \mathbb{C}^p | \mathrm{Re}\,\langle z, u_j\rangle \geq 0, j = 1, 2, \cdots, k\}$$

where $H(u_j), j = 1, 2, \cdots, k$ are closed half-spaces involving the point u_j, the polar (or dual) of the set S is given by

$$S^* = \{u \in \mathbb{C}^p | \mathrm{Re}\,\langle z, u\rangle \geq 0 \text{ for all } z \in S\}.$$

Clearly $S^{**} = S$, and $S^* = S$ in the finite dimensional space.

For each $\xi \in X \subset \mathbb{C}^{2n}$, if Y is a compact subset in $\mathbb{C}^{2m}$, the set

$$Y(\xi) = \{\eta \in Y | \mathrm{Re}\,\varphi(\xi, \eta) = \sup_{\zeta\in Y} \mathrm{Re}\,\varphi(\xi, \zeta)\}$$

is a nonempty compact subset of $\mathbb{C}^{2m}$ when $\varphi(\xi, \bullet)$ is analytic on $Y \subset \mathbb{C}^{2m}$.

For each $\eta = (w, \widetilde{w}) \in Y$, the mappings

$$\varphi(\bullet, \eta) : \mathbb{C}^{2n} \to \mathbb{C} \text{ and } g(\bullet) : \mathbb{C}^{2n} \to \mathbb{C}^p$$

are analytic on $\xi = (z, \bar{z}) \in X \subset \mathbb{C}^{2n}$, and by the general Mean Value theorem,

$$\varphi(\xi, \eta) - \varphi(\xi_0, \eta) = \varphi'_\xi(\xi_0, \eta)(\xi - \xi_0) + o(|\xi - \xi_0|)$$

and

$$g(\xi) - g(\xi_0) = g'(\xi_0)(\xi - \xi_0) + o(|\xi - \xi_0|),\ \xi_0 = (z_0, \bar{z}_0) \in X$$

where

$$\begin{aligned}
\varphi'_\xi(\xi_0, \eta)(\xi - \xi_0) &= (\bigtriangledown_z\varphi(\xi_0, \eta), \bigtriangledown_{\bar{z}}\varphi(\xi_0, \eta)) \begin{pmatrix} z - z_0 \\ \bar{z} - \overline{z_0} \end{pmatrix} \\
&= \bigtriangledown_z\varphi(\xi_0, \eta)(z - z_0) + \bigtriangledown_{\bar{z}}\varphi(\xi_0, \eta)(\bar{z} - \overline{z_0}) \\
&\quad \in \mathbb{C}, \\
g'(\xi_0)(\xi - \xi_0) &= (\bigtriangledown_z g(\xi_0), \bigtriangledown_{\bar{z}} g(\xi_0)) \begin{pmatrix} z - z_0 \\ \bar{z} - \overline{z_0} \end{pmatrix} \\
&= \bigtriangledown_z g(\xi_0)(z - z_0) + \bigtriangledown_{\bar{z}} g(\xi_0)(\bar{z} - \overline{z_0}) \\
&\quad \in \mathbb{C}^p.
\end{aligned}$$

In view of the expression (1.3), it behooves us to recall the definition of $(\mathcal{F}, \rho, \theta)$-convexity concerning analytic functions. Suppose that the functional $\mathcal{F} : \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{R}$ is sublinear with respect to the 3rd argument. That is,

$$\mathcal{F}(z_1, z_2; u_1 + u_2) \leq \mathcal{F}(z_1, z_2; u_1) + \mathcal{F}(z_1, z_2; u_2)$$

and

$$\mathcal{F}(z_1, z_2, \alpha u) = \alpha \mathcal{F}(z_1, z_2; u) \text{ for } \alpha \geq 0.$$

Let $\theta : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{R}_+$ be such that $\theta(z_1, z_2) = 0$ only if $z_1 = z_2$, and let $\rho \in \mathbb{R}$.

Now for simplicity, given any $\eta \in Y \subset \mathbb{C}^{2m}$, $\xi = (z, \bar{z})$ and $\xi_0 = (z_0, \overline{z_0})$ in X, we use the notations:

$$\left.\begin{aligned} A &= \mathrm{Re}\,[\varphi(\xi, \eta) - \varphi(\xi_0, \eta)] \\ B &= \mathcal{F}(z, z_0; \overline{\nabla_z \varphi}(\xi_0, \eta) + \nabla_{\bar{z}} \varphi(\xi_0, \eta)) + \rho\theta(z, z_0) \end{aligned}\right\} \tag{2.1}$$

and for any $\mu \in S$, the polyhedral cone in $\mathbb{C}^p$, we use:

$$\left.\begin{aligned} \widetilde{A} &= \mathrm{Re}\,\langle \mu, g(\xi) - g(\xi_0) \rangle \\ \widetilde{B} &= \mathcal{F}(z, z_0; \mu^\top \overline{\nabla_z g}(\xi_0) + \mu^H \nabla_{\bar{z}} g(\xi_0)) + \rho\theta(z, z_0) \end{aligned}\right\} \tag{2.2}$$

where $\mu^\top$ stands for transpose of μ, and μ^H denotes the hermitian of μ, that is, $\mu^H = (\bar{\mu})^\top$.

Then the $(\mathcal{F}, \rho, \theta)$-convexity, -quasiconvexity, -pseudoconvexity of the analytic functions $\varphi(\bullet, \eta)$ and $g(\bullet)$ on X are defined as follows (cf. Lai et al. [7,8]).

Definitions

1. (a) The real part $\mathrm{Re}\ \varphi(\bullet, \eta)$ of the function $\varphi(\bullet, \eta) : X \subset \mathbb{C}^{2n} \to \mathbb{C}$ is $(\mathcal{F}, \rho, \theta)$- **convex** [resp. **strictly** $(\mathcal{F}, \rho, \theta)$-**convex**] with respect to (=w.r.t. for abbreviation) $\mathbb{R}_+$,
if $A \geq B$ [resp. $A > B$].

(b) The mapping $g : X \subset \mathbb{C}^{2n} \to \mathbb{C}^p$ is $(\mathcal{F}, \rho, \theta)$-convex [resp. **strictly** $(\mathcal{F}, \rho, \theta)$-**convex**] w.r.t. the polyhedral cone $S \subset \mathbb{C}^p$,
if $\widetilde{A} \geq \widetilde{B}$ [resp $\widetilde{A} > \widetilde{B}$].

2. (a) $\mathrm{Re}\ \varphi(\bullet, \eta)$ is $(\mathcal{F}, \rho, \theta)$-**quasiconvex** [resp. **strictly** $(\mathcal{F}, \rho, \theta)$-**quasiconvex**],
if $A \leq 0 \Rightarrow B \leq 0$ [resp. $A < 0 \Rightarrow B \leq 0$].
Equivalently, $B > 0 \Rightarrow A > 0$ [resp. $B > 0 \Rightarrow A \geq 0$].

(b) The mapping $g : X \subset \mathbb{C}^{2n} \to \mathbb{C}^p$ is $(\mathcal{F}, \rho, \theta)$-**quasiconvex** [resp. **strictly** $(\mathcal{F}, \rho, \theta)$-**quasiconvex**] w.r.t. polyhedral cone S in $\mathbb{C}^p$
if $\widetilde{A} \leq 0 \Rightarrow \widetilde{B} \leq 0$ [resp. $\widetilde{A} < 0 \Rightarrow \widetilde{B} \leq 0$].
Equivalently, $\widetilde{B} > 0 \Rightarrow \widetilde{A} > 0$ [resp. $\widetilde{B} > 0 \Rightarrow \widetilde{A} \geq 0$].

3. (a) $\mathrm{Re}\,\varphi(\bullet,\eta)$ is $(\mathcal{F},\rho,\theta)$-**pseudoconvex** [resp. **strictly** $(\mathcal{F},\rho,\theta)$-**pseudoconvex**] w.r.t. $\mathbb{R}_+$,
if $$B \geq 0 \Rightarrow A \geq 0 \text{ [resp. } B \geq 0 \Rightarrow A > 0].$$
Equivalently, $A < 0 \Rightarrow B < 0$ [resp. $A \leq 0 \Rightarrow B < 0$].

(b) The mapping $g : X \subset \mathbb{C}^{2n} \to \mathbb{C}^p$ is $(\mathcal{F},\rho,\theta)$-**pseudoconvex** [resp. **strictly** $(\mathcal{F},\rho,\theta)$-**pseudoconvex**] w.r.t. the polyhedral cone $S \subset \mathbb{C}^p$,
if $$\widetilde{B} \geq 0 \Rightarrow \widetilde{A} \geq 0 \text{ [resp. } \widetilde{B} \geq 0 \Rightarrow \widetilde{A} > 0].$$
Equivalently, $\widetilde{A} < 0 \Rightarrow \widetilde{B} < 0$ [resp. $\widetilde{A} \leq 0 \Rightarrow \widetilde{B} < 0$].

3. Optimality Conditions

By Fritz John type optimality conditions in the complex programming problem (P_c), the following theorem will be useful in discussion of optimality.

Theorem 3.1. (*Necessary optimality condition*). *Let $\xi_0 = (z_0, \overline{z_0})$ be an optimal solution of (P_c), and ξ_0 is a regular point for the constraint function $g : \mathbb{C}^{2n} \to \mathbb{C}^p$, that is, the gradient components $g_1'(\xi_0), g_2'(\xi_0), \cdots, g_p'(\xi_0)$ of $g(\xi_0)$ are linearly independent, equivalently*

$$\langle g'(\xi_0), \mu\rangle = 0 \Rightarrow \mu = 0 \text{ in } \mathbb{C}^p.$$

Then for $k \in \mathbb{N}$, the set of natural numbers, there exist multipliers $\lambda \in \mathbb{R}^k_+$ with $|\lambda| = \sum_{i=1}^k \lambda_i = 1$, a nonzero $\mu \in S^* \subset \mathbb{C}^p$ and vectors $\eta_i \in Y(\xi_0), i = 1, 2, \cdots, k$ such that

$$\sum_{i=1}^k \lambda_i(\overline{\nabla_z \varphi}(\xi_0,\eta_i) + \nabla_{\bar{z}}\varphi(\xi_0,\eta_i)) + \langle \mu, \nabla_z g(\xi_0)\rangle + \overline{\langle \mu, \nabla_{\bar{z}} g(\xi_0)\rangle} = 0 \tag{3.1}$$

$$\mathrm{Re}\,\langle \mu, g(\xi_0)\rangle = 0 \tag{3.2}$$

where $\langle\bullet,\bullet\rangle$ stands for inner product in $\mathbb{C}^p$ and $\overline{\langle \mu, \nabla_{\bar{z}} g(\xi_0)\rangle} \equiv \mu^H \nabla_{\bar{z}}\, g(\xi_0)$.

The existence of optimal solution in a programming problem can be deduced from the converse of necessary optimality conditions by extra assumptions. Thus, instead of convexity, many authors explore the possibility of the extra conditions. e.g. the invexity, pseudo/quasi convexity etc. Recently, Lai and Liu defined the $(\mathcal{F},\rho,\theta)$-convexity (See [7, 8], cf also [12]), and employed such generalized convexity to treat with the sufficient optimality conditions on complex fractional programming. We will use the crucial role that convexity plays in the complex minimax programming problem (P_c) and its duality problem.

Based on the above Definitions, we can establish the following results.

Theorem 3.2. (*Sufficient optimality condition I*). *Suppose that*

$(A1)$ *Let* $\xi_0 = (z_0, \overline{z_0}) \in X_{P_c} \subset \mathbb{C}^{2n}$ *be a feasible solution of* (P_c). *For a positive integer* k, *there exist* $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k) \in \mathbb{R}^k_+$ *with* $\Sigma_{i=1}^k \lambda_i = 1$, *a nonzero multiplier* $\mu \in S^*$, *polar of* S *in* $\mathbb{C}^p$, *and vectors* $\eta_i \in Y(\xi_0),\ i = 1, 2, \cdots, k$ *such that the results* (*3.1*) *and* (*3.2*) *hold in Theorem 3.1.*
In addition, suppose that

$(A2)$ (i) $\mathrm{Re}\left[\sum_{i=1}^k \lambda_i \varphi(\bullet, \eta_i)\right]$ *is a strict* $(\mathcal{F}, \rho_1, \theta)$*-quasiconvex function w.r.t.* $\mathbb{R}^+$ *on* X,

(ii) $g(\bullet) : X \subset \mathbb{C}^{2n} \to \mathbb{C}^p$ *is a strict* $(\mathcal{F}, \rho_2, \theta)$*-pseudoconvex map w.r.t. the polyhedral cone* $S \subset \mathbb{C}^p$ *on* X,

(iii) $\rho_1 + \rho_2 \geq 0$.

Then $\xi_0 = (z_0, \overline{z_0})$ *is an optimal solution of* (P_c).

Proof. Suppose on the contrary that $\xi_0 = (z, z_0)$ were not an optimal solution of (P_c). Then there is a $\xi = (z, \overline{z})$ such that

$$\sup_{\eta \in Y} \mathrm{Re}\ \varphi(\xi, \eta) < \sup_{\eta \in Y} \mathrm{Re}\ \varphi(\xi_0, \eta).$$

It follows that, for a positive $k \in \mathbb{N}$, $\eta_i \in Y(\xi_0),\ i = 1, 2, \cdots, k$, we have

$$\mathrm{Re}\ \varphi(\xi, \eta_i) < \mathrm{Re}\ \varphi(\xi_0, \eta_i),\ i = 1, 2, \cdots, k.$$

Now for $\lambda = (\lambda_i),\ \lambda_i \geq 0,\ |\lambda| = \sum_{i=1}^k \lambda_i = 1$, we have

$$\mathrm{Re}\left[\sum_{i=1}^k \lambda_i(\varphi(\xi, \eta_i) - \varphi(\xi_0, \eta_i))\right] < 0. \tag{3.3}$$

By hypothesis, $\mathrm{Re} \sum_{i=1}^k \lambda_i \varphi(\bullet, \eta_i)$ is strictly $(\mathcal{F}, \rho_1, \theta)$-quasiconvex, then (3.3) implies that

$$\mathcal{F}\left(z, z_0; \sum_{i=1}^k \lambda_i \left(\overline{\nabla_z \varphi}(\xi_0, \eta_i) + \nabla_{\overline{z}} \varphi(\xi_0, \eta_i)\right)\right) \leq -\rho_1 \theta(z, z_0). \tag{3.4}$$

On the other hand $\xi \in X_{P_c},\ \xi_0$ satisfies (2.2) and so for $\mu \in S^*$, we see that

$$\begin{aligned} &\mathrm{Re}\ \langle \mu, g(\xi) \rangle \leq 0 = \mathrm{Re}\ \langle \mu, g(\xi_0) \rangle, \\ &\mathrm{Re}\ \langle \mu, g(\xi) - g(\xi_0) \rangle \leq 0. \end{aligned} \tag{3.5}$$

By assumption (ii), $g(\bullet)$ is a strict $(\mathcal{F}, \rho_2, \theta)$ -pseudoconvex map w.r.t. the polyhedral cone S, then (3.5) implies that

$$\mathcal{F}\left(z, z_0; \mu^\top \overline{\nabla_z g}(\xi_0) + \mu^H \nabla_{\overline{z}}\, g(\xi_0)\right) < -\rho_2 \theta(z, z_0). \tag{3.6}$$

Adding (3.4) and (3.6), we conclude from sublinearity of $\mathcal{F}(z, z_0; 0)$ and condition (3.1) that

$$0 < -(\rho_1 + \rho_2)\theta(z, z_0),$$

that is,

$$\rho_1 + \rho_2 < 0 \text{ since } \theta(z, z_0) > 0 \text{ if } z \neq z_0.$$

This inequality contradicts the fact of $\rho_1 + \rho_2 \geq 0$. Hence ξ_0 is an optimal solution of (P_c). ■

Theorem 3.3. (*Sufficient optimality condition II*). *Suppose that the assumptions* $(A1)$ *and* $(A2)(i)$ *in Theorem 3.2 are fulfilled. Further assume that*

$(A3)$ (i) *the same as (A2)(i),*

(ii) $g(\bullet)$ *is* $(\mathcal{F}, \rho_2, \theta)$*-quasiconvex w.r.t.* $S \subset \mathbb{C}_p$ *on* X *and*

(iii) $\rho_1 + \rho_2 > 0$.

Then $\xi_0 = (z_0, \overline{z_0})$ *is an optimal solution of* (P_c).

Proof. If $\xi_0 = (z_0, \overline{z_0})$ were not an optimal of (P_c) then like the proof given in the first part of Theorem 3.2, we have the inequalities (3.3) and (3.4). Here, by the strict $(\mathcal{F}, \rho_1, \theta)$-quasiconvexity, (3.3) implies (3.4), that is,

$$\mathcal{F}\left(z, z_0; \sum_{i=1}^{k} \lambda_i \left(\overline{\nabla_z \varphi}(\xi_0, \eta_i) + \nabla_{\overline{z}} \varphi(\xi_0, \eta_i)\right)\right) \leq -\rho_1 \theta(z, z_0). \tag{3.4}$$

Now if $g(\bullet)$ is (F, ρ_2, θ)-quasiconvex w.r.t. the polyhedral cone S, then (3.5) would imply

$$\mathcal{F}\left(z, z_0; \mu^\top \overline{\nabla_z g}(\xi_0) + \mu^H \nabla_{\overline{z}}\, g(\xi_0)\right) \leq -\rho_2 \theta(z, z_0). \tag{3.7}$$

Adding (3.4) and (3.7), and by the sublinearity of $\mathcal{F}(z, z_0; \bullet)$ and the condition (3.1), we obtain

$$0 \leq -(\rho_1 + \rho_2)\theta(z, z_0),$$

or

$$\rho_1 + \rho_2 \leq 0 \text{ whenever } \theta(z, z_0) > 0 \ (z \neq z_0).$$

This result contradicts the fact of $\rho_1 + \rho_2 > 0$. Hence ξ_0 is an optimal solution of (P_c). ■

Theorem 3.4. (*Sufficient optimality condition III*).
Suppose that the assumption A1 in Theorem 3.2 holds. In addition, assume that

$(A4)$ (i) $\mathrm{Re}\left(\sum_{i=1}^{k}\lambda_i\varphi(\bullet,\eta_i)+\mu^H g(\bullet)\right)$ *is a strictly* (F,ρ,θ)*-quasiconvex w.r.t.* $\mathbb{R}_+$ *on* X, *and*

(ii) $\rho > 0$.

Then $\xi_0 = (z_0, \overline{z_0})$ *is an optimal solution of* (P_c).

Proof. Suppose on the contrary that, there is another feasible point $\xi = (z, \overline{z}) \in X_{P_c}$ such that

$$\sup_{\eta\in Y}\mathrm{Re}\ \varphi(\xi,\eta) < \sup_{\eta\in Y}\mathrm{Re}\ \varphi(\xi_0,\eta).$$

Since Y is compact, there are finite points $\eta_1, \eta_2, \cdots, \eta_k$ in $Y(\xi_0)$ such that

$$\mathrm{Re}\ \varphi(\xi,\eta_i) < \mathrm{Re}\ \varphi(\xi_0,\eta_i) \text{ for } i = 1, 2, \cdots, k.$$

Then, for $\lambda \in \mathbb{R}_+^k$ with $|\lambda| = \sum_{i=1}^{k}\lambda_i = 1$, we have

$$\mathrm{Re}\left[\sum_{i=1}^{k}\lambda_i(\varphi(\xi,\eta_i)-\varphi(\xi_0,\eta_i))\right] < 0. \tag{3.9}$$

On the other hand, the λ, ξ_0 and $\mu \in S^* \subset \mathbb{C}^p$ satisfy (3.1), and ξ is a feasible point of (P_c), we then obtain

$$\begin{aligned} &\mathrm{Re}\,\langle\mu,\ g(\xi)\rangle \le \mathrm{Re}\,\langle\mu,\ g(\xi_0)\rangle \\ \text{or }\ &\mathrm{Re}\,\langle\mu,\ g(\xi)-g(\xi_0)\rangle \le 0. \end{aligned} \tag{3.10}$$

Thus adding (3.9) and (3.10), we have

$$\mathrm{Re}\left[\sum_{i=1}^{k}\lambda_i\left(\varphi(\xi,\eta_i)-\varphi(\xi_0,\eta_i)\right)+\mu^H\left(g(\xi)-g(\xi_0)\right)\right] < 0. \tag{3.11}$$

Since $\mathrm{Re}\,(\sum_{i=1}^{k}\lambda_i\varphi(\bullet,\eta_i)+\mu^H g(\bullet))$ is strictly $(\mathcal{F},\rho,\theta)$-quasiconvex w.r.t. $\mathbb{R}_+$ on X , thus (3.11) implies that

$$\begin{aligned} &\mathcal{F}\left(z, z_0; \overline{\nabla_z\varphi}(\xi_0,\eta)+\nabla_{\overline{z}}\varphi(\xi_0,\eta)+\mu^\top\overline{\nabla_z g}(\xi_0)\right. \\ &\qquad \left.+\mu^H\,\nabla_{\overline{z}}\,g(\xi_0)\right) \le -\rho\theta(z,z_0). \end{aligned} \tag{3.12}$$

Since $\mathcal{F}$ is sublinear, and by condition (2.1), we have

$$0 \le -\rho\theta(z,z_0) \text{ or } \rho\theta(z,z_0) \le 0,$$

it leads $\rho \le 0$ since $\theta(z,z_0) > 0$. This contradicts the assumption (A3)(ii).

Hence $\xi_0 = (z_0, \overline{z_0})$ is an optimal solution. ■

4. Duality Model

The minimax programming problem (P_c) can be rewritten as the following form:

$$(P_c) \qquad \underset{\xi\in X}{\text{Min}}\ f(\xi), \quad f(\xi)=\sup_{\eta\in Y}\ \text{Re}\,\varphi(\xi,\eta)$$
$$\text{subject to } -g(\xi)\in S\subset\mathbb{C}^p$$

where $X=\{\xi=(z,\bar{z})\in\mathbb{C}^{2n}, z\in\mathbb{C}^n\}$, Y is compact in $\mathbb{C}^{2m}$, and S is a polyhedral cone in $\mathbb{C}^p$.

Since Y is compact, for each $\xi\in X$, the $\sup_{\eta\in Y}\text{Re}\,\varphi(\xi,\eta)$ is attainable if $\varphi(\xi,\bullet)$ is analytic in $\mathbb{C}^{2m}$. Let

$$Y(\xi)=\{\eta\in Y|\text{Re}\,\varphi(\xi,\eta)=\sup_{\zeta\in Y}\text{Re}\,\varphi(\xi,\zeta)\}, \tag{4.1}$$

$$W(\xi)=\Big\{(k,\lambda,\eta^k)\in\mathbb{N}\times\mathbb{R}^k_+\times\mathbb{C}^{2mk}\Big|\lambda\in\mathbb{R}^k_+ \text{ with } |\lambda|=\textstyle\sum_{j=1}^k\lambda_j=1, \text{ and } \eta^k=(\eta_1,\eta_2,\cdots,\eta_k),\ \eta_j\in Y(\xi)\Big\}. \tag{4.2}$$

Then by employing the sufficient optimality conditions for problem (P_c), we can constitute a duality model as the form

$$(D) \qquad \max_{(k,\lambda,\eta^k)\in W(\xi)}\ \sup_{(\xi,\mu,\overline{\mu})\in\widetilde{X}(k,\lambda,\eta^k)} f(\xi)$$

where $W(\xi)$ is defined by (4.2), and for $(k,\lambda,\eta^k)\in W(\xi)$,

$$\widetilde{X}(k,\lambda,\eta^k)=\big\{(\xi,\mu,\overline{\mu})\in\mathbb{C}^{2n}\times\mathbb{C}^p\times\mathbb{C}^p\mid \mu\neq 0 \text{ in } S^*,\ \text{Re}\,\langle\mu,g(\xi)\rangle\geq 0\big\}$$

is the set of points $(\xi,\mu,\bar{\mu})$ satisfying the condition (3.1) and

$$\text{Re}\,\langle\mu,g(\xi)\rangle\geq 0. \tag{4.3}$$

Here for $(k,\lambda,\eta^k)\in W(\xi)$, if the set $\widetilde{X}(k,\lambda,\eta^k)=\emptyset$, we define the supremun over $\widetilde{X}$ to be -∞.

We prove that (D) plays a duality to the primary problem (P_c), and the duality theorems are established in the next section.

5. Duality Theorems

Theorem 5.1. $($*Weak Duality*$)$. *Let* $\zeta=(z,\bar{z})\in X_{P_c}$ *and* $(k,\lambda,\eta^k,\xi,\mu,\overline{\mu})$ *be the feasible solutions of* (P_c) *and* (D), *respectively. Suppose that any one of the conditions (a)∼(c) is satisfied*:

(a) *(A2) in Theorem 3.2*,

(b) *(A3) in Theorem 3.3*,

(c) *(A4) in Theorem 3.4.*

Then $f(\xi) \leq f(\zeta)$. *Equivalently*,

$$\sup_{\eta \in Y} \operatorname{Re} \varphi(\xi, \eta) \leq \sup_{\eta \in Y} \operatorname{Re} \varphi(\zeta, \eta).$$

Proof. Suppose on the contrary that there is another feasible point $\xi = (z, \overline{z})$ such that

$$f(\zeta) < f(\xi),$$

equivalently

$$\sup \operatorname{Re} \varphi(\zeta, \eta) < \sup_{\eta \in Y} \operatorname{Re} \varphi(\xi, \eta). \tag{5.1}$$

Then we have

$$\operatorname{Re} \varphi(\zeta, \widetilde{\eta}) < \operatorname{Re} \varphi(\xi, \widetilde{\eta}) \text{ for all } \widetilde{\eta} \in Y(\xi) \subset Y. \tag{5.2}$$

Since $Y(\xi)$ is compact, there exist finite points $\widetilde{\eta}_j \in Y(\xi),\ j = 1, 2, \cdots, k$ such that

$$\sup_{\eta \in Y} \operatorname{Re} \varphi(\xi, \eta) = \operatorname{Re} \varphi(\xi, \widetilde{\eta}_j),\ j = 1, 2, \cdots, k. \tag{5.3}$$

By (5.2) and (5.3), we get

$$\operatorname{Re} \varphi(\xi, \widetilde{\eta}_j) < \operatorname{Re} \varphi(\xi, \widetilde{\eta}_j),\ j = 1, 2, \cdots, k.$$

Multiplying $\lambda_j \geq 0$ on the both sides of the last inequality, and summing up with condition $\sum_{j=1}^{k} \lambda_j = 1$, then we have

$$\sum_{j=1}^{k} \operatorname{Re} [\lambda_j \varphi(\zeta, \widetilde{\eta}_j)] < \sum_{j=1}^{k} \operatorname{Re} [\lambda_j \varphi(\xi, \widetilde{\eta}_j)]. \tag{5.4}$$

Since $\zeta \in X_{P_c}$, thus for a nonzero $\mu \in S^*$ and condition (4.3) we have

$$\begin{aligned} &\operatorname{Re} \langle \mu, g(\zeta) \rangle \leq 0 \leq \operatorname{Re} \langle \mu, g(\xi) \rangle, \\ &\operatorname{Re} \langle \mu, g(\zeta) - g(\xi) \rangle \leq 0. \end{aligned} \tag{5.5}$$

If hypothesis (a) holds, by (A2)(i), the function $\mathrm{Re}\,[\sum_{j=1}^{k}\lambda_j\varphi(\bullet,\eta_j)]$ is **strictly** $(\mathcal{F},\rho_1,\theta)$-**quasiconvex** w.r.t. $\mathbb{R}_+$ on X, it follows that the inequality (5.4) implies

$$\mathcal{F}\left(z,z_0;\sum_{j=1}^{k}\lambda_j\overline{\nabla_z\varphi}(\xi,\widetilde{\eta_j})+\sum_{j=1}^{k}\lambda_j\nabla_{\bar z}\varphi(\xi,\widetilde{\eta})\right)\le-\rho_1\theta(z,z_1). \tag{5.6}$$

By (A2)(ii), $g(\bullet)$ is **strict** $(\mathcal{F},\rho_2,\theta)$-**pseudoconvex** w.r.t. the polyhedral cone S, the inequality (5.5) implies that

$$\mathcal{F}(z,z_1;\mu^{\top}\overline{\nabla_z g}(\xi)+\mu^{H}\nabla_{\bar z}g(\xi))<-\rho_2\theta(z,z_1). \tag{5.7}$$

Adding (5.6) and (5.7), and by the sublinearity of $\mathcal{F}(z,z_1;\bullet)$ and condition (3.1), we obtain

$$0<-(\rho_1+\rho_2)\theta(z,z_1).$$

It leads to $\rho_1+\rho_2<0$ when $\theta(z,z_1)>0$ $(z\neq z_1)$. This contradicts the fact (A2)(iii): $\rho_1+\rho_2\ge 0$. Hence $f(\xi)\le f(\zeta)$.

If hypothesis (b) holds, it can be proved by the same line as the case in (a), and get the expressions (5.1)$\sim$(5.6). While the condition (A3)(ii): $g(\bullet)$ **is** $(\mathcal{F},\rho_2,\theta)$-**quasiconvex**, it follows that (5.5) implies

$$\mathcal{F}(z,z_1;\mu^{\top}\overline{\nabla_z g}(\xi)+\mu^{H}\nabla_{\bar z}g(\xi))\le-\rho_2\theta(z,z_1). \tag{5.8}$$

By adding (5.6) and (5.8), the condition of (3.1) and the sublinearity of $\mathcal{F}(z,z_1:\bullet)$ reduce to

$$0\le-(\rho_1+\rho_2)\theta(z,z_1).$$

It leads to $\rho_1+\rho_2\le 0$. This contradicts the fact of $\rho_1+\rho_2>0$.

If hypothesis (c) holds, the proof can be carried by the same line (a) as well as (b). ∎

Suppose that the assumptions of Theorem 5.1 are fulfilled. Then the optimal solution ξ_0 of problem (P_c) could reduce to an optimal solution of the dual problem (D) and the two optimal values of (P) and (D) are equal. More specifically, we summarize the result in the following theorem.

Theorem 5.2. (*Strong Duality*). *Let ξ_0 be an optimal solution of problem (P_c). If ξ_0 is also a regular point for the mapping $g:\mathbb{C}^{2m}\to\mathbb{C}^p$. Then there exist $(k,\lambda,\eta^k)\in W(\xi_0)$ and $(\xi_0,\mu,\overline{\mu})\in\widetilde{X}(k,\lambda,\eta^k)$ such that $(k,\lambda,\eta^k,\xi_0,\mu,\overline{\mu})$ is a feasible solution for (D). If the assumptions of Theorem 5.1 are fulfilled, then*

$(k, \lambda, \eta^k, \xi_0, \mu, \overline{\mu})$ is an optimal solution of (D), and the two optimal values of (P_c) and (D) are equal.

Proof. If ξ_0 is an optimal solution of (P_c), by Theorem 2.1, there exist $(k, \lambda, \eta) \in W(\xi_0)$ and $(\xi_0, \mu, \overline{\mu}) \in \widetilde{X}(k, \lambda, \eta)$ satisfying (2.1), that is, $(k, \lambda, \eta, \xi_0, \mu, \overline{\mu})$ is a feasible solution of (D). Since (P_c) and (D) have the same objective function, it follows that the feasible point $(k, \lambda, \eta, \xi_0, \mu, \overline{\mu})$ is an optimal solution for (D), and the optimal values of (P_c) and (D) are equal. ■

Now, if both problems (P_c) and (D) have their own optimal solutions respectively, then Theorem 5.2 implies that the optimal solution of (P_c) induces to an optimal solution of (D). The question rises that does the induced optimal solution of (D) coincide with the original (D)-optimal ? This argument is the converse duality of (D).

Actually, the optimal solution of (D) can also reduce to an optimal solution of (P_c), and the induced optimal solution coincide with the original solution of (P_c) provided that assumptions of Theorem 5.1 are fulfilled. Precisely, we can state this result as the following theorem.

Theorem 5.3. (*Stirct Converse Duality*). *Let $\widehat{\zeta}$ and $(\hat{k}, \hat{\lambda}, \hat{\eta}, \hat{\xi}, \hat{\mu}, \hat{\overline{\mu}})$ be optimal solutions of (P_c) and (D) respectively. Suppose that the assumptions of Theorem 5.2 are fulfilled. Then the optimal solution of (D) reduces to the optimal solution $\widehat{\zeta}$ such that $\widehat{\xi} = \widehat{\zeta}$, and their optimal values are equal.*

Remark. One can reduce the sufficient optimality conditions by using the appropriate combination for the generalized convexity of $\mathrm{Re}\,\varphi(\bullet, \eta)$ as well as the analytic map $g(\bullet)$ in the framework of the paper, and the relative duality theorems are also established.

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