TAIWANESE JOURNAL OF MATHEMATICS
Vol. 8, No. 4, pp. 639-652, December 2004
This paper is available online at http://www.math.nthu.edu.tw/tjm/

POSITIVE PERIODIC SOLUTIONS OF COUPLED DELAY DIFFERENTIAL SYSTEMS DEPENDING ON TWO PARAMETERS

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Abstract. A coupled functional differential systems depending on two parameters is considered. It is shown that there are three mutually exclusive and exhaustive subsets Θ_1, Γ and Θ_2 of the parameter space such that there exist at least two positive periodic solutions associated with pairs in $\Theta_1,$ at least one positive periodic solution associated with Γ and none associated with Θ_2.

1. INTRODUCTION

Coupled differential systems arise in a number of biological, ecological, economical and other models which describe interactions. In particular, predator and prey differential systems are good examples.

In this article, we are concerned with the existence and nonexistence of positive periodic solutions for a class of first order functional differential systems of the form

$$(1)\qquad \begin{cases} x'(t) = -a(t)\,x(t) + \lambda k(t)\, f\left(x\left(t-\tau_1(t)\right), y\left(t-\sigma_1(t)\right)\right), \\ y'(t) = -b(t)y(t) + \mu h(t) g\left(x\left(t-\tau_2(t)\right), y\left(t-\sigma_2(t)\right)\right), \end{cases}$$

where $a = a(t), b = b(t), k = k(t), h = h(t)\ \tau_1 = \tau_1(t), \tau_2 = \tau_2(t), \sigma_1 = \sigma_1(t)$ and $\sigma_2 = \sigma_2(t)$ are continuous ω-periodic functions. To avoid trivial cases, we will assume that the period ω is a positive number. The functions $f = f(u,v)$, $g = g(u,v)$, $k = k(t)$ as well as $h = h(t)$ are positive continuous functions, and the functions $a(t)$ and $b(t)$ are continuous functions such that $\int_0^\omega a(t)\,dt > 0$ and

Received May 21, 2003; revised June 17, 2003.
Communicated by Valentin Afraimovich.
2000 *Mathematics Subject Classification*: 34B15.
Key words and phrases: Positive periodic solution, Coupled differential equations, Monotone method, Topological degree.
*Project supported by Natural Science Foundation of Shanxi Province and the High Science and Technology Development Foundation of Education Department of Shanxi Province.

$\int_0^\omega b\,(t)\,dt > 0$. All functions, here and in the sequel, are defined on R or R^2. The numbers λ, μ will be assumed to be nonnegative and treated as parameters.

Note that when $\lambda = 0$, the first equation in (1) reduces to

$$x'\,(t) = -a\,(t)\,x\,(t)\,,$$

which is well known in Malthusian population models. Note also that when $\lambda = \mu = 0$, our system reduces to a pair of decoupled equations. For this reason, the case $\lambda = \mu = 0$ will be avoided in our subsequent discussions. Therefore we may regard our system as a two-species interactive population model depending on (λ, μ) in the set $\{(x, y)|x, y \geq 0\} \setminus \{(0, 0)\}$.

Similar systems or equations have been studied by a number of authors, see for examples [1-21] and the references therein. In this paper, however, we will prove that there exists a continuous curve Γ spliting $\{(x, y)|x, y \geq 0\} \setminus \{(0, 0)\}$ into disjoint subsets Θ_1, Γ and Θ_2 such that the system (1) has at least two, at least one, or no positive ω-periodic solutions according to whether (λ, μ) is in Θ_1, Γ, or Θ_2, respectively. Such results are new and derived by means of the method of upper and lower solutions as well as the degree theory. Furthermore, since the curve Γ is defined by the shooting method, it can be computed numerically.

For any (a_1, b_1) and (a_2, b_2) in R^2, we will write $(a_1, b_1) \geq (a_2, b_2)$ if $a_1 \geq a_2$ and $b_1 \geq b_2$. If either $a_1 > a_2$ and $b_1 \geq b_2$ or $a_1 \geq a_2$ and $b_1 > b_2$, we will write $(a_1, b_1) > (a_2, b_2)$. A vector function $(x\,(t)\,, y\,(t))$ defined on R is said to be positive if $(x\,(t)\,, y\,(t)) \geq (0, 0)$ for all $t \in R$ and $(x\,(t_0)\,, y\,(t_0)) > (0, 0)$ for some $t_0 \in R$. Finally, the interval $[0, \infty)$ will also be denoted by R_+.

2. Preliminary Considerations

By a solution of (1) associated with the pair (α, β), we mean a vector function of the form $(x(t), y(t))$, where $t \in R$, such that $x = x(t)$ and $y = y(t)$ are continuously differentiable everywhere and satisfy (1) for $\lambda = \alpha$ and $\mu = \beta$. Assume that $(x\,(t)\,, y\,(t))$ is a ω-periodic solution of (1), then

$$\begin{cases} \left[x(t)\exp\left(\int_0^t a(s)ds\right)\right]' = \lambda\exp\left(\int_0^t a(s)ds\right)k(t)f\left(x\left(t-\tau_1\left(t\right)\right), y\left(t-\sigma_1\left(t\right)\right)\right), \\ \left[y(t)\exp\left(\int_0^t b(s)ds\right)\right]' = \mu\exp\left(\int_0^t b(s)ds\right)h(t)g\left(x\left(t-\tau_2\left(t\right)\right), y\left(t-\sigma_2\left(t\right)\right)\right). \end{cases}$$

After integrating the above equations from t to $t + \omega$, we obtain

$$\begin{cases} x(t) = \lambda\int_t^{t+\omega} K(t,s)k(s)f\left(x\left(s-\tau_1\left(s\right)\right), y\left(s-\sigma_1\left(s\right)\right)\right)ds, \\ y(t) = \mu\int_t^{t+\omega} H(t,s)h(s)g\left(x\left(s-\tau_2\left(s\right)\right), y\left(s-\sigma_2\left(2\right)\right)\right)ds, \end{cases} \tag{2}$$

where

$$K(t,s)=\frac{\exp\left(\int_t^s a(u)du\right)}{\exp\left(\int_0^\omega a(u)du\right)-1}$$

and

$$H(t,s)=\frac{\exp\left(\int_t^s b(u)du\right)}{\exp\left(\int_0^\omega b(u)du\right)-1}$$

for $t\le s\le t+\omega$. Note that the denominators in $K(t,s)$ and $H(t,s)$ are not zero since we have assumed that $\int_0^\omega a\left(t\right)dt>0$ and $\int_0^\omega b\left(t\right)dt>0$.

It is not difficult to check that any ω-periodic vector function $\left(x\left(t\right),y\left(t\right)\right)$ that satisfies (2) is also a ω-periodic solution of (1). Thus, system (1) has a ω-periodic solution $\left(x\left(t\right),y\left(t\right)\right)$ if, and only if $\left(x\left(t\right),y\left(t\right)\right)$ is a ω-periodic solution of (2). Therefore, we may transform our existence problem into a fixed point problem. To this end, we first note that

$$N\equiv\min_{0\le t,s\le\omega}K(t,s)\le K(t,s)\le\max_{0\le t,s\le\omega}K(t,s)\equiv M,\ t\le s\le t+\omega,$$

$$1\ge\frac{K(t,s)}{\max_{0\le t,s\le\omega}K(t,s)}\ge\frac{\min_{0\le t,s\le\omega}K(t,s)}{\max_{0\le t,s\le\omega}K(t,s)}=\frac{N}{M},$$

$$N'\equiv\min_{0\le t,s\le\omega}H(t,s)\le H(t,s)\le\max_{0\le t,s\le\omega}H(t,s)\equiv M',\ t\le s\le t+\omega,$$

and

$$1\ge\frac{H(t,s)}{\max_{0\le t,s\le\omega}H(t,s)}\ge\frac{\min_{0\le t,s\le\omega}H(t,s)}{\max_{0\le t,s\le\omega}H(t,s)}=\frac{N'}{M'},\ t\le s\le t+\omega.$$

Now let X be the set of all real ω-periodic continuous functions defined on R which is endowed with the usual linear structure as well as the norm

$$\|y\|=\sup_{t\in[0,\omega]}|y(t)|\,.$$

Then X^2 is also a Banach space with the norm $\|(u,v)\|=\|u\|+\|v\|$. Furthermore, let Φ and Ω be defined respectively by

$$\Phi=\left\{(u,v)\in X^2:u(t),v(t)\ge0,t\in R\right\}$$

and

$$\Omega=\left\{(u,v)\in\Phi:u\left(t\right)+v\left(t\right)\ge\alpha^*\left\|(u,v)\right\|,t\in R\right\},$$

where

$$\alpha^*=\min\left\{N/M,N'/M'\right\}.$$

Then Φ and Ω are cones in X^2.

Define, for each $(x, y) \in X^2$,

$$T_{\lambda,\mu}(x, y)(t) = (A_\lambda(x, y)(t), B_\mu(x, y)(t)),$$

where

$$A_\lambda(x, y)(t) = \lambda \int_t^{t+\omega} K(t, s)k(s)f(x(s - \tau_1(s)), y(s - \sigma_1(s)))\,ds,$$

and

$$B_\mu(x, y)(t) = \mu \int_t^{t+\omega} H(t, s)h(s)g(x(s - \tau_2(s)), y(s - \sigma_2(2)))\,ds.$$

By standard arguments, it is not difficult to see that $T_{\lambda,\mu}$ is completely continuous. Furthermore, for $(x, y) \in \Phi$,

$$\begin{aligned} A_\lambda(x, y)(t) &= \lambda \int_t^{t+\omega} K(t, s)k(s)f(x(s - \tau_1(s)), y(s - \sigma_1(s)))\,ds \\ &\le \lambda M \int_t^{t+\omega} k(s)f(x(s - \tau_1(s)), y(s - \sigma_1(s)))\,ds \\ &= \lambda M \int_0^{\omega} k(s)f(x(s - \tau_1(s)), y(s - \sigma_1(s)))\,ds, \end{aligned}$$

so that

$$\frac{1}{M}\|A_\lambda(x, y)\| \le \lambda \int_0^{\omega} k(s)f(x(s - \tau_1(s)), y(s - \sigma_1(s)))\,ds$$

and

$$\begin{aligned} A_\lambda(x, y)(t) &= \lambda \int_t^{t+\omega} K(t, s)k(s)f(x(s - \tau_1(s)), y(s - \sigma_1(s)))\,ds \\ &\ge \lambda N \int_t^{t+\omega} k(s)f(x(s - \tau_1(s)), y(s - \sigma_1(s)))\,ds \\ &= \lambda N \int_0^{\omega} k(s)f(x(s - \tau_1(s)), y(s - \sigma_1(s)))\,ds \\ &\ge \alpha^* \|A_\lambda(x, y)\|. \end{aligned}$$

Similarly, we have

$$B_\mu(x, y)(t) \ge \alpha^* \|B_\mu(x, y)\|.$$

That is, $T_{\lambda,\mu}\Phi$ is contained in Ω.

Let us say that a real function F is nondecreasing on $R_+^2 = [0, \infty) \times [0, \infty)$ if $F(u_1, v_1) \le F(u_2, v_2)$ for $(u_1, v_1) \le (u_2, v_2)$. We will need the following two assumptions.

(H1) $f(u,v)$ and $g(u,v)$ are nondecreasing and $f(0,0)>0$ and $g(0,0)>0$.

(H2) $\lim_{u,v\to\infty}\dfrac{f(u,v)}{u+v}=\infty$ and $\lim_{u,v\to\infty}\dfrac{g(u,v)}{u+v}=\infty$.

Lemma 1. *Suppose (H2) holds. For any compact subset D of $R_+^2\setminus\{(0,0)\}$, there exists a constant $b_D>0$ such that any positive ω-periodic solution (u,v) of (1) associated with $(\lambda,\mu)\in D$ will satisfy $\|(u,v)\|<b_D$.*

Proof. Suppose to the contrary that there is a sequence $\{(u_n,v_n)\}_{n=1}^{\infty}$ of positive ω-periodic solutions of (1) associated with (λ_n,μ_n) such that $(\lambda_n,\mu_n)\in D$ for all n and $\lim_{n\to\infty}\|(u_n,v_n)\|=\infty$. Note that (u_n,v_n) satisfies equation (2) so that $(u_n,v_n)\in\Omega$. That is,

$$u_n(t)+v_n(t)\geq\alpha^*\|(u_n,v_n)\|$$

for $n\geq 1$. Now assume $\lambda_n>0$ and $\mu_n\geq 0$ for sufficiently large n. Then in view of (H2), we may choose $R_f>0$, η and $n_0\geq 1$ such that $f(x,y)\geq\eta(x+y)$ for all nonnegative x,y which satisfy $x+y\geq R_f$, $u_{n_0}+v_{n_0}\geq R_f$, and

$$\alpha^*\eta N\lambda_{n_0}\int_0^{\omega}k(s)\,ds>1.$$

Thus, we have

$$\begin{aligned}\|u_{n_0}\|\geq u_{n_0}(t)&=\lambda_{n_0}\int_t^{t+\omega}K(t,s)\,k(s)\,f\left(u_{n_0}(s-\tau_1(s)),v_{n_0}(s-\sigma_1(s))\right)ds\\&\geq\alpha^*\eta N\lambda_{n_0}\int_0^{\omega}k(s)\left(\|u_{n_0}\|+\|v_{n_0}\|\right)ds>\|u_{n_0}\|.\end{aligned}$$

This is a contradiction. The case where $\lambda_n\geq 0$ and $\mu_n>0$ for sufficiently large n can similarly be proved by using $g_\infty=\infty$. The proof is complete.

Lemma 2. *Suppose (H1) holds. If (1) has a positive ω-periodic solution associated with $\left(\overline{\lambda},\overline{\mu}\right)>(0,0)$, then for any $(\lambda,\mu)\in R^2\setminus\{(0,0)\}$ that satisfies $(\lambda,\mu)\leq\left(\overline{\lambda},\overline{\mu}\right)$, equation (1) also has a positive ω -periodic solution associated with (λ,μ) .*

Proof. Let $(\overline{u},\overline{v})$ be a positive ω-periodic solution of (1) associated with $\left(\overline{\lambda},\overline{\mu}\right)$. In view of (2) and (H1), we have

$$\begin{aligned}\overline{u}(t)&=\overline{\lambda}\int_t^{t+\omega}K(t,s)k(s)f(\overline{u}(s-\tau_1(s)),\overline{v}(s-\sigma_1(s)))ds\\&\geq\lambda\int_t^{t+\omega}K(t,s)k(s)f(\overline{u}(s-\tau_1(s)),\overline{v}(s-\sigma_1(s)))ds\end{aligned}$$

and

$$\overline{v}(t) \geq \mu \int_t^{t+\omega} H(t,s)h(s)g(\overline{u}(s-\tau_2(s)), \overline{v}\left(s-\sigma_2\left(s\right)\right))ds.$$

Let $\left(\overline{u}_0\left(t\right), \overline{v}_0\left(t\right)\right) = \left(\overline{u}\left(t\right), \overline{v}\left(t\right)\right)$,

(3) $$\left(\overline{u}_{n+1}\left(t\right), \overline{v}_{n+1}\left(t\right)\right) = T_{\lambda,\mu}\left(\overline{u}_n, \overline{v}_n\right)\left(t\right),\ n = 0, 1, 2, ...$$

Clearly, we have

$$\left(\overline{u}_0\left(t\right), \overline{v}_0\left(t\right)\right) \geq \left(\overline{u}_1\left(t\right), \overline{v}_1\left(t\right)\right) \geq ... \geq \left(\overline{u}_n\left(t\right), \overline{v}_n\left(t\right)\right) \geq (0,0).$$

Let $\left(u\left(t\right), v\left(t\right)\right) = \lim_{n\to\infty}\left(\overline{u}_n\left(t\right), \overline{v}_n\left(t\right)\right)$. In view of the Lebsegue dominated convergence theorem, we see from (3) that $\left(u\left(t\right), v\left(t\right)\right)$ is a nonnegative ω-periodic solution of (1). Since $(\overline{u}, \overline{v}) > (0,0)$ and $(0,0)$ is not a solution of (1) associated with (λ, μ), we have $(u, v) > (0,0)$. The proof is complete.

Lemma 3. *Suppose (H1) holds. Then (1) has a positive ω-periodic solution associated with some (λ_*, μ_*) satisfying $\lambda_*, \mu_* > 0$.*

Proof. Let

$$\alpha\left(t\right) = \int_t^{t+\omega} K\left(t,s\right)k\left(s\right)ds,$$

$$\beta\left(t\right) = \int_t^{t+\omega} H\left(t,s\right)h\left(s\right)ds,$$

and

$$M_f = \max_{0\leq t\leq\omega} f\left(\alpha\left(t-\tau_1\left(t\right)\right), \beta\left(t-\sigma_1\left(t\right)\right)\right),$$

$$M_g = \max_{0\leq t\leq\omega} g\left(\alpha\left(t-\tau_2\left(t\right)\right), \beta\left(t-\sigma_2\left(t\right)\right)\right).$$

Then clearly $M_f, M_g > 0$. Let $(\lambda_*, \mu_*) = (1/M_f, 1/M_g)$. We have

$$\begin{aligned}\alpha\left(t\right) &= \int_t^{t+\omega} K\left(t,s\right)k\left(s\right)ds \\ &\geq \lambda_* \int_t^{t+\omega} K\left(t,s\right)k\left(s\right)f\left(\alpha\left(s-\tau_1\left(s\right)\right), \beta\left(s-\sigma_1\left(s\right)\right)\right)ds,\end{aligned}$$

and

$$\begin{aligned}\beta\left(t\right) &= \int_t^{t+\omega} H\left(t,s\right)h\left(s\right)ds \\ &\geq \mu_* \int_t^{t+\omega} H\left(t,s\right)h\left(s\right)g\left(\alpha\left(s-\tau_2\left(s\right)\right), \beta\left(s-\sigma_2\left(s\right)\right)\right)ds.\end{aligned}$$

Let us define a sequence $\{(\overline{u}_n, \overline{v}_n)\}_{n=0}^{\infty}$ of vector functions by $(\overline{u}_0(t), \overline{v}_0(t)) = (\alpha(t), \beta(t))$, and

(4) $$(\overline{u}_{n+1}(t), \overline{v}_{n+1}(t)) = T_{\lambda,\mu}(\overline{u}_n, \overline{v}_n)(t), \ n = 0, 1, 2, ...$$

Clearly, we have

$$(\overline{u}_0(t), \overline{v}_0(t)) \geq (\overline{u}_1(t), \overline{v}_1(t)) \geq ... \geq (\overline{u}_n(t), \overline{v}_n(t)) \geq (0,0).$$

Let $(u(t), v(t)) = \lim_{n\to\infty}(\overline{u}_n(t), \overline{v}_n(t))$. In view of the Lebsegue dominated convergence theorem, we see from (4) that $(u(t), v(t))$ is a nonnegative ω-periodic solution of (1). Since $(\alpha, \beta) > (0,0)$ and $(0,0)$ is not a solution of (1) associated with (λ^*, μ^*), we have $(u, v) > (0,0)$. The proof is complete.

Let Π be the set of $(\lambda, \mu) \in R_+^2 \setminus \{(0,0)\}$ such that (1) has a positive ω-periodic solution associated with (λ, μ). Then by Lemma 3, Π contains the solution of (1) associated with (λ_*, μ_*). Therefore, by Lemma 2, it contains the subset

(5) $$\Pi_* = \{(\lambda, \mu) |\ (\lambda, \mu) > (0,0), \lambda \leq \lambda_*, \mu \leq \mu_*\}.$$

We may show further that Π is bounded above under the conditions (H1) and (H2).

Lemma 4. *Suppose (H1) and (H2) hold. Then* Π *is bounded above.*

Proof. Suppose to the contrary that there is a sequence $\{(u_n, v_n)\}$ of positive ω-periodic solutions of (1) associated with $\{(\lambda_n, \mu_n)\}$ such that $\lim_{n\to\infty}\lambda_n = \infty$ or $\lim_{n\to\infty}\mu_n = \infty$. If $\lim_{n\to\infty}\lambda_n = \infty$, then either there exists a subsequence $\left\{\left(u_{n_j}, v_{n_j}\right)\right\}$ such that $\left\|\left(u_{n_j}, v_{n_j}\right)\right\| \to +\infty$ as $j \to \infty$ or there is $\overline{M} > 0$ such that $\|(u_n, v_n)\| \leq \overline{M}$ for all n. Note that $(u_n, v_n) \in \Omega$, thus

$$u_n(t) + v_n(t) \geq \alpha^* \|(u_n, v_n)\|.$$

By (H2), we may choose $R_f > 0$ such that $f(x, y) \geq \eta_1(|x| + |y|)$ for all $|x|+|y| \geq R_f$ and some $\eta_1 > 0$. In view of (H1), there exists $\eta_2 > 0$ such that $f(0,0) \geq \eta_2 M$. Let $\eta = \min\{\eta_1, \eta_2\}$. On the other hand, there exists a sequence $\{t_n\} \subset [0, \omega]$ such that $u_n'(t_n) = 0$ and $u_n(t_n) = \max_{t\in[0,\omega]} u_n(t)$ by the periodicity and differentiability of $\{u_n(t)\}$. Thus, we have

$$\begin{aligned} a(t_n)\|u_n\| = a(t_n)u(t_n) &= \lambda_n k(t_n) f(u_n(t_n - \tau_1(t_n)), v_n(t_n - \sigma_1(t_n))) \\ &\geq \lambda_n \eta \alpha^* k(t_n) \|(u_n, v_n)\| \geq \lambda_n \eta \alpha^* k(t_n) \|u_n\|. \end{aligned}$$

But this is a contradiction since the continuous function $a(t) / (\eta\alpha^* k(t))$ is bounded. If $\lim_{n\to\infty}\mu_n = \infty$, we can get a contradiction in a similar manner. The proof is complete.

We also need the following lemmas for arguments involving the topological degree. One may refer to Guo and Lakshmikantham [22] for proofs and further discussion of the topological degree.

Lemma 5. *Let X be a Banach space with cone K. Let Ω be a bounded and open subset in X. Let $0 \in \Omega$ and $T : K \cap \overline{\Omega} \to K$ be condensing. Suppose that $Tx \neq \nu x$ for all $x \in K \cap \partial\Omega$ and all $\nu \geq 1$. Then $i\,(T, K \cap \Omega, K) = 1$.*

Lemma 6. *Let X be a Banach space and K a cone in X. For $r > 0$, define $K_r = \{x \in K : \|x\| < r\}$. Assume that $T : \overline{K}_r \to K$ is a compact map such that $Tx \neq x$ for $x \in \partial K_r$. If $\|x\| \leq \|Tx\|$ for $x \in \partial K_r$, then $i\,(T, K_r, K) = 0$.*

3. Main Theorems

For each $\theta \in [0, \pi/2]$, consider the half ray

$$L_\theta = \left\{(\lambda, \mu) \in R_+^2 \backslash \{(0,0)\} |\ (\lambda, \mu) = t(\sin\theta, \cos\theta),\ t > 0\right\}.$$

Near one end of this ray are points which belong to Π_* defined by (5) and near the other end are points outside Π (in view of Lemma 4), that is, the set $\{t > 0|\ t(\sin\theta, \cos\theta) \in \Pi\}$ is nonempty and bounded. Thus we are led to define

$$t_\theta^* = \sup\{t > 0|\ t(\sin\theta, \cos\theta) \in \Pi\},$$

and

$$(\lambda_\theta^*, \mu_\theta^*) = t_\theta^*(\sin\theta, \cos\theta)$$

for each $\theta \in [0, \pi/2]$.

We first assert that for each $\theta \in [0, \pi/2]$, $(\lambda_\theta^*, \mu_\theta^*) \in \Pi$. Indeed, let $\{(\lambda_n, \mu_n)\}_{n=1}^\infty$ be a sequence which satisfies $\lambda_n < \lambda_{n+1}, \mu_n < \mu_{n+1}$ for $n \geq 1$ and converges to $(\lambda_\theta^*, \mu_\theta^*)$. For each n, let (u_n, v_n) be a positive ω-periodic solution of (1) associated with (λ_n, μ_n). In view of Lemma 1, we know that the set $\{(u_n, v_n)\}$ is uniformly bounded in X^2. Thus, the sequence $\{(u_n, v_n)\}$ has a subsequence converging to $(u, v) \in X^2$. Then we can easily show, by the Lebesgue dominated convergence theorem, that (u, v) is a positive ω-periodic solution of (1) at $(\lambda_\theta^*, \mu_\theta^*)$.

Let the function $\rho : [0, \pi/2] \to (0, \infty)$ be defined by

$$\rho(\theta) = \left\{(\lambda_\theta^*)^2 + (\mu_\theta^*)^2\right\}^{1/2} = t_\theta^*.$$

We assert that ρ is continuous. Indeed, without loss of generality, let us assume $\phi \in (0, \pi/2)$ and let B be an open neighborhood containing $(\lambda_\phi^*, \mu_\phi^*)$ and contained in the interior of R_+^2. For any half ray L_θ that passes through B, it is (geometrically) clear that there will be some point $(\tilde{\lambda}, \tilde{\mu})$ in L_θ such that $(\tilde{\lambda}, \tilde{\mu}) \le (\lambda_\phi^*, \mu_\phi^*)$. In view of Lemma 2, there will be a positive ω-periodic solution $(\tilde{u}, \tilde{v})$ of (1) associated with $(\tilde{\lambda}, \tilde{\mu})$. If we pick the neighborhood B such that in polar coordinates it is of the form

$$\{(r, \theta) |\ \rho(\phi) - \varepsilon < r < \rho(\phi) + \varepsilon, \phi - \delta < \theta < \phi + \delta\},$$

where ε, δ are sufficiently small positive numbers, then we see that $\rho(\theta) > \rho(\phi) - \varepsilon$. By symmetric arguments, we may also show that $\rho(\theta) < \rho(\phi) + \varepsilon$. These arguments show that when θ and ϕ are sufficiently close, so is $\rho(\theta)$ and $\rho(\phi)$.

We summarize the above considerations as follows: Under the conditions (H1) and (H2), there exists a continuous curve Γ (defined by ρ) joining some point $(\rho(0), 0)$ on the positive λ-axis and some point $(0, \rho(\pi/2))$ on the positive μ-axis and separating $R_+^2 \setminus \{(0,0)\}$ into two disjoint subsets Θ_1 and Θ_2 such that $(0,0)$ is a boundary point of Θ_1 and (1) has at least one positive ω-periodic solution for $(\lambda, \mu) \in \Theta_1 \cup \Gamma$ and no positive ω-periodic solution for $(\lambda, \mu) \in \Theta_2$.

We intend to show that there are at least one more solution for each (λ, μ) in Θ_1. To this end, we suppose that condition (H1) holds and suppose (1) has a positive ω-periodic solution $(\overline{u}, \overline{v})$ associated with $(\overline{\lambda}, \overline{\mu}) > (0,0)$. Then equation (1) also has a positive ω-periodic solution $(u, v) < (\overline{u}, \overline{v})$ associated with $(\lambda, \mu) \in R^2 \setminus \{(0,0)\}$ and $(\lambda, \mu) < (\overline{\lambda}, \overline{\mu})$. Let (u^*, v^*) be a positive ω-periodic solution of (1) associated with $(\lambda^*, \mu^*) \in \Gamma$. Then for $(\lambda, \mu) < (\lambda^*, \mu^*)$ and $(\lambda, \mu) \in R^2 \setminus \{(0,0)\}$, by the uniform continuity of f and g on a compact set, there exists $\varepsilon_0 > 0$ such that

$$\begin{aligned} &f\left(u^*\left(s-\tau_1(s)\right)+\varepsilon, v^*\left(s-\sigma_1(s)\right)+\varepsilon\right) - f\left(u^*\left(s-\tau_1(s)\right), v^*\left(s-\sigma_1(s)\right)\right) \\ &\qquad < \frac{f(0,0)\left(\lambda^*-\lambda\right)}{\lambda}, \end{aligned}$$

and

$$\begin{aligned} &g\left(u^*\left(s-\tau_2(s)\right)+\varepsilon, v^*\left(s-\sigma_2(s)\right)+\varepsilon\right) - g\left(u^*\left(s-\tau_2(s)\right), v^*\left(s-\sigma_2(s)\right)\right) \\ &\qquad < \frac{g(0,0)\left(\lambda^*-\lambda\right)}{\lambda} \end{aligned}$$

for $s \in R$ and $0 < \varepsilon \le \varepsilon_0$. Thus, we have

$$\begin{aligned} &\lambda \int_t^{t+\omega} K(t,s)k(s) f\left(u^*\left(s-\tau_1(s)\right)+\varepsilon, v^*\left(s-\sigma_1(s)\right)+\varepsilon\right) ds \\ &-\lambda^* \int_t^{t+\omega} K(t,s)k(s) f\left(u^*\left(s-\tau_1(s)\right), v^*\left(s-\sigma_1(s)\right)\right) ds \\ = {} &\lambda \int_t^{t+\omega} K(t,s)k(s) \left[f\left(u^*\left(s-\tau_1(s)\right)+\varepsilon, v^*\left(s-\sigma_1(s)\right)+\varepsilon\right)\right. \end{aligned}$$

$$\begin{aligned}
&-f\left(u^*\left(s-\tau_1\left(s\right)\right),v^*\left(s-\sigma_1\left(s\right)\right)\right)\right]ds\\
&-\left(\lambda^*-\lambda\right)\int_t^{t+\omega}K(t,s)k(s)f\left(u^*\left(s-\tau_1\left(s\right)\right),v^*\left(s-\sigma_1\left(s\right)\right)\right)ds\\
<\ & f\left(0,0\right)\left(\lambda^*-\lambda\right)\int_t^{t+\omega}K(t,s)k(s)ds\\
&-\left(\lambda^*-\lambda\right)\int_t^{t+\omega}K(t,s)k(s)f\left(u^*\left(s-\tau_1\left(s\right)\right),v^*\left(s-\sigma_1\left(s\right)\right)\right)ds\\
=\ &\left(\lambda^*-\lambda\right)\int_t^{t+\omega}K(t,s)k(s)\left[f\left(0,0\right)-f\left(u^*\left(s-\tau_1\left(s\right)\right),v^*\left(s-\sigma_1\left(s\right)\right)\right)\right]ds\le 0
\end{aligned}$$

and

$$\begin{aligned}
\lambda\int_t^{t+\omega}&K(t,s)k(s)f\left(u^*\left(s-\tau_1\left(s\right)\right)+\varepsilon,v^*\left(s-\sigma_1\left(s\right)\right)+\varepsilon\right)ds\\
&\le\lambda^*\int_t^{t+\omega}K(t,s)k(s)f\left(u^*\left(s-\tau_1\left(s\right)\right),v^*\left(s-\sigma_1\left(s\right)\right)\right)ds\\
&=u^*\left(t\right)<u^*\left(t\right)+\varepsilon.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\mu\int_t^{t+\omega}&H(t,s)h(s)g\left(u^*\left(s-\tau_1\left(s\right)\right)+\varepsilon,v^*\left(s-\sigma_1\left(s\right)\right)+\varepsilon\right)ds\\
&\le\mu^*\int_t^{t+\omega}H(t,s)h(s)g\left(u^*\left(s-\tau_1\left(s\right)\right),v^*\left(s-\sigma_1\left(s\right)\right)\right)ds\\
&=v^*\left(t\right)<v^*\left(t\right)+\varepsilon.
\end{aligned}$$

Let

$$u_\varepsilon^*\left(t\right)=u^*\left(t\right)+\varepsilon,v_\varepsilon^*\left(t\right)=v^*\left(t\right)+\varepsilon$$

and

$$\Psi=\left\{(u,v)\in X^2:-\varepsilon<u\left(t\right)<u_\varepsilon^*\left(t\right),-\varepsilon<v\left(t\right)<v_\varepsilon^*\left(t\right),t\in R\right\}.$$

Then Ψ is bounded and open in X, $0\in\Psi$ and $T:\Omega\cap\overline{\Psi}\to\Omega$ is condensing (since it is completely continuous). Let $(u,v)\in\Omega\cap\partial\Psi$. Then there exists t_0 such that either $u(t_0)=u_\varepsilon^*\left(t_0\right)$ or $v\left(t_0\right)=v_\varepsilon^*\left(t_0\right)$. Suppose that $u\left(t_0\right)=u_\varepsilon^*\left(t_0\right)$. Then by (H1),

$$\begin{aligned}
A_\lambda\left(u,v\right)\left(t_0\right)&=\lambda\int_{t_0}^{t_0+\omega}K(t_0,s)k\left(s\right)f(u\left(s-\tau_1\left(s\right)\right),v\left(s-\sigma_1\left(s\right)\right))ds\\
&\le\lambda\int_{t_0}^{t_0+\omega}K(t_0,s)k\left(s\right)f(u_\varepsilon^*\left(s-\tau_1\left(s\right)\right),v_\varepsilon^*\left(s-\sigma_1\left(s\right)\right))ds\\
&<u_\varepsilon^*\left(t_0\right)=u\left(t_0\right)\le\nu u\left(t_0\right)
\end{aligned}$$

for all $\nu \geq 1$. Similarly, for the case $v(t_0) = v_\varepsilon^*(t_0)$, we can get $B_\mu(u,v)(t_0) < \nu v(t_0)$ for all $\nu \geq 1$. Thus $T_{\lambda,\mu}(u,v) \neq \nu(u,v)$ for $(u,v) \in \Omega \cap \partial\Psi$ and $\nu \geq 1$. In view of Lemma 5, we have $i(F, \Omega \cap \Psi, \Omega) = 1$.

By (H2), we may choose $R_f > 0$ such that $f(u,v) \geq \eta(u,v)$ for all $u+v \geq R_f$, where η satisfies

$$\alpha\eta N\lambda \int_0^\omega k(s)\,ds > 1.$$

Let $R = \max\{b_D, R_f/\sigma, \|(u_\varepsilon^*, v_\varepsilon^*)\|\}$, where b_D is given in Lemma 1 with D a closed rectangle in $R_+^2 \setminus \{(0,0)\}$ containing (λ,μ). Let $\Omega_R = \{(u,v) \in \Omega : \|(u,v)\| < R\}$. Then in view of Lemma 1, $(u,v) \neq T_{\lambda,\mu}(u,v)$ for $(u,v) \in \partial\Omega_R$. Furthermore, if $(u,v) \in \partial\Omega_R$, then $u(t) + v(t) \geq \alpha\|(u,v)\| \geq R_f$. Thus, we have

$$\begin{aligned} A_\lambda(u,v)(t) &= \lambda \int_t^{t+\omega} K(t,s)\,k(s)\,f(u(s-\tau_1(s)), v(s-\sigma_1(s)))\,ds \\ &\geq \alpha\eta N\lambda \int_0^\omega k(s)\,\|(u,v)\|\,ds > \|(u,v)\|. \end{aligned}$$

Therefore $\|T_{\lambda,\mu}(u,v)\| \geq \|A_\lambda(u,v)\| > \|(u,v)\|$ and Lemma 6 then implies $i(T_{\lambda,\mu}, \Omega_R, \Omega) = 0$. Consequently by the additivity of the topological degree,

$$0 = i(T_{\lambda,\mu}, \Omega_R, \Omega) = i(T_{\lambda,\mu}, \Omega \cap \Psi, \Omega) + i\left(T_{\lambda,\mu}, \Omega_R \backslash \overline{\Omega \cap \Psi}, \Omega\right).$$

Since $i(T_{\lambda,\mu}, \Omega \cap \Psi, \Omega) = 1$, $i\left(T_{\lambda,\mu}, \Omega_R \backslash \overline{\Omega \cap \Psi}, \Omega\right) = -1$ and $T_{\lambda,\mu}$ has a fixed point on $\Omega \cap \Psi$ and another on $\Omega_R \backslash \overline{\Omega \cap \Psi}$. Thus, we have the following result.

Theorem 1. *Under the conditions (H1) and (H2), there exists a continuous curve Γ joining some point on the positive λ-axis and some point on the positive μ-axis and separating $\{(\lambda,\mu) |\ \lambda,\mu \geq 0\} \setminus \{(0,0)\}$ into two disjoint subsets Θ_1 (which is bounded) and Θ_2 (which is unbounded) such that (1) has at least two positive ω-periodic solutions for $(\lambda,\mu) \in \Theta_1$, at least one positive ω-periodic solution for $(\lambda,\mu) \in \Gamma$, and no positive ω-periodic solution for $(\lambda,\mu) \in \Theta_2$.*

4. Remarks

First of all, note that the curve Γ is defined by the shooting method. Therefore, numerically it can be computed. Furthermore, since the qualitative behavior of the solution sets changes as Γ is crossed, it can be regarded as a 'bifurcation curve'.

Next, the conditions in Theorem 1 are not vacuous. As an example, consider the system

$$(6) \qquad \begin{cases} x'(t) + a(t)\,x(t) = \lambda k(t)\,\{x(t-\tau_1(t)) + y(t-\sigma_1(t)) + 1\}^\gamma, \\ y'(t) + b(t)\,y(t) = \mu h(t)\,\{x(t-\tau_2(t)) + y(t-\sigma_2(t)) + 1\}^\delta, \end{cases}$$

where $a, b, k, h, \tau_1, \tau_2, \sigma_1$ and σ_2 satisfy the same assumptions stated above, and $\gamma, \delta > 1$. Then all conditions of Theorem 1 are satisfied.

As another example, consider the system

$$\text{(7)} \quad \begin{cases} x'(t) = -a(t)\,x(t) + \lambda k(t) \left\{ (x(t-\tau_1(t)) + y(t-\sigma_1(t)))^2 + \varepsilon \right\}, \\ y'(t) = -b(t)\,y(t) + \mu h(t) \left\{ (x(t-\tau_2(t)) + y(t-\sigma_2(t)))^2 + \varepsilon \right\}, \end{cases}$$

where $a, b, k, h, \tau_1, \tau_2, \sigma_1$ and σ_2 satisfy the same assumptions stated above, and ε is a positive constant. Clearly, all conditions of Theorem 1 is satisfied. Thus, there exists a continuous curve Γ separating $R_+^2 \setminus \{(0,0)\}$ into two disjoint subsets Θ_1 and Θ_2 such that (7) has at least two positive ω-periodic solution for $(\lambda, \mu) \in \Theta_1$, at least one positive ω-periodic solution for $(\lambda, \mu) \in \Gamma$, and no positive ω-periodic solution for $(\lambda, \mu) \in \Theta_2$ respectively.

Next, we remark that similar discussions can be carried out for systems of the form

$$\text{(8)} \quad \begin{cases} x'(t) = \pm a(t)\,x(t) \mp \lambda k(t)\, f\left(x(t-\tau_1(t)), y(t-\sigma_1(t))\right), \\ y'(t) = \pm b(t) y(t) \mp \mu h(t) g\left(x(t-\tau_2(t)), y(t-\sigma_2(t))\right), \end{cases}$$

where $a, b, k, h, f, g, \tau_1, \tau_2, \sigma_1, \sigma_2$ are the same functions defined in the introduction, and even to systems such as

$$\begin{cases} x'(t) = \pm a(t)\,x(t) \mp \lambda k(t)\, f\left(x(t - l_1(x(t), y(t))), y(t - l_2(x(t), y(t)))\right), \\ y'(t) = \pm b(t) y(t) \mp \mu h(t) g\left(x(t - m_1(x(t), y(t))), y(t - m_2(x(t), y(t)))\right). \end{cases}$$

Theorem 2. *Suppose (H1) and (H2) hold. There exists a continuous curve Γ joining some point on the positive λ-axis and some point on the positive μ-axis and separating $\{(\lambda, \mu) |\ \lambda, \mu \geq 0\} \setminus \{(0,0)\}$ into two disjoint subsets Θ_1 (which is bounded) and Θ_2 (which is unbounded) such that (8) has at least two positive ω-periodic solutions for $(\lambda, \mu) \in \Theta_1$, at least one positive ω-periodic solution for $(\lambda, \mu) \in \Gamma$, and no positive ω-periodic solution for $(\lambda, \mu) \in \Theta_2$ respectively.*

As our final remark, note that when $f = f(x, y)$ is odd with respect to x and to y, and $g = g(x, y)$ is odd with respect to x and to y, then letting $x(t) = -u(t)$ and $y(t) = -v(t)$ in (1), we have

$$\begin{cases} u'(t) = -a(t)\,u(t) + \lambda k(t)\, f\left(u(t-\tau_1(t)), v(t-\sigma_1(t))\right), \\ v'(t) = -b(t) v(t) + \mu h(t) g\left(u(t-\tau_2(t)), v(t-\sigma_2(t))\right), \end{cases}$$

which is the original system. This shows that when (x, y) is a positive periodic solution of (1), then $(-x, -y)$ is a 'negative' periodic solution of (1). There are other possibilities such as $f = f(x, y)$ is odd with respect to x but even with respect to y, or $g = g(x, y)$ is even with respect to x and odd with respect to y. The principle, however, remains the same.

ACKNOWLEDGMENT

The first author is grateful to Professor Anping Liao from Hunan Normal University for some helpful discussions.

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