MULTIPLICATIVE RENORMALIZATION AND GENERATING FUNCTIONS II

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Abstract. Let μ be a probability measure on the real line with finite moments of all orders. Suppose the linear span of polynomials is dense in $L^2(\mu)$. Then there exists a sequence $\{P_n\}_{n=0}^{\infty}$ of orthogonal polynomials with respect to μ such that P_n is a polynomial of degree n with leading coefficient 1 and the equality $(x - \alpha_n)P_n(x) = P_{n+1}(x) + \omega_n P_{n-1}(x)$ holds, where α_n and ω_n are Szegö-Jacobi parameters. In this paper we use the concepts of pre-generating function, multiplicative renormalization, and generating function to derive $\{P_n, \alpha_n, \omega_n\}$ from a given μ. Two types of pre-generating functions are studied. We apply our method to the special distributions such as Gaussian, Poisson, gamma, uniform, arcsine, semi-circle, and beta-type to derive $\{P_n, \alpha_n, \omega_n\}$. Moreover, we show that the corresponding polynomials P_n's are exactly the classical polynomials such as Hermite, Charlier, Laguerre, Legendre, Chebyshev of the first kind, Chebyshev of the second kind, and Gegenbauer. We also apply our method to study the negative binomial distributions.

1. Introduction

The theory of orthogonal polynomials has a long history (see for example [7] [8] [11] [13] and the references therein) with a wide range of applications. Its connection with the theory of interacting Fock space has recently been discovered by Accardi and Bożejko [1]. Let μ be a probability measure on $\mathbb{R}$ with finite moments of all orders such that the linear span of the monomials $x^n, n \geq 0$, is dense in $L^2(\mu)$. It is well-known [13] that there exists a complete system $\{P_n\}_{n=0}^{\infty}$ of orthogonal polynomials such that P_n is a polynomial of degree n with leading coefficient 1 and the following recursion formula is satisfied:

(1.1) $\qquad (x - \alpha_n)P_n(x) = P_{n+1}(x) + \omega_n P_{n-1}(x), \qquad n \geq 0,$

where $\alpha_n \in \mathbb{R}, \omega_n \geq 0$ for $n \geq 0$ and by convention $\omega_0 = 1, P_{-1} = 0$. The numbers α_n and ω_n are called *Szegő-Jacobi parameters* of μ.

Define a sequence $\lambda = \{\lambda_n\}_{n=0}^{\infty}$ by

(1.2) $\qquad\qquad\qquad\qquad \lambda_n = \omega_0 \omega_1 \cdots \omega_n, \qquad n \geq 0.$

Assume that the sequence λ satisfies the condition

(1.3) $\qquad\qquad\qquad\qquad \inf_{n \geq 0} \lambda_n^{1/n} > 0.$

For such a sequence λ we define a Hilbert space Γ_λ by

$$\Gamma_\lambda = \left\{ (a_0, a_1, \ldots, a_n, \ldots) \,\Big|\, a_n \in \mathbb{C}, \sum_{n=0}^{\infty} \lambda_n |a_n|^2 < \infty \right\}$$

with norm $\| \cdot \|$ given by

$$\|(a_n)\|_\lambda = \left(\sum_{n=0}^{\infty} \lambda_n |a_n|^2 \right)^{1/2}.$$

The Hilbert space Γ_λ is called the *one-mode interacting Fock space* associated with λ. The *annihilation operator* A is the densely defined operator on Γ_λ given by

$$A\Phi_0 = 0, \quad A\Phi_n = \omega_n \Phi_{n-1}, \quad n \geq 1,$$

where $\Phi_n = (0, \ldots, 0, 1, 0, \ldots)$ with 1 in the $(n+1)$st entry. The *creation operator* is defined to be the adjoint A^* of A. It can be easily checked that

$$A^* \Phi_n = \Phi_{n+1}, \qquad n \geq 0.$$

Define the number operator N and another operator α_N on Γ_λ by

$$N\Phi_n = n\Phi_n, \quad \alpha_N \Phi_n = \alpha_n \Phi_n, \qquad n \geq 0.$$

In the paper [1] Accardi and Bożejko proved that there exits a unitary isomorphism $U : \Gamma_\lambda \to L^2(\mu)$ such that

(1) $U\Phi_0 = 1$,
(2) $UA^*U^*P_n = P_{n+1}$,
(3) $U(A + A^* + \alpha_N)U^* = X$, where X is the multiplication operator by x.

This unitary isomorphism U is canonical in the sense of condition (3), namely, under U, the multiplication operator X by the independent variable x on $L^2(\mu)$

corresponds to a linear combination of the annihilation, creation, and number operators on the interacting Fock space Γ_λ.

Thus for a given probability measure μ on $\mathbb{R}$ it is important for the study of interacting Fock space to find the associated orthogonal polynomials $\{P_n(x)\}_{n=0}^\infty$ (for $L^2(\mu)$ and the Szegö-Jacobi parameters $\{\alpha_n, \omega_n\}_{n=0}^\infty$ (for Γ_λ).

In part I of our paper [5] we described a new method for deriving the polynomials $\{P_n(x)\}_{n=0}^\infty$ in terms of generating functions. In this paper we will provide the detail of the results announced in [5] and compute the Szegö-Jacobi parameters from the corresponding generating function (see Sections 2 and 3.) Furthermore, we will derive generating functions for certain measures and use them to produce several classical orthogonal polynomials (see Section 4). Our method can also be used to derive orthogonal polynomials for negative binomial distributions (see Section 5.)

In this paper, we present a systematic method to obtain generating functions of orthogonal polynomials and to directly derive Szegö-Jacobi parameters from generating functions. In some cases, we need to calculate complicated integrals.

In the forthcoming paper [6], we will give an alternative way to evaluate Szegö-Jacobi parameters from generating functions. Moreover, by using generating functions, we will introduce a general method to find representations of orthogonal polynomials in terms of differential operators such as in Theorem 4.1.

2. Generating Functions of Orthogonal Polynomials

Let μ be a fixed probability measure on $\mathbb{R}$ satisfying the conditions in Section 1. For convenience we make the following definitions.

Definition 2.1. By a *pre-generating function* for μ we mean a function $\varphi(t, x)$ with a power series expansion in t

$$(2.1) \qquad \varphi(t, x) = \sum_{n=0}^\infty g_n(x) t^n,$$

satisfying the following conditions:

(a) $g_n(x)$ is a polynomial of degree n for each $n \geq 0$.

(b) $\lim\sup_{n\to\infty} \|g_n\|_{L^2(\mu)}^{1/n} < \infty$.

We point out that conditions (a) and (b) imply the following fact:

• There exists $\tau_0 > 0$ such that $E_\mu \varphi(t, \cdot) \neq 0$ for all $|t| < \tau_0$.

To verify this fact, note that by (b),

$$R_0 = \liminf_{n\to\infty} \|g_n\|_{L^2(\mu)}^{-1/n} > 0.$$

Hence the series

$$\varphi(t, x) = \sum_{n=0}^\infty g_n(x) t^n$$

converges in $L^2(\mu)$ for $t \in \mathbb{C}$ with $|t| < R_0$. Therefore,

$$E_\mu[|\varphi(t, \cdot)|] \leq \sum_{n=0}^\infty \|g_n\|_{L^2(\mu)} |t|^n < \infty$$

for $t \in \mathbb{C}, |t| < R_0$. This implies that $E_\mu[\varphi(t, \cdot)]$ is analytic on $\{t \in \mathbb{C}; |t| < R_0\}$. On the other hand, by (a)

$$E_\mu[\varphi(0, \cdot)] = g_0 \neq 0.$$

Thus there exists τ_0 such that $0 < \tau_0 < R_0$ and $E_\mu[\varphi(t, \cdot)] \neq 0$ for all $t \in \mathbb{C}, |t| < \tau_0$.

Definition 2.2. By a *generating function* for μ we mean a pre-generating function $\psi(t, x)$ given by

$$(2.2) \qquad\qquad \psi(t, x) = \sum_{n=0}^\infty Q_n(x) t^n,$$

where the polynomials $Q_n, n \geq 0$, are orthogonal in $L^2(\mu)$.

Definition 2.3. The *multiplicative renormalization* of a pre-generating function $\varphi(t, x)$ is defined to be the function

$$\psi(t, x) = \frac{\varphi(t, x)}{E_\mu \varphi(t, \cdot)}.$$

Note that a generating function is very different from moment generating function. A moment generating function is used to find moments of a probability measure, while a generating function is used to find a sequence of orthogonal polynomials as we showed in [5].

Lemma 2.4. *Let $\varphi(t, x)$ be a pre-generating function given by*

$$\varphi(t, x) = \sum_{n=0}^\infty g_n(x) t^n$$

and its renormalization factor be expanded as

$$C(t) = C(\varphi, \mu, t) = \frac{1}{E_\mu \varphi(t, \cdot)} = \sum_{n=0}^\infty b_n t^n.$$

Let $\psi(t, x) = C(t)\varphi(t, x)$ be the multiplicative renormalization of $\varphi(t, x)$ and

$$\psi(t, x) = \sum_{n=0}^\infty Q_n(x) t^n.$$

Then $\psi(t, x)$ is also a pre-generating function, $Q_0(x) = 1$ and $E_\mu \psi(t, \cdot) = 1$ for all
t where $\psi(t, x)$ is defined. Moreover, for each $n \geq 0$, Q_n is a linear combination
of $g_0, g_1, \ldots, g_n$

$$Q_n(x) = b_0 g_n + \cdots + b_n g_0$$

and vice versa g_n is also a linear combination of $Q_0, Q_1, \ldots, Q_n$.

Proof. Since $E_\mu \varphi(t, \cdot)$ is analytic and $E_\mu \varphi(0, \cdot) = g_0 \neq 0$, we have the power
series expansion of $C(t)$ around $t = 0$ with $b_0 = 1/g_0 \neq 0$. Set

$$Q_n(x) = b_0 g_n(x) + \cdots + b_{n-1} g_1(x) + b_n g_0.$$

Then $Q_n(x)$ is a polynomial of degree n. In particular $Q_0(x) = b_0 g_0 = 1$. It is
easily seen that $g_n(x)$ is a linear combination of $\{Q_k(x) : 0 \leq k \leq n\}$. Since
$\|g_n\|_{L^2(\mu)} \leq C_1 c_1^n$ and $|b_n| \leq C_2 c_2^n$ hold for suitable $C_1, C_2, c_1, c_2 > 0$, we have

$$\limsup_{n \to \infty} \|Q_n\|_{L^2(\mu)}^{1/n} \leq \limsup_{n \to \infty} \left(C_1 C_2 \sum_{k=0}^{n} c_2^{n-k} c_1^k \right)^{1/n}$$
$$\leq \limsup_{n \to \infty} \left(n C_1 C_2 \max\{c_1, c_2\}^n \right)^{1/n}$$
$$= \max\{c_1, c_2\} < \infty.$$

Therefore,

$$\psi(t, x) = \sum_{n=0}^{\infty} Q_n(x) t^n$$

is a pre-generating function. Since

$$\sum_{n=0}^{\infty} b_n t^n \sum_{n=0}^{\infty} g_n(x) t^n = \sum_{n=0}^{\infty} \sum_{k=0}^{n} b_{n-k} g_k(x) t^n = \psi(t, x),$$

$\psi(t, x)$ is the multiplicative renormalization of $\varphi(t, x)$. $E_\mu \psi(t, x) = 1$ is obvious.■

Theorem 2.5. *Let $\psi(t, x)$ be the multiplicative renormalization of a pre-*
generating function and

(2.3) $$\psi(t, x) = \sum_{n=0}^{\infty} Q_n(x) t^n.$$

Then the polynomials $Q_n, n \geq 0$, are orthogonal in $L^2(\mu)$ if and only if $E_\mu \psi(t, \cdot) \psi(s, \cdot)$
depends only on ts. (Hence $\psi(t, x)$ is a generating function for μ.)

Proof. Suppose the polynomials $Q_n, n \geq 0$, are orthogonal. Then

$$
\begin{aligned}
E_\mu \psi(t, \cdot) \psi(s, \cdot) &= \sum_{n,m=0}^\infty E_\mu Q_n Q_m t^n s^m \\
&= \sum_{n=0}^\infty E_\mu Q_n^2 (ts)^n.
\end{aligned}
$$

Hence $E_\mu \psi(t, \cdot) \psi(s, \cdot)$ depends only on ts. Conversely, suppose a double series in t and s depends only on ts, namely,

$$
(2.4) \qquad \sum_{n,m=0}^\infty a_{nm} t^n s^m = \theta(ts).
$$

Let $n > m$. Differentiate Equation (2.4) n times in t to get

$$
(2.5) \qquad \partial_t^n \sum_{n,m=0}^\infty a_{nm} t^n s^m = s^n \theta^{(n)}(ts).
$$

Then differentiate Equation (2.5) m times in s and put $t = s = 0$ to show that $a_{nm} = 0$. In case $n < m$, we first differentiate Equation (2.4) m times in s, then n times in t and put $t = s = 0$ to show that $a_{nm} = 0$. Hence we have $a_{nm} = 0$ if $n \neq m$. Therefore, if $E_\mu \psi(t, \cdot) \psi(s, \cdot)$ depends only on ts, then

$$
E_\mu Q_n Q_m = 0, \qquad \forall n \neq m
$$

and so the polynomials $Q_n, n \geq 0$, are orthogonal in $L^2(\mu)$. ∎

For a given probability measure μ on $\mathbb{R}$, Lemma 2.4 and Theorem 2.5 provide a method to derive the corresponding orthogonal polynomials $\{P_n\}_{n=0}^\infty$ in Equation (1.1). Try a certain form $\varphi(t, x)$ of pre-generating function and take the multiplicative renormalization $\psi(t, x) = \varphi(t, x)/E_\mu \varphi(t, \cdot)$. Then use the condition in Theorem 2.5 to find the exact form of $\varphi(t, x)$. With this $\varphi(t, x)$ we can compute $\psi(t, x)$ and by the series expansion we can obtain the polynomials Q_n in Equation (2.3). Let a_n be the leading coefficient of $Q_n(x)$. Then the polynomials $P_n(x) = Q_n(x)/a_n, n \geq 0$, are those associated with μ as given in Equation (1.1). This can easily be seen from the Gram-Schmidt orthogonalization process to the sequence $\{1, x, x^2, \ldots, x^n, \ldots\}$.

Thus a critical question is how to find an appropriate form of a pre-generating function. We will address this issue in Section 3.

Recall that the interacting Fock space Γ_λ associated with μ, the annihilation, creation, and α_N operators acting on Γ_λ are defined in terms of the Szegö-Jacobi

parameters in Equation (1.1). Therefore it is desirable to find a way to derive these parameters from the corresponding generating function. In order to do that we rewrite the generating function in Theorem 2.5 as follows:

$$(2.6) \qquad \psi(t,x) = \sum_{n=0}^{\infty} a_n P_n(x) t^n,$$

where a_n is the leading coefficient of $Q_n(x)$ in Equation (2.3) and $P_n(x) = Q_n(x)/a_n$. Thus the polynomials P_n's are those in Equation (1.1).

There are two ways to compute the Szegö-Jacobi parameters. After deriving the polynomials P_n's from $\psi(t,x)$, we can use Equation (1.1) to compare the coefficients (e.g., of x^n and x^0) in both sides to find α_n and ω_n.

Another way to compute the Szegö-Jacobi parameters is a classical one. Multiply both sides of Equation (1.1) by P_n and take the expectation to get

$$(2.7) \qquad \alpha_n = \frac{E_\mu(xP_n^2)}{E_\mu P_n^2}.$$

On the other hand, multiply both sides of Equation (1.1) by P_{n-1} and take the expectation to get

$$\omega_n = \frac{E_\mu(xP_nP_{n-1})}{E_\mu P_{n-1}^2}.$$

But from Equation (1.1) with n being replaced by $n-1$ we get

$$
\begin{aligned}
E_\mu(xP_nP_{n-1}) &= E_\mu\big(P_n(xP_{n-1})\big) \\
&= E_\mu\big(P_n(P_n + \alpha_{n-1}P_{n-1} + \omega_{n-1}P_{n-2})\big) \\
&= E_\mu P_n^2.
\end{aligned}
$$

Hence ω_n and λ_n are given by

$$(2.8) \qquad \omega_n = \frac{E_\mu P_n^2}{E_\mu P_{n-1}^2} \quad \text{and} \quad \lambda_n = E_\mu P_n^2.$$

The formulas in Equations (2.7) and (2.8) are well-known (see, e.g., the book [7].) Hence the question now is how to find the quantities in the right hand sides of Equations (2.7) and (2.8) from the generating function $\psi(t,x)$ in Equation (2.6). The answer is given by the next theorem.

Theorem 2.6. *Let $\psi(t,x) = \sum_{n=0}^{\infty} a_n P_n(x) t^n$ be a generating function for μ. Then*

$$(2.9) \qquad \lim_{t\to 0} \psi\left(t, \frac{x}{t}\right) = \sum_{n=0}^{\infty} a_n x^n,$$

$$(2.10) \qquad E_\mu \psi(t, \cdot)^2 = \sum_{n=0}^{\infty} a_n^2 \lambda_n t^{2n},$$

$$(2.11) \qquad E_\mu x \psi(t, \cdot)^2 = \sum_{n=0}^{\infty} \left(a_n^2 \lambda_n \alpha_n t^{2n} + 2 a_n a_{n-1} \lambda_n t^{2n-1} \right),$$

where $a_{-1} = 0$ by convention.

Proof. Equation (2.10) follows from the orthogonality of the polynomials P_n and Equation (2.8). To show Equation (2.11) note that

$$(2.12) \qquad \begin{aligned} E_\mu x \psi(t, \cdot)^2 &= \sum_{n,m=0}^{\infty} a_n a_m E_\mu (x P_n P_m) t^n t^m \\ &= \sum_{n=0}^{\infty} a_n^2 E_\mu (x P_n^2) t^{2n} + 2 \sum_{n>m} a_n a_m E_\mu (x P_n P_m) t^{n+m}. \end{aligned}$$

But when $n > m$ we have

$$E_\mu (x P_n P_m) = E_\mu \big((P_{n+1} + \alpha_n P_n + \omega_n P_{n-1}) P_m \big) = \delta_{m,n-1} \omega_n E_\mu (P_{n-1}^2).$$

Therefore,

$$(2.13) \qquad \sum_{n>m} a_n a_m E_\mu (x P_n P_m) t^{n+m} = \sum_{n=1}^{\infty} a_n a_{n-1} E_\mu (P_n^2) t^{2n-1}.$$

Hence Equations (2.7), (2.8), (2.12) and (2.13) yield Equation (2.11). ∎

Once we have a generating function $\psi(t, x)$ for μ, we can find the power series of $E_\mu \psi(t, \cdot)^2$ and $E_\mu x \psi(t, \cdot)^2$. Then by the above Theorem 2.6 we can find a_n and the Szegö-Jacobi parameters α_n and ω_n.

3. Two Types of Pre-generating Functions

In part I of our paper [5] we mentioned two types of pre-generating functions

$$(3.1) \qquad \varphi(t, x) = e^{\rho(t)x} = \sum_{n=0}^{\infty} \frac{1}{n!} \big(\rho(t) x \big)^n,$$

$$(3.2) \qquad \varphi(t, x) = \left(1 - \rho(t)x\right)^c = \sum_{n=0}^{\infty} \binom{c}{n} (-1)^n \left(\rho(t)x\right)^n,$$

where the function $\rho(t)$ and constant c are to be derived so that the multiplicative renormalization $\psi(t, x)$ satisfies the condition that $E_\mu \psi(t, \cdot)\psi(s, \cdot)$ depends only on ts according to Theorem 2.5. These two types of functions cover many classical examples of orthogonal polynomials as shown in [5]. In fact, the first type in Equation (3.1) can be applied to more general cases.

In both cases in Equations (3.1) and (3.2), $\rho(t)$ must be analytic around $t = 0$, $\rho(0) = 0$, and $\rho'(0) \neq 0$ in order to get a polynomial $g_n(x)$ of degree n in Equation (2.1). For these cases, the coefficients $\{a_n\}$ in Equation (2.6) can be obtained as follows. We can easily see that

$$\lim_{t \to 0} \psi\left(t, \frac{x}{t}\right) = \lim_{t \to 0} \frac{\varphi(t, \frac{x}{t})}{E_\mu \varphi(t, \cdot)} = \lim_{t \to 0} \varphi\left(t, \frac{x}{t}\right).$$

For the case in Equation (3.1), we have

$$\lim_{t \to 0} \varphi\left(t, \frac{x}{t}\right) = \lim_{t \to 0} e^{\rho(t)x/t} = e^{\rho(0)'x} = \sum_{n=0}^{\infty} \frac{\rho'(0)^n}{n!} x^n,$$

and hence

$$(3.3) \qquad\qquad a_n = \frac{\rho'(0)^n}{n!}.$$

For the case in Equation (3.2)

$$\lim_{t \to 0} \varphi\left(t, \frac{x}{t}\right) = (1 - \rho'(0)x)^c = \sum_{n=0}^{\infty} \binom{c}{n} (-\rho'(0))^n x^n$$

and hence

$$(3.4) \qquad\qquad a_n = \binom{c}{n} (-\rho'(0))^n.$$

Definition 3.1. A probability measure μ on $\mathbb{R}$ is said to be of *exponential type* if there exists a constant $0 < a < \infty$ such that $\int_{\mathbb{R}} e^{a|x|} \, d\mu(x) < \infty$.

For an exponential type probability measure μ, define its Laplace transform by

$$\ell(r) = \int_{\mathbb{R}} e^{rx} \, d\mu(x), \qquad |r| \leq a.$$

Theorem 3.2. *Let μ be an exponential type probability measure on $\mathbb{R}$. Let ℓ be its Laplace transform and $g(r) = \ell'(r)/\ell(r)$. Suppose $\rho(t)$ has a power series expansion near 0 such that $\rho(0) = 0$, $\rho'(0) = 1$ and satisfies the following equation*

$$(3.5) \qquad g\big(\rho(t) + \rho(s)\big)\big(t\rho'(t) - s\rho'(s)\big) = g\big(\rho(t)\big)t\rho'(t) - g\big(\rho(s)\big)s\rho'(s).$$

Then the multiplicative renormalization of $e^{\rho(t)x}$

$$\psi(t, x) = \frac{e^{\rho}(t)x}{E_\mu e^{\rho}(t)x} = \frac{e^{\rho}(t)x}{\ell\big(\rho(t)\big)}$$

is a generating function for μ.

Note: Let $t, s > 0$ and put $\rho(t) = \theta(\log t)$. Then Equation (3.5) can be reduced to an equation for $\theta(r)$, $r < -K$, (K: a positive constant)

$$g\big(\theta(r) + \theta(u)\big)\big(\theta'(r) - \theta'(u)\big) = g\big(\theta(r)\big)\theta'(r) - g\big(\theta(u)\big)\theta'(u)$$

and $\lim_{r\to-\infty} \theta(r) = 0, \lim_{r\to-\infty} e^{-r}\theta'(r) = 1$.

Proof. Let $\rho(t)$ be a function with a power series expansion near 0 and $\rho(0) = 0$, $\rho'(0) = 1$. Then it is easy to see that the function $\varphi(t, x) = e^{\rho(t)x}$ is a pre-generating function. Note that $E_\mu\varphi(t, \cdot) = \ell\big(\rho(t)\big)$ and so the multiplicative renormalization of $\varphi(t, x)$ is given by

$$\psi(t, x) = \frac{e^{\rho}(t)x}{\ell\big(\rho(t)\big)}.$$

Then for any t, s, we have

$$(3.6) \qquad\qquad E_\mu\psi(t, \cdot)\psi(s, \cdot) = \frac{\ell\big(\rho(t) + \rho(s)\big)}{\ell\big(\rho(t)\big)\ell\big(\rho(s)\big)}.$$

Observe that $E_\mu\psi(t, \cdot)\psi(s, \cdot)$ in Equation (3.6) depends only on ts if and only if after we put $s = r/t$ the following function is independent of t

$$E_\mu\psi(t, \cdot)\psi(r/t, \cdot) = \frac{\ell\big(\rho(t) + \rho(r/t)\big)}{\ell\big(\rho(t)\big)\ell\big(\rho(r/t)\big)}.$$

Therefore,

$$\frac{\partial}{\partial t}\log\frac{\ell\big(\rho(t) + \rho(r/t)\big)}{\ell\big(\rho(t)\big)\ell\big(\rho(r/t)\big)} = 0,$$

which is equivalent to

$$g\big(\rho(t) + \rho(r/t)\big)\Big(\rho'(t) - \rho'(r/t)\frac{r}{t^2}\Big) - g\big(\rho(t)\big)\rho'(t) + g\big(\rho(r/t)\big)\rho'(r/t)\frac{r}{t^2} = 0,$$

where $g = \ell'/\ell$, the logarithmic derivative of ℓ. Putting $r/t = s$ back, we see that this equation is equivalent to Equation (3.5) in the theorem. Thus if $\rho(t)$ satisfies Equation (3.5) then $E_\mu \psi(t, \cdot)\psi(s, \cdot)$ depends only on ts and so $\psi(t, x)$ is a generating function. ∎

4. Classical Orthogonal Polynomials

In this section we will use our method of generating function developed in Sections 2 and 3 to find several generating functions and derive the corresponding orthogonal polynomials together with the Szegő-Jacobi parameters. In addition, we will verify that our orthogonal polynomials derived from generating functions are indeed the classical ones (up to a scaling constant). However, we do not assume any classical formula on generating functions. In fact, this is the whole point. Our technique is first find a generating function then use it to derive orthogonal polynomials and other quantities. Thus, a priori, our orthogonal polynomials are not defined by the classical definitions.

4.1. Gaussian Measure and Hermite Polynomials

Let μ be the Gaussian measure with mean 0 and variance σ^2

$$d\mu(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2\sigma^2}x^2}\, dx.$$

Try the type of pre-generating function $\varphi(t, x) = e^{\rho(t)x}$ in Equation (3.1). It is easily checked that $E_\mu \varphi(t, \cdot) = e^{\frac{1}{2}\sigma^2 \rho(t)^2}$ and so the multiplicative renormalization of φ is given by

$$\psi(t, x) = e^{\rho(t)x - \frac{1}{2}\sigma^2 \rho(t)^2}.$$

We can find $\rho(t)$ by using Theorem 2.5 as in [5] or by using Theorem 3.2 as follows. The Laplace transform of μ and its logarithmic derivative are given by $\ell(r) = e^{\frac{1}{2}\sigma^2 r^2}$ and $g(r) = \sigma^2 r$, respectively. Therefore, Equation (3.5) in Theorem 3.2 is reduced to the following equation:

$$\rho(t)s\rho'(s) = \rho(s)t\rho'(t).$$

and so $t\rho'(t)/\rho(t) = s\rho'(s)/\rho(s) = c$, a constant. Thus $\rho(t) = c_1 t^c$. Choose $c_1 = c = 1$ to get $\rho(t) = t$ and we have the following pre-generating and generating functions:

$$\varphi(t, x) = e^{tx}, \quad \psi(t, x) = e^{tx - \frac{1}{2}\sigma^2 t^2}.$$

To derive the orthogonal polynomials, note that

$$e^{tx}e^{-\frac{1}{2}\sigma^2 t^2} = \left(\sum_{n=0}^{\infty} \frac{1}{n!}t^n x^n\right)\left(\sum_{m=0}^{\infty} \frac{(-\sigma^2)^m}{m!2^m}t^{2m}\right)$$

$$= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{[n/2]} \frac{(-\sigma^2)^k}{(n-2k)!k!2^k}\, x^{n-2k}\right)t^n.$$

Therefore, we have

$$(4.1) \qquad \psi(t,x) = e^{tx-\frac{1}{2}\sigma^2 t^2} = \sum_{n=0}^{\infty} \frac{1}{n!}P_n(x)t^n,$$

where the polynomial $P_n(x)$ is defined by

$$(4.2) \qquad P_n(x) = \sum_{k=0}^{[n/2]} \frac{n!}{(n-2k)!k!2^k}(-\sigma^2)^k x^{n-2k}.$$

To find the Szegö-Jacobi parameters, first note that $P_n(x)$ is even or odd when n is even or odd, respectively. Hence $P_n(x)^2$ is even for any n and so $E_\mu x P_n(x)^2 = 0$. Hence by Equation (2.7) we have

$$\alpha_n = 0, \quad n \geq 0.$$

(In fact, it is well-known that μ is symmetric if and only if $\alpha_n = 0$ for all $n \geq 0$.) To find ω_n we first check that $E_\mu \psi(t,\cdot)^2 = e^{\sigma^2 t^2}$ and so

$$E_\mu \psi(t,\cdot)^2 = \sum_{n=0}^{\infty} \frac{\sigma^{2n}}{n!}t^{2n}.$$

Compare this equation to Equation (2.10) with $a_n = \frac{1}{n!}$ by Equation (4.1) to get

$$\lambda_n = \sigma^{2n}n!,$$

which satisfies the condition in Equation (1.3). Therefore, by Equation (2.8) we get

$$\omega_n = \sigma^2 n, \quad n \geq 1. \quad (\omega_0 = 1).$$

Finally we show that the polynomials defined by Equation (4.2) are the classical Hermite polynomials with parameter σ^2 (see, e.g., page 354 in [12].)

Theorem 4.1. *Let P_n be the polynomial defined by Equation (4.2). Then*

$$P_n(x) = (-\sigma^2)^n e^{\frac{x^2}{2\sigma^2}} D_x^n e^{-\frac{x^2}{2\sigma^2}}.$$

Proof. From Equation (4.1) we have

$$e^{tx - \frac{1}{2}\sigma^2 t^2} = \sum_{n=0}^{\infty} \frac{1}{n!} P_n(x) t^n.$$

By completing the square of the exponent in t we can rewrite this equation as

(4.3)
$$e^{\frac{x^2}{2\sigma^2}} f(x - \sigma^2 t) = \sum_{n=0}^{\infty} \frac{1}{n!} P_n(x) t^n,$$

where $f(u) = e^{-\frac{1}{2\sigma^2} u^2}$. Note that

$$D_t^n f(x - \sigma^2 t) = f^{(n)}(x - \sigma^2 t)(-\sigma^2)^n.$$

Hence if we differentiate both sides of Equation (4.3) n-times in t and then let $t = 0$ then we get

$$P_n(x) = e^{\frac{x^2}{2\sigma^2}} f^{(n)}(x)(-\sigma^2)^n = (-\sigma^2)^n e^{\frac{x^2}{2\sigma^2}} D_x^n e^{-\frac{x^2}{2\sigma^2}}. \qquad \blacksquare$$

4.2. Poisson Measure and Charlier Polynomials

Let μ be the Poisson measure with parameter $\lambda > 0$

$$\mu(\{k\}) = e^{-\lambda} \frac{\lambda^k}{k!}, \qquad k = 0, 1, 2, \ldots.$$

Try the type of pre-generating function $\varphi(t, x) = e^{\rho(t)x}$ in Equation (3.1). It is easily checked that $E_\mu \varphi(t, \cdot) = \exp\left(\lambda(e^{\rho(t)} - 1)\right)$. Then we can either use Theorem 2.5 as in [5] or apply Theorem 3.2 to derive that $e^{\rho(t)} = 1 + t$. Thus the multiplicative renormalization of $\varphi(t, x)$

(4.4)
$$\psi(t, x) = e^{-\lambda t}(1 + t)^x$$

is a generating function for μ. To derive the corresponding orthogonal polynomials we need to use the binomial series

$$(1 + t)^x = \sum_{n=0}^{\infty} \frac{p_{x,n}}{n!} t^n,$$

where $p_{x,0} = 1$ by convention and $p_{x,n} = x(x-1)\cdots(x-n+1)$ for $n \geq 1$. Hence we have

$$e^{-\lambda t}(1+t)^x = \left(\sum_{n=0}^{\infty} \frac{(-\lambda)^n}{n!} t^n \right) \left(\sum_{m=0}^{\infty} \frac{p_{x,m}}{m!} t^m \right)$$

$$= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{(-\lambda)^{n-k}}{(n-k)!k!} p_{x,k} \right) t^n$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\sum_{k=0}^{n} \binom{n}{k} (-\lambda)^{n-k} p_{x,k} \right) t^n.$$

Therefore, we have

(4.5) $$\psi(t,x) = e^{-\lambda t}(1+t)^x = \sum_{n=0}^{\infty} \frac{1}{n!} P_n(x) t^n,$$

where the polynomial $P_n(x)$ is defined by

(4.6) $$P_n(x) = \sum_{k=0}^{n} \binom{n}{k} (-\lambda)^{n-k} p_{x,k}.$$

One way to find the Szegö-Jacobi parameters is to use Equation (2.11) with $a_n = 1/n!$ in view of Equation (4.5). First we can easily compute that

$$E_\mu x \psi(t,\cdot)^2 = \lambda(1+t)^2 e^{\lambda t^2}.$$

Therefore, by Equation (2.11),

$$\lambda(1+t)^2 e^{\lambda t^2} = \sum_{n=0}^{\infty} \left(\frac{1}{(n!)^2} \lambda_n \alpha_n t^{2n} + 2 \frac{1}{n!(n-1)!} \lambda_n t^{2n-1} \right).$$

By comparing the coefficients of t^{2n} and t^{2n-1} we get

$$\lambda_n \alpha_n = \lambda^n (\lambda + n) n!, \quad \lambda_n = \lambda^n n!.$$

Hence by Equation (2.8) the Szegö-Jacobi parameters are given by

$$\alpha_n = \lambda + n, \quad n \geq 0,$$

$$\omega_n = \lambda n, \quad n \geq 1, \quad (\omega_0 = 1).$$

Moreover the parameter λ_n satisifes the condition in Equation (1.3).

Now we will show that the polynomials defined by Equation (4.6) are the classical Charlier polynomials with parameter λ [7] [9].

Theorem 4.2. *Let P_n be the polynomial defined by Equation (4.6). Then*

$$C_n(x; \lambda) = (-1)^n \lambda^{-x} \Gamma(x+1) \Delta_x^n \left(\frac{\lambda^x}{\Gamma(x-n+1)} \right),$$

where Δ is the difference operator $\Delta f(x) = f(x+1) - f(x)$ and $\Gamma(\cdot)$ is the Gamma function.

Proof. Let $\psi(t, x) = e^{-\lambda t}(1+t)^x$. Then

$$(4.7) \qquad \partial_t \psi(t, x) = e^{-\lambda t}(1+t)^{x-1}\big(x - \lambda(1+t)\big).$$

On the other hand, we can easily check that

$$(4.8) \qquad \Delta_x \left(\frac{\big(\lambda(1+t)\big)^x}{\Gamma(x)} \right) = -\frac{\big(\lambda(1+t)\big)^x}{\Gamma(x+1)}\big(x - \lambda(1+t)\big).$$

Cancel out the common last factor in Equations (4.7) and (4.8) to get

$$\partial_t \psi(t, x) = -e^{-\lambda t}(1+t)^{x-1} \frac{\Gamma(x+1)}{\big(\lambda(1+t)\big)^x} \Delta_x \left(\frac{\big(\lambda(1+t)\big)^x}{\Gamma(x)} \right)$$

$$= -e^{-\lambda t} \lambda^{-x}(1+t)^{-1}\Gamma(x+1) \Delta_x \left(\frac{\big(\lambda(1+t)\big)^x}{\Gamma(x)} \right).$$

Bring the factor $e^{-\lambda t}(1+t)^{-1}$ inside the difference operator Δ_x to get

$$\partial_t \psi(t, x) = -\lambda^{-x}\Gamma(x+1) \Delta_x \left(\frac{\lambda^x}{\Gamma(x)} e^{-\lambda t}(1+t)^{x-1} \right)$$

$$= -\lambda^{-x}\Gamma(x+1) \Delta_x \left(\frac{\lambda^x}{\Gamma(x)} \psi(t, x-1) \right).$$

Then apply induction to show that for any n we have

$$\partial_t^n \psi(t, x) = (-1)^n \lambda^{-x} \Gamma(x+1) \Delta_x^n \left(\frac{\lambda^x}{\Gamma(x-n+1)} \psi(t, x-n) \right).$$

Finally put $t = 0$ to get

$$P_n(x) = \partial_t^n \psi(t, x)\Big|_{t=0} = (-1)^n \lambda^{-x} \Gamma(x+1) \Delta_x^n \left(\frac{\lambda^x}{\Gamma(x-n+1)} \right). \qquad \blacksquare$$

4.3. Gamma Distribution and Laguerre Polynomials

Let μ be the Gamma distribution with parameter $\alpha > 0$

$$d\mu(x) = \frac{1}{\Gamma(\alpha)} x^{\alpha-1} e^{-x} \, dx, \qquad x > 0.$$

Try the type of pre-generating function $\varphi(t, x) = e^{\rho(t)x}$ in Equation (3.1). The Laplace transform ℓ and multiplicative renormalization $\psi(t, x)$ of μ are given by

$$(4.9) \qquad \qquad \ell(r) = \frac{1}{(1 - r)^{\alpha}},$$

$$(4.10) \qquad \qquad \psi(t, x) = \frac{e^{\rho(t)x}}{\ell\big(\rho(t)\big)}.$$

We can use Theorem 2.5 as in [5] to derive the function $\rho(t)$. On the other hand, we can apply Theorem 3.2 to derive $\rho(t)$ as follows. From Equation (4.9) we see that the logarithmic derivative of ℓ is given by

$$g(r) = \frac{\ell'(r)}{\ell(r)} = \frac{\alpha}{1 - r}$$

and so Equation (3.5) in Theorem 3.2 becomes

$$\frac{t\rho'(t) - s\rho'(s)}{1 - \rho(t) - \rho(s)} = \frac{t\rho'(t)}{1 - \rho(t)} - \frac{s\rho'(s)}{1 - \rho(s)}.$$

By letting $\xi(t) = 1 - \rho(t)$ we see that this equation is equivalent to

$$t\xi'(t)\big(\xi(s)^2 - \xi(s)\big) - s\xi'(s)\big(\xi(t)^2 - \xi(t)\big) = 0.$$

Therefore, we have

$$\frac{t\xi'(t)}{\xi(t)^2 - \xi(t)} = \frac{s\xi'(s)}{\xi(s)^2 - \xi(s)} = c,$$

which can be easily solved

$$\xi(t) = \frac{1}{1 - c_1 t^c}.$$

Hence $\rho(t)$ is given by

$$\rho(t) = -\frac{c_1 t^c}{1 - c_1 t^c}.$$

Since $\rho(0) = 0$ and $\rho'(0) = 1$ as required in Theorem 3.2, we get $\rho(t) = \frac{t}{1+t}$. Hence the resulting generating function for μ in Equation (4.10) is given by

$$\psi(t, x) = (1 + t)^{-\alpha} e^{\frac{tx}{1+t}}.$$

To derive the corresponding orthogonal polynomials, first note that

$$e^{\frac{tx}{1+t}} = \sum_{n=0}^{\infty} \frac{x^n}{n!} t^n (1+t)^{-n}$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{n!} t^n \sum_{k=0}^{\infty} \binom{-n}{k} t^k$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{n!} \sum_{k=0}^{\infty} \binom{-n}{k} t^{n+k}$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{n!} \sum_{m=n}^{\infty} \binom{-n}{m-n} t^m.$$

Next, change the order of summation to get

$$e^{\frac{tx}{1+t}} = \sum_{m=0}^{\infty} \left[\sum_{n=0}^{m} \frac{x^n}{n!} \binom{-n}{m-n} \right] t^m.$$

Therefore,

$$(1+t)^{-\alpha} e^{\frac{tx}{1+t}} = \sum_{n=0}^{\infty} \binom{-\alpha}{n} t^n \sum_{m=0}^{\infty} \left[\sum_{k=0}^{m} \frac{x^k}{k!} \binom{-k}{m-k} \right] t^m$$

$$= \sum_{n=0}^{\infty} \left[\sum_{j=0}^{n} \binom{-\alpha}{n-j} \sum_{k=0}^{j} \frac{x^k}{k!} \binom{-k}{j-k} \right] t^n.$$

In the double summation inside $[\cdots]$ change the order of summation to get

$$(1+t)^{-\alpha} e^{\frac{tx}{1+t}} = \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} \frac{x^k}{k!} \sum_{j=k}^{n} \binom{-\alpha}{n-j} \binom{-k}{j-k} \right] t^n$$

$$= \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} \frac{x^k}{k!} \sum_{m=0}^{n-k} \binom{-\alpha}{n-k-m} \binom{-k}{m} \right] t^n.$$

In the summation over m we apply the formula

$$\sum_{m=0}^{j} \binom{a}{j-m} \binom{b}{m} = \binom{a+b}{j}$$

and get the following the equality

$$(1+t)^{-\alpha} e^{\frac{tx}{1+t}} = \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} \frac{1}{k!} \binom{-\alpha-k}{n-k} x^k \right] t^n.$$

Therefore we have shown that

$$(4.11) \qquad \psi(t, x) = (1 + t)^{-\alpha} e^{\frac{tx}{1+t}} = \sum_{n=0}^{\infty} \frac{1}{n!} P_n(x) t^n,$$

where the polynomial $P_n(x)$ is defined by

$$(4.12) \qquad P_n(x) = \sum_{k=0}^{n} \frac{n!}{k!} \begin{pmatrix} -\alpha - k \\ n - k \end{pmatrix} x^k.$$

Next we find the Szegö-Jacobi parameters. Direct computations show that

$$E_\mu \psi(t, \cdot)^2 = (1 - t^2)^{-\alpha}$$

and so we have the power series expansion

$$E_\mu \psi(t, \cdot)^2 = \sum_{n=0}^{\infty} (-1)^n \begin{pmatrix} -\alpha \\ n \end{pmatrix} t^{2n}.$$

Hence use Equation (2.10) in Theorem 2.6 with $a_n = 1/n!$ by Equation (4.1) to get

$$\lambda_n = (n!)^2 (-1)^n \begin{pmatrix} -\alpha \\ n \end{pmatrix} = \frac{n! \Gamma(\alpha + n)}{\Gamma(\alpha)}, \quad n \geq 1,$$

which satisfies the condition in Equation (1.3). Therefore, by Equation (2.8) we have

$$\omega_n = n(\alpha + n - 1), \quad n \geq 1, \quad (\omega_0 = 1).$$

On the other hand it can be easily checked that

$$E_\mu x \psi(t, \cdot)^2 = \alpha(1 + t)^{\alpha+1}(1 - t)^{-\alpha-1}.$$

Thus we have the power series expansion

$$E_\mu x \psi(t, \cdot)^2 = \alpha(1 - t^2)^{-\alpha} \frac{1 + t}{1 - t}$$

$$= \alpha \left[\sum_{n=0}^{\infty} (-1)^n \begin{pmatrix} -\alpha \\ n \end{pmatrix} t^{2n} \right] \left(1 + 2 \sum_{m=1}^{\infty} t^m \right),$$

whose coefficient for t^{2n} is given by

$$\alpha(-1)^n \begin{pmatrix} -\alpha \\ n \end{pmatrix} + 2\alpha \left[(-1)^{n-1} \begin{pmatrix} -\alpha \\ n-1 \end{pmatrix} + (-1)^{n-2} \begin{pmatrix} -\alpha \\ n-2 \end{pmatrix} \right.$$

$$\left. + \cdots + (-1)^0 \begin{pmatrix} -\alpha \\ 0 \end{pmatrix} \right].$$

By using the formula

$$(-1)^0 \begin{pmatrix} -\alpha \\ 1 \end{pmatrix} + (-1)^1 \begin{pmatrix} -\alpha \\ 1 \end{pmatrix} + \cdots + (-1)^{n-1} \begin{pmatrix} -\alpha \\ n-1 \end{pmatrix}$$

$$= (-1)^{n-1} \begin{pmatrix} -\alpha - 1 \\ n-1 \end{pmatrix}$$

we see that the coefficient of t^{2n} is given by

$$\alpha(-1)^n \begin{pmatrix} -\alpha \\ n \end{pmatrix} + 2\alpha(-1)^{n-1} \begin{pmatrix} -\alpha - 1 \\ n-1 \end{pmatrix} = \frac{\Gamma(n+\alpha)}{\Gamma(\alpha)n!}(\alpha + 2n).$$

Therefore, by Equation (2.11)

$$\lambda_n\alpha_n = (n!)^2 \frac{\Gamma(n+\alpha)}{\Gamma(\alpha)n!}(\alpha + 2n) = \frac{n!\Gamma(n+\alpha)}{\Gamma(\alpha)}(\alpha + 2n).$$

Hence by Equation (2.7) we have

$$\alpha_n = \alpha + 2n, \quad n \geq 0.$$

Finally we show that the polynomials defined in Equation (4.12) are the classical Laguerre polynomials up to a constant multiple.

Theorem 4.3. *Let $P_n(x)$ be the polynomial defined by Equation (4.12). Then*

$$P_n(x) = (-1)^n n! L_n^{(\alpha)}(x),$$

where $L_n^{(\alpha)}(x)$ is the classical Laguerre polynomial defined by

$$L_n^{(\alpha)}(x) = \frac{1}{n!} x^{-\alpha+1} e^x D_x^n \left(x^{\alpha+n-1} e^{-x} \right).$$

Proof. Since we will use the parameter α in the proof, we denote the generating function $\psi(t, x)$ in Equation (4.11) by $\psi_\alpha(t, x)$, namely, let

$$\psi_\alpha(t, x) = (1+t)^{-\alpha} e^{\frac{tx}{1+t}}.$$

Differentiate $\psi_\alpha(t, x)$ in t to get

(4.13) $$\partial_t \psi_\alpha(t, x) = (1+t)^{-\alpha-1} e^{\frac{tx}{1+t}} \left(-\alpha + \frac{x}{1+t} \right).$$

On the other hand, we have

(4.14) $$D_x \left(x^\alpha e^{-\frac{x}{1+t}} \right) = -x^{\alpha-1} e^{-\frac{x}{1+t}} \left(-\alpha + \frac{x}{1+t} \right).$$

Cancel out the common last factor in Equations (4.13) and (4.14) to get

$$\begin{aligned}
\partial_t \psi_\alpha(t, x) &= -x^{-\alpha+1} e^x (1+t)^{-\alpha-1} D_x\left(x^\alpha e^{-\frac{x}{1+t}}\right) \\
&= -x^{-\alpha+1} e^x D_x\left(x^\alpha (1+t)^{-\alpha-1} e^{-\frac{x}{1+t}}\right) \\
&= -x^{-\alpha+1} e^x D_x\left(x^\alpha e^{-x} \psi_{\alpha+1}(t, x)\right).
\end{aligned}$$

Inductively we have for any $n \geq 1$

$$\partial_t^n \psi_\alpha(t, x) = (-1)^n x^{-\alpha+1} e^x D_x^n\left(x^{\alpha+n-1} e^{-x} \psi_{\alpha+n}(t, x)\right).$$

Put $t = 0$ to get

$$P_n(x) = \partial_t^n \psi_\alpha(t, x)\Big|_{t=0} = (-1)^n x^{-\alpha+1} e^x D_x^n\left(x^{\alpha+n-1} e^{-x}\right).$$

Hence we conclude that $P_n(x) = (-1)^n n! L_n^{(\alpha)}(x)$.   ∎

4.4. Uniform Distribution and Legendre Polynomials

Let μ be the uniform distribution on the interval $[-1, 1]$

$$d\mu(x) = \frac{1}{2} dx, \qquad -1 \leq x \leq 1.$$

Try the type of pre-generating function in Equation (3.2) with $c = -1/2$

$$\varphi(t, x) = \frac{1}{\sqrt{1 - \rho(t)x}}.$$

The expectation and the multiplicative renormalization of $\varphi(t, x)$ are given by

(4.15)
$$E_\mu \varphi(t, \cdot) = \frac{\sqrt{1 + \rho(t)} - \sqrt{1 - \rho(t)}}{\rho(t)},$$

$$\psi(t, x) = \frac{\rho(t)}{\sqrt{1 + \rho(t)} - \sqrt{1 - \rho(t)}} \frac{1}{\sqrt{1 - \rho(t)x}}.$$

Consider small $t, s > 0$ so that $\rho(t), \rho(s) > 0$. We can check that

(4.16)
$$\begin{aligned}
E_\mu \psi(t, \cdot) \psi(s, \cdot) = &\left[\frac{\sqrt{\rho(t)}}{\sqrt{1 + \rho(t)} - \sqrt{1 - \rho(t)}} \frac{\sqrt{\rho(s)}}{\sqrt{1 + \rho(s)} - \sqrt{1 - \rho(s)}}\right] \\
&\times \log\left(\frac{\sqrt{\rho(s)}\sqrt{1 - \rho(t)} - \sqrt{\rho(t)}\sqrt{1 - \rho(s)}}{\sqrt{\rho(s)}\sqrt{1 + \rho(t)} - \sqrt{\rho(t)}\sqrt{1 + \rho(s)}}\right).
\end{aligned}$$

In order for $E_\mu \psi(t, \cdot)\psi(s, \cdot)$ to be a function of ts the quantity inside $[\cdots]$ must be a function of ts. Hence

$$\frac{\sqrt{\rho(t)}}{\sqrt{1 + \rho(t)} - \sqrt{1 - \rho(t)}} = at^b,$$

which can be easily solved for $\rho(t)$ to be

$$\rho(t) = \frac{4a^2 t^{2b}}{1 + 4a^4 t^{4b}}.$$

Choose $a = 1/\sqrt{2}, b = 1/2$ to get

(4.17) $$\rho(t) = \frac{2t}{1 + t^2}.$$

Now, with this $\rho(t)$, we can check that the $\log$ factor in Equation (4.16) is indeed a function of ts. Moreover, the other cases than $t > 0, s > 0$ can be handled by similar arguments. Thus we can conclude from Theorem 2.5 that for the choice $\rho(t)$ in Equation (4.17) the corresponding function from Equation (4.15), namely,

(4.18) $$\psi(t, x) = \frac{1}{\sqrt{1 - 2tx + t^2}}$$

is a generating function for the uniform measure μ on $[-1, 1]$. To derive the power series expansion of $\psi(t, x)$ we first use the binomial series to get

$$\psi(t, x) = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} (-1)^n (2tx - t^2)^n$$

and apply the binomial theorem to expand $(2tx - t^2)^n$. Then observe that the powers in each expansion have the pattern

$$\{0\}, \{1, 2\}, \{2, 3, 4\}, \{3, 4, 5, 6\}, \ldots, \{n, n+1, \ldots, 2n\}, \ldots.$$

Therefore, the coefficient of t^n in the series expansion of $\psi(t, x)$ is given by

$$\sum_{k=0}^{[n/2]} (-1)^n 2^{n-2k} \binom{-\frac{1}{2}}{n-k} \binom{n-k}{k} x^{n-2k},$$

which is a polynomial of degree n in x with the leading coefficient

$$(-1)^n 2^n \binom{-\frac{1}{2}}{n} = \frac{(2n-1)!!}{n!},$$

where $(2n-1)!! = (2n-1)(2n-3)\cdots 3\cdot 1$ and by convention $(-1)!! = 1$. Thus we have obtained the power series expansion of the function in Equation (4.18)

$$(4.19) \qquad \psi(t,x) = \frac{1}{\sqrt{1-2tx+t^2}} = \sum_{n=0}^{\infty} \frac{(2n-1)!!}{n!} P_n(x)t^n,$$

where $P_n(x)$ is defined by

$$(4.20) \qquad P_n(x) = \frac{n!}{(2n-1)!!} \sum_{k=0}^{[n/2]} (-1)^n 2^{n-2k} \begin{pmatrix} -\frac{1}{2} \\ n-k \end{pmatrix} \begin{pmatrix} n-k \\ k \end{pmatrix} x^{n-2k}.$$

To find the Szegö-Jacobi parameters, first note that the uniform measure μ on $[-1,1]$ is symmetric and so

$$\alpha_n = 0, \qquad n \geq 0.$$

Next we can easily evaluate

$$E_\mu \psi(t,x)^2 = \frac{1}{2t}\big(\log(1+t) - \log(1-t)\big)$$

and so we have the series expansion

$$E_\mu \psi(t,x)^2 = \sum_{n=0}^{\infty} \frac{1}{2n+1} t^{2n}.$$

Thus by Equation (2.10) in Theorem 2.6 with $a_n = (2n-1)!!/n!$ in view of Equation (4.19) we get

$$\left(\frac{(2n-1)!!}{n!}\right)^2 \lambda_n = \frac{1}{2n+1},$$

which yields that

$$\lambda_n = \frac{1}{2n+1} \frac{(n!)^2}{\big((2n-1)!!\big)^2} \qquad n \geq 1$$

satisfying the condition in Equation (1.3). Then by Equation (2.8) we have

$$\omega_n = \frac{n^2}{4n^2-1}, \quad n \geq 1, \quad (\omega_0 = 1).$$

Finally we show that the polynomials defined in Equation (4.20) are the classical Legendre polynomials up to a constant multiple.

Theorem 4.4. *Let $P_n(x)$ be the polynomial defined by Equation (4.20). Then*

$$P_n(x) = \frac{n!}{(2n-1)!!} L_n(x),$$

where $L_n(x)$ is the classical Legendre polynomial defined by

$$L_n(x) = \frac{1}{2^n n!} D_x^n (x^2 - 1)^n.$$

Proof. The coefficient of x^{n-2k} in the summation of Equation (4.20)

$$(-1)^n 2^{n-2k} \begin{pmatrix} -\frac{1}{2} \\ n-k \end{pmatrix} \begin{pmatrix} n-k \\ k \end{pmatrix}$$

can be easily simplified to

$$(-1)^k \frac{1}{2^n n!} \begin{pmatrix} n \\ k \end{pmatrix} (2n-2k)(2n-2k-1)\cdots(n-2k+1).$$

Hence $P_n(x)$ can be rewritten as

$$P_n(x) = \frac{n!}{(2n-1)!!} \frac{1}{2^n n!}$$
$$\times \sum_{k=0}^{[n/2]} (-1)^k \begin{pmatrix} n \\ k \end{pmatrix} (2n-2k)(2n-2k-1)\cdots(n-2k+1)x^{n-2k}.$$

Now, observe that

$$(2n-2k)(2n-2k-1)\cdots(n-2k+1)x^{n-2k} = D_x^n x^{2n-2k}.$$

Therefore

$$P_n(x) = \frac{n!}{(2n-1)!!} \frac{1}{2^n n!} \sum_{k=0}^{[n/2]} (-1)^k \begin{pmatrix} n \\ k \end{pmatrix} D_x^n x^{2n-2k}$$

$$= \frac{n!}{(2n-1)!!} \frac{1}{2^n n!} D_x^n \left[\sum_{k=0}^{[n/2]} (-1)^k \begin{pmatrix} n \\ k \end{pmatrix} x^{2n-2k} \right].$$

Note that $D_x^n x^{2n-2k} = 0$ for any $[n/2] < k \leq n$. Hence we have

$$P_n(x) = \frac{n!}{(2n-1)!!} \frac{1}{2^n n!} D_x^n \left[\sum_{k=0}^{n} (-1)^k \begin{pmatrix} n \\ k \end{pmatrix} (x^2)^{n-k} \right]$$

$$= \frac{n!}{(2n-1)!!} \frac{1}{2^n n!} D_x^n (x^2 - 1)^n$$

$$= \frac{n!}{(2n-1)!!} L_n(x). \qquad \blacksquare$$

4.5. Arcsine Distribution and Chebyshev Polynomials of the First Kind

Let μ be the arcsine distribution given by

$$d\mu(x) = \frac{1}{\pi} \frac{1}{\sqrt{1 - x^2}} \, dx, \qquad |x| < 1.$$

Try the type of pre-generating function in Equation (3.2) with $c = -1$

(4.21) $$\varphi(t, x) = \frac{1}{1 - \rho(t)x}.$$

The expectation of $\varphi(t, x)$ can be checked to be

$$E_\mu \varphi(t, \cdot) = \frac{1}{\sqrt{1 - \rho(t)^2}}$$

and so the multiplicative renormalization of $\varphi(t, x)$ is given by

(4.22) $$\psi(t, x) = \sqrt{1 - \rho(t)^2} \frac{1}{1 - \rho(t)x}.$$

Direct computation shows that

(4.23) $$E_\mu \psi(t, \cdot)\psi(s, \cdot) = \frac{\rho(t)\sqrt{1 - \rho(s)^2} - \rho(s)\sqrt{1 - \rho(t)^2}}{\rho(t) - \rho(s)}.$$

In order to find a function $\rho(t)$ so that $E_\mu \psi(t, \cdot)\psi(s, \cdot)$ depends only on ts, let

$$\theta(t) = \frac{1 + \sqrt{1 - \rho(t)^2}}{1 - \sqrt{1 - \rho(t)^2}}, \quad \rho(t) = \frac{2\sqrt{\theta(t)}}{1 + \theta(t)}$$

Then Equation (4.23) becomes

$$E_\mu \psi(t, \cdot)\psi(s, \cdot) = \frac{\sqrt{\theta(t)\theta(s)} + 1}{\sqrt{\theta(t)\theta(s)} - 1}.$$

Hence $\theta(t)$ is given by

$$\theta(t) = at^b$$

and so we have

$$\rho(t) = \frac{2\sqrt{a}t^{b/2}}{1 + at^b}.$$

Choose $a = 1$ and $b = 2$ to get

$$\rho(t) = \frac{2t}{1 + t^2}.$$

Thus the resulting function from Equation (4.22)

$$(4.24) \qquad \psi(t,x) = \frac{1-t^2}{1-2tx+t^2}$$

is a generating function for the arcsine distribution μ. To derive the power series expansion of $\psi(t,x)$, first use the similar argument for Equation (4.18) to obtain

$$(4.25) \qquad \frac{1}{1-2tx+t^2} = \sum_{n=0}^{\infty} \left[\sum_{k=0}^{[n/2]} (-1)^k \left(\begin{array}{c} n-k \\ k \end{array} \right) 2^{n-2k} x^{n-2k} \right] t^n.$$

With this equality we can easily derive the power series expansion of the function $\psi(t,x)$ in Equation (4.24)

$$(4.26) \qquad \psi(t,x) = \frac{1-t^2}{1-2tx+t^2} = \sum_{n=0}^{\infty} 2^n P_n(x) t^n,$$

where $P_n(x)$ is defined by

$$(4.27) \qquad P_n(x) = x^n + \frac{1}{2^n} \sum_{k=1}^{[n/2]} (-1)^k \frac{n}{k} \left(\begin{array}{c} n-k-1 \\ k-1 \end{array} \right) 2^{n-2k} x^{n-2k}.$$

Next we derive the Szegö-Jacobi parameters. Since μ is symmetric, we have

$$\alpha_n = 0, \qquad n \geq 0.$$

Direct computation shows that

$$E_\mu \psi(t,\cdot)^2 = \frac{1+t^2}{1-t^2}$$

and so we have the series expansion

$$E_\mu \psi(t,\cdot)^2 = 1 + 2 \sum_{n=1}^{\infty} t^{2n}.$$

Therefore, by Equation (2.10) with $a_n = 2^n$ in view of Equation (4.26)

$$\lambda_n = 2^{1-2n}, \qquad n \geq 1 \quad \text{and} \quad \lambda_0 = 1,$$

which satisfy the condition in Equation (1.3). Recall that $P_0 = 1$ and so by Equation (2.8) we get

$$\omega_n = \begin{cases} 1, & \text{if } n = 0; \\ 1/2, & \text{if } n = 1; \\ 1/4, & \text{if } n \geq 2. \end{cases}$$

Now we show that the polynomials defined in Equation (4.27) are the classical Chebyshev polynomials of the first kind up to a constant multiple.

Theorem 4.5. *Let $P_n(x)$ be the polynomial defined by Equation (4.27). Then*

$$P_n(x) = \frac{1}{2^{n-1}} T_n(x), \quad n \geq 1, \quad (P_0 = 1),$$

where $T_n(x)$ is the classical Chebyshev polynomial of the first kind defined by

$$T_n(x) = \cos(n \arccos x), \quad n \geq 0.$$

Proof. It is well-known that $\cos(n\theta)$ is a polynomial of $\cos \theta$ as given by

$$\cos(n\theta) = 2^{n-1} \cos^n \theta + \sum_{k=1}^{[n/2]} (-1)^k \frac{n}{k} \binom{n-k-1}{k-1} 2^{n-2k-1} \cos^{n-2k} \theta, \quad n \geq 1.$$

Let $x = \cos \theta$ and divide both sides by 2^{n-1} to get

$$\frac{1}{2^{n-1}} \cos(n \arccos x) = x^n + \frac{1}{2^n} \sum_{k=1}^{[n/2]} (-1)^k \frac{n}{k} \binom{n-k-1}{k-1} 2^{n-2k} x^{n-2k}.$$

Thus from the definition of $P_n(x)$ in Equation (4.27) we see that for $n \geq 1$,

$$P_n(x) = \frac{1}{2^{n-1}} \cos(n \arccos x) = \frac{1}{2^{n-1}} T_n(x). \qquad \blacksquare$$

4.6. Semi-circle Distribution and Chebyshev Polynomials of the Second Kind

Let μ be the semi-circle distribution given by

$$d\mu(x) = \frac{2}{\pi} \sqrt{1 - x^2} \, dx, \qquad |x| < 1.$$

Try the type of pre-generating function in Equation (3.2) with $c = -1$

$$\varphi(t, x) = \frac{1}{1 - \rho(t)x}.$$

The expectation of $\varphi(t, x)$ can be checked to be

$$E_\mu \varphi(t, \cdot) = \frac{2}{1 + \sqrt{1 - \rho(t)^2}}$$

and so the multiplicative renormalization of $\varphi(t, x)$ is given by

$$(4.28) \qquad \psi(t, x) = \frac{1 + \sqrt{1 - \rho(t)^2}}{2} \frac{1}{1 - \rho(t)x}.$$

Direct computation shows that

$$E_\mu \psi(t, \cdot) \psi(s, \cdot) = \frac{1}{2} + \frac{1}{2} \frac{\rho(t)\sqrt{1 - \rho(s)^2} - \rho(s)\sqrt{1 - \rho(t)^2}}{\rho(t) - \rho(s)}.$$

Thus in view of Equation (4.23) we get

$$\rho(t) = \frac{2t}{1 + t^2}.$$

and the resulting function from Equation (4.28)

$$\psi(t, x) = \frac{1}{1 - 2tx + t^2}$$

is a generating function for the semi-circle distribution μ. Its power series expansion is already given in Equation (4.25). Hence we have

$$(4.29) \qquad \psi(t, x) = \frac{1}{1 - 2tx + t^2} = \sum_{n=0}^{\infty} 2^n P_n(x) t^n,$$

where $P_n(x)$ is defined by

$$(4.30) \qquad P_n(x) = \frac{1}{2^n} \sum_{k=0}^{[n/2]} (-1)^k \binom{n-k}{k} 2^{n-2k} x^{n-2k}.$$

To find the Szegö-Jacobi parameters, first we have

$$\alpha_n = 0, \qquad n \geq 0$$

since the measure μ is symmetric. On the other hand, it is easy to check that

$$E_\mu \psi(t, \cdot)^2 = \frac{1}{1 - t^2} = \sum_{n=0}^{\infty} t^{2n}.$$

Hence by Equation (2.10) with $a_n = 2^n$ in view of Equation (4.29)

$$\lambda_n = \frac{1}{4^n}, \qquad n \geq 0,$$

and so by Equation (2.8) we get

$$\omega_n = \frac{1}{4}, \qquad n \geq 0.$$

Obviously, the condition in Equation (1.3) is satisfied.

Next we show that the polynomials defined in Equation (4.30) are the classical Chebyshev polynomials of the second kind up to a constant multiple.

Theorem 4.6. *Let $P_n(x)$ be the polynomial defined by Equation (4.30). Then*

$$P_n(x) = \frac{1}{2^n} U_n(x), \quad n \geq 0,$$

where $U_n(x)$ is the classical Chebyshev polynomial of the second kind defined by

(4.31) $$U_n(x) = \frac{\sin\left[(n+1)\arccos x\right]}{\sin(\arccos x)}, \quad n \geq 0.$$

Proof. It is well-known that $\sin[(n+1)\theta]/\sin\theta$ is a polynomial of $\cos\theta$ as given by

$$\frac{\sin[(n+1)\theta]}{\sin\theta} = \sum_{k=0}^{[n/2]} (-1)^k \binom{n-k}{k} 2^{n-2k} \cos^{n-2k}\theta, \quad n \geq 0.$$

Let $x = \cos\theta$ to get the function $U_n(x)$ in Equation (4.31)

(4.32) $$U_n(x) = \frac{\sin\left[(n+1)\arccos x\right]}{\sin(\arccos x)} = \sum_{k=0}^{[n/2]} (-1)^k \binom{n-k}{k} 2^{n-2k} x^{n-2k}.$$

By comparing Equations (4.30) and (4.32) we see that $P_n(x) = \frac{1}{2^n} U_n(x)$, $n \geq 0$.∎

4.7. Beta-type Distribution and Gegenbauer Polynomials

Let μ be the beta-type distribution with parameter $\beta > -1/2$ given by

$$d\mu(x) = \frac{1}{\sqrt{\pi}} \frac{\Gamma(\beta+1)}{\Gamma(\beta+1/2)} (1-x^2)^{\beta-\frac{1}{2}}, \qquad |x| < 1,$$

where $\Gamma(\cdot)$ is the Gamma function. Note the following special cases

(a) $\beta = 1/2$: uniform distribution.
(b) $\beta = 0$: arcsine distribution.

(c) $\beta = 1$: semi-circle distribution.

To find a generating function for μ we try the type of pre-generating function in Equation (3.2) with $c = -\beta$

$$\varphi(t, x) = \frac{1}{(1 - \rho(t)x)^{\beta}}.$$

Observe that although the measure μ reduces to the arcsine distribution when $\beta = 0$, this pre-generating function does not reduce to the one in Equation (4.21). Thus we assume that $\beta \neq 0$ from now on.

From page 56 in [11] we have the formula

$$\sum_{n=0}^{\infty} \frac{\Gamma(r + n)\Gamma(r + \frac{1}{2} + n)}{\Gamma(2r + n)} \frac{z^n}{n!} = \frac{\sqrt{\pi}}{\sqrt{1 - z}} \left(\frac{1}{1 + \sqrt{1 - z}} \right)^{2r-1}.$$

This formula is equivalent to the following one by integration and differentiation

$$(4.33) \qquad \sum_{n=0}^{\infty} \frac{\Gamma(r - 1 + n)\Gamma(r - \frac{1}{2} + n)}{\Gamma(2r - 1 + n)} \frac{z^n}{n!} = \frac{\sqrt{\pi}}{r - 1} \left(\frac{1}{1 + \sqrt{1 - z}} \right)^{2r-2}.$$

We also have the formula

$$(4.34) \qquad 2 \int_0^{\pi/2} \cos^{2p-1}\theta \sin^{2q-1}\theta \, d\theta = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p + q)}.$$

By using the binomial series and the formulas in Equations (4.33) and (4.34) we can derive that

$$E_{\mu}\varphi(t, \cdot) = \left(\frac{2}{1 + \sqrt{1 - \rho(t)^2}} \right)^{\beta}.$$

and so the multiplicative renormalization of $\varphi(t, x)$ is given by

$$(4.35) \qquad \psi(t, x) = \frac{\varphi(t, x)}{E_{\mu}\varphi(t, \cdot)} = \left(\frac{1 + \sqrt{1 - \rho(t)^2}}{2} \right)^{\beta} \frac{1}{(1 - \rho(t)x)^{\beta}}.$$

Although the computation is much more complicated we can show as in the cases of uniform and semi-circle distributions that $E_{\mu}\psi(t, \cdot)\psi(s, \cdot)$ depends only on ts if and only if $\rho(t)$ is given by

$$\rho(t) = \frac{2at^b}{1 + a^2 t^{2b}}.$$

Choose $a = b = 1$ to get $\rho(t) = \frac{2t}{1+t^2}$. The resulting function from Equation (4.35)

$$(4.36) \qquad \psi(t, x) = \frac{1}{(1 - 2tx + t^2)^{\beta}}$$

is a generating function for μ. Note that by the binomial series we have

$$\psi(t,x) = \sum_{n=0}^{\infty} \binom{-\beta}{n}(-1)^n(2tx - t^2)^n.$$

Then we use the same argument as in the derivation of Equation (4.19) to show that the coefficient of t^n in the series expansion of $\psi(t,x)$ is given by

$$\sum_{k=0}^{[n/2]}(-1)^n 2^{n-2k}\binom{-\beta}{n-k}\binom{n-k}{k}x^{n-2k},$$

which is a polynomial of degree n in x with the leading coefficient

$$(-1)^n 2^n \binom{-\beta}{n} = \frac{2^n \Gamma(\beta+n)}{\Gamma(\beta)n!},$$

Therefore we have the following power series expansion

$$(4.37) \qquad \psi(t,x) = \frac{1}{(1-2tx+t^2)^\beta} = \sum_{n=0}^{\infty}\frac{2^n \Gamma(\beta+n)}{\Gamma(\beta)n!}P_n(x)t^n,$$

where $P_n(x)$ is defined by

$$(4.38) \quad P_n(x) = \frac{\Gamma(\beta)n!}{2^n \Gamma(\beta+n)}\sum_{k=0}^{[n/2]}(-1)^n 2^{n-2k}\binom{-\beta}{n-k}\binom{n-k}{k}x^{n-2k}.$$

For the Szegö-Jacobi parameters, since the measure μ is symmetric we have

$$\alpha_n = 0, \qquad n \geq 0.$$

To find ω_n, observe that

$$E_\mu \psi(t,\cdot)^2 = E_\mu(1-2tx+t^2)^{-2\beta}$$

$$= \frac{1}{\sqrt{\pi}}\frac{\Gamma(\beta+1)}{\Gamma(\beta+\frac{1}{2})}(1+t^2)^{-2\beta}\int_{-1}^{1}(1-\rho(t)x)^{-\gamma}(1-x^2)^{\beta-\frac{1}{2}}dx.$$

Define

$$I(\rho) = \int_{-1}^{1}(1-\rho x)^{-2\beta}(1-x^2)^{\beta-\frac{1}{2}}dx.$$

Then we have the following

$$I(\rho) = \sum_{k=0}^{\infty}\binom{-2\beta}{2k}(-\rho)^{2k}\int_{-1}^{1}x^{2k}(1-x^2)^{\beta-\frac{1}{2}}dx$$

$$= \frac{2^{2\beta-1}\Gamma(\beta+\frac{1}{2})}{\Gamma(2\beta)}\sum_{k=0}^{\infty}\frac{\Gamma(k+\beta+\frac{1}{2})}{(k+\beta)k!}\rho^{2k}.$$

Therefore,

$$\frac{d}{d\rho}\left(I(\rho)\rho^{2\beta}\right) = \frac{2^{2\beta}\Gamma(\beta+\frac{1}{2})}{\Gamma(2\beta)}\sum_{k=0}^{\infty}\frac{\Gamma(k+\beta+\frac{1}{2})}{k!}\rho^{2k+2\beta-1}$$

$$= \frac{2\sqrt{\pi}\Gamma(\beta+\frac{1}{2})}{\Gamma(\beta)}(1-\rho^2)^{-\beta-\frac{1}{2}}\rho^{2\beta-1}.$$

By Equation (2.11), we see that

$$\sum_{n=0}^{\infty} 2(n+\beta)a_n^2\lambda_n t^{2n+2\beta-1}$$

$$= \frac{d}{dt}\left(E_\mu\psi^2(t\cdot)t^{2\beta}\right)$$

$$= \frac{d}{dt}\left(\frac{2^{-2\beta}1}{\sqrt{\pi}}\frac{\Gamma(\beta+1)}{\Gamma(\beta+\frac{1}{2})}\left(\frac{2t}{1+t^2}\right)^{2\beta}I(\rho(t))\right)$$

$$= \frac{2^{-2\beta}1}{\sqrt{\pi}}\frac{\Gamma(\beta+1)}{\Gamma(\beta+\frac{1}{2})}\frac{d}{dt}\left(I(\rho(t))\rho^{2\beta}(t)\right)$$

$$= \frac{2^{-2\beta}1}{\sqrt{\pi}}\frac{\Gamma(\beta+1)}{\Gamma(\beta+\frac{1}{2})}\rho'(t)\frac{d}{d\rho}\left(I(\rho)\rho^{2\beta}\right)\Big|_{\rho=\rho(t)}$$

$$= \frac{2^{-2\beta}1}{\sqrt{\pi}}\frac{\Gamma(\beta+1)}{\Gamma(\beta+\frac{1}{2})}\frac{2(1-t^2)}{(1+t^2)^2}\frac{2\sqrt{\pi}\Gamma(\beta+\frac{1}{2})}{\Gamma(\beta)}(1-\rho^2(t))^{-\beta-\frac{1}{2}}\rho^{2\beta-1}(t)$$

$$= 2\beta t^{2\beta-1}(1-t^2)^{-2\beta}$$

$$= \sum_{n=0}^{\infty}\frac{2\beta\Gamma(n+2\beta)}{\Gamma(2\beta)n!}t^{2n+2\beta-1}.$$

By comparing the coefficients we get

$$\lambda_n = \frac{2\beta\Gamma(n+2\beta)}{2(n+\beta)\Gamma(2\beta)n!a_n^2} = \frac{\beta\Gamma(n+2\beta)\Gamma(\beta)^2n!}{2^{2n}(n+\beta)\Gamma(2\beta)\Gamma(n+\beta)^2}$$

and hence

$$\omega_n = \frac{\lambda_n}{\lambda_{n-1}} = \frac{n(n-1+2\beta)}{4(n+\beta)(n-1+\beta)}, \quad n \geq 1, \quad (\omega_0 = 1).$$

Finally we show that the polynomials defined in Equation (4.38) are the classical Gegenbauer polynomials up to a constant multiple.

Theorem 4.7. *Let $P_n(x)$ be the polynomial defined by Equation (4.38). Then*

(4.39) $$P_n(x) = \frac{\Gamma(\beta)n!}{2^n\Gamma(n+\beta)}G_n^{(\beta)}(x), \quad n \geq 0,$$

where $G_n^{(\beta)}(x)$ is the classical Gegenbauer polynomial defined by

$$G_n^{(\beta)}(x) = \frac{(-1)^n}{2^n} \frac{\Gamma(\beta + \frac{1}{2})\Gamma(n + 2\beta)}{\Gamma(2\beta)\Gamma(n + \beta + \frac{1}{2})} \frac{(1 - x^2)^{\frac{1}{2} - \beta}}{n!} D_x^n \left((1 - x^2)^{n + \beta - \frac{1}{2}} \right).$$

Proof. It is straightforward to check that Equation (4.39) is equivalent to

$$\sum_{k=0}^{[n/2]} (-1)^k \frac{(2\beta)(2\beta + 2) \cdots (2\beta + 2n - 2k - 2)}{2^k k!(n - 2k)!} x^{n - 2k}$$

$$= (-1)^n \frac{(2\beta + n - 1)(2\beta + n - 2) \cdots (2\beta)}{(2\beta + 2n - 1)(2\beta + 2n - 3) \cdots (2\beta + 1)} \frac{(1 - x^2)^{\frac{1}{2} - \beta}}{n!}$$

$$D_x^n \left((1 - x^2)^{n + \beta - \frac{1}{2}} \right).$$

This equality can be verified by mathematical induction, but the computation is rather lengthy and tedious. ∎

5. NEGATIVE BINOMIAL DISTRIBUTIONS

Let μ be the negative binomial distribution with parameters $r > 0$ and $0 < p < 1$

$$\mu(\{k\}) = p^r \begin{pmatrix} -r \\ k \end{pmatrix} (-1)^k (1 - p)^k, \qquad k = 0, 1, 2, \ldots.$$

When $r = 1$, μ is the geometric distribution. To find a generating function for μ, try the type of pre-generating function

$$\varphi(t, x) = e^{\rho(t)x} = \theta(t)^x$$

in Equation (3.1). For simplicity, let $q = 1 - p$. The expectation of $\varphi(t, \cdot)$ is easily evaluated to be

$$E_\mu \varphi(t, \cdot) = \left(\frac{p}{1 - q\theta(t)} \right)^r$$

and so the multiplicative renormalization of $\varphi(t, x)$ is given by

$$\psi(t, x) = \left(\frac{1 - q\theta(t)}{p} \right)^r \theta(t)^x.$$

Then we can derive that for any t, s

$$E_\mu \psi(t, \cdot)\psi(s, \cdot) = \left(\frac{q}{p} \right)^r \left(-1 + \frac{1}{1 - q\theta(t)} + \frac{1}{1 - q\theta(s)} - \frac{p}{\left(1 - q\theta(t)\right)\left(1 - q\theta(s)\right)} \right)^{-r}.$$

This form is exactly the same except for the power r as the geometric distribution case given in Example 3.7 of our part I paper [5]. Hence by that example we can take

$$\theta(t) = \frac{1+t}{1+qt}$$

to get a generating function for μ

$$\psi(t,x) = (1+t)^x(1+qt)^{-x-r}.$$

Apply the binomial series expansion to show that

$$(1+t)^x(1+qt)^{-x-r} = \sum_{n=0}^{\infty} \binom{x}{n} t^n \sum_{m=0}^{\infty} \binom{-x-r}{m} q^m t^m$$

$$= \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} \binom{x}{n-k} \binom{-x-r}{k} q^k \right] t^n.$$

Note that $\binom{-x-r}{k} = (-1)^k \binom{x+r+k-1}{k}$. Hence

$$(1+t)^x(1+qt)^{-x-r} = \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} (-1)^k \binom{x}{n-k} \binom{x+r+k-1}{k} q^k \right] t^n.$$

Note that the leading coefficient of the polynomial in the summation inside $[\cdots]$ is given by

$$\sum_{k=0}^{n} \frac{1}{(n-k)!} \frac{(-1)^k}{k!} q^k = \frac{p^n}{n!}.$$

Therefore, we have

(5.1) $$\psi(t,x) = (1+t)^x(1+qt)^{-x-r} = \sum_{n=0}^{\infty} \frac{p^n}{n!} P_n(x) t^n,$$

where $P_n(x)$ is the polynomial defined by

(5.2) $$P_n(x) = \frac{n!}{p^n} \sum_{k=0}^{n} (-1)^k \binom{x}{n-k} \binom{x+r+k-1}{k} q^k.$$

Here are the first few polynomials:

$$P_0(x) = 1,$$

$$P_1(x) = \frac{1}{p}\big(x - (x+r)q\big),$$

$$P_2(x) = \frac{2}{p^2}\left[\begin{pmatrix} x \\ 2 \end{pmatrix} - \begin{pmatrix} x \\ 1 \end{pmatrix}\begin{pmatrix} x+r \\ 1 \end{pmatrix}q + \begin{pmatrix} x+r+1 \\ 2 \end{pmatrix}q^2\right],$$

$$P_3(x) = \frac{3!}{p^3}\left[\begin{pmatrix} x \\ 3 \end{pmatrix} - \begin{pmatrix} x \\ 2 \end{pmatrix}\begin{pmatrix} x+r \\ 1 \end{pmatrix}q + \begin{pmatrix} x \\ 1 \end{pmatrix}\begin{pmatrix} x+r+1 \\ 2 \end{pmatrix}q^2 \right.$$
$$\left. - \begin{pmatrix} x+r+2 \\ 3 \end{pmatrix}q^3\right].$$

Now we find the Szegö-Jacobi parameters. First we compute the expectation and then use the binomial series to get

$$E_\mu \psi(t,\cdot)^2 = (1 - qt^2)^{-r} = \sum_{n=0}^{\infty} \begin{pmatrix} -r \\ n \end{pmatrix}(-1)^n q^n t^{2n}.$$

Thus by Equation (2.10) in Theorem 2.6 with $a_n = p^n/n!$ in view of Equation (5.1)

$$a_n^2 \lambda_n = \begin{pmatrix} -r \\ n \end{pmatrix}(-1)^n q^n = \frac{\Gamma(n+r)}{\Gamma(r)n!}q^n.$$

Hence we have

$$\lambda_n = \frac{n!\Gamma(n+r)}{\Gamma(r)}\frac{q^n}{p^{2n}},$$

which satisfies condition in Equation (1.3), and so by Equation (2.8) we get

$$\omega_n = n(r+n-1)\frac{q}{p^2}, \quad n \geq 1, \quad (\omega_0 = 1).$$

To find the other Szegö-Jacobi parameter α_n, let us apply Equations (2.12) to

$$E_\mu x\psi(t,\cdot)^2 = \frac{rq}{p}(1+t)^2(1-qt^2)^{-r-1}.$$

Then we have

$$a_n^2 \lambda_n \alpha_n = \frac{rq}{p}\left(\frac{\Gamma(n+r+1)}{\Gamma(r+1)n!}q^n + \frac{\Gamma(n+r)}{\Gamma(r+1)(n-1)!}\right)q^{n-1}$$
$$= \frac{q^n}{p}\frac{\Gamma(n+r)}{\Gamma(r)n!}((n+r)q+n).$$

Therefore

$$\alpha_n = \frac{(n+r)q + n}{p}.$$

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