A POWER SERIES IN SMALL ENERGY FOR THE PERIOD OF THE LOTKA-VOLTERRA SYSTEM

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Abstract. The classical Lotka-Volterra system of two first-order nonlinear differential equations was investigated by Lotka [25] on chemical reactions, Lotka [26] on rhythmical reactions in physiology, Lotka [27] on parasitology, Volterra [46] on fishing activity in the upper Adriatic Sea, and Kozyreff [23] and Erneux and Kozyreff [10] on laser dynamics in the vicinity of the Hopf bifurcation. A functional relation between two dependent variables has been known to describe its periodic behavior in the phase plane. We first solve this equation explicitly for one dependent variable in terms of the other, and then obtain two integral representations of the period having a singularity of the square root type at each endpoint of the integration. Our notations are based on Lambert's W functions, which are two inverse functions of $x \to x \exp(x)$ restricted to $(-\infty, -1]$ and $[-1, \infty)$, respectively. A power series of the period is constructed in small energy for arbitrary number of terms by virtue of expansions of Lambert's W functions near the branch point $x = -\exp(-1)$. Our result settles the discrepancy of various approximate results in the literature, and are further compared with numerical results of computing the period by applying the Gauss-Tschebyscheff integration rule of the first kind.

1. INTRODUCTION

This paper considers periodic solutions of the Lotka-Volterra equation (1.2). A power series of the period is given for small energy. The series settles the discrepancy between other approximate results, and is shown to have excellent agreement with numerical calculations of the period for larger values of the energy.

For two competing species model

(1.1) $u'(t) = u\, f(u, v), \qquad v'(t) = v\, g(u, v);$

Alfred J. Lotka [27] and Vito Volterra [46] studied independently the predator prey model

(1.2) $u'(t) = u(t)\{a - b\, v(t)\}, \qquad v'(t) = v(t)\{c\, u(t) - d\};$

where $u(t)$ is the prey population, $v(t)$ that of the predator at time t, and a, b, c, d are positive constants, see also Rubinow [36] and Murray [29]. Lotka [25] derived this system for the chemical reactions which exhibit periodic behavior in the chemical concentrations, see also Murray [28]. Lotka [26], moreover, used this system for rhythmical reactions such as heartbeat in physiology. The system (1.2) is a classical but nontrivial problem. Periodic solutions of some Lotka-Volterra systems are further investigated by Davis [7], Grasman and Veling [17], Frame [14], Lauwerier [24], Dutt [8], Hsu [20], Rothe [33], Waldvogel [48], [49], and Vasil'eva and Belyanin [44]. Non-periodic solutions of some Lotka-Volterra systems are studied by Abdelkader [1], Varma [43], Willson [50], Burnside [4], Murty and Rao [30], and Olek [32]. Several approximate solutions of the Lotka-Volterra equations with oscillatory behavior were reviewed in detail by Grasman [16].

From the viewpoint of population dynamics, the system (1.2) has been generalized in a number of different ways to model real-world oscillatory phenomena, for example, from the unbounded growth to the bounded growth, from two species to three or more species, from the autonomous system of differential equations to the non-autonomous system of differential equations, from the system of first-order ordinary differential equations to the system of parabolic differential equations, from differential equations to differential-difference or integro-differential equations, or a combination of the above. Two examples are given now. On the hypothesis of heredity of duration T_0, Volterra [45], [47] replaced the two differential equations (1.2) by two integro-differential equations (1.1) with

$$f(u, v) = a - bv - \int_{t-T_0}^{t} K_1(t-s)v(s)\, ds, \quad g(u, v) = cu - d + \int_{t-T_0}^{t} K_2(t-s)u(s)\, ds;$$

where a, b, c, d are positive constants, see also Davis [7]. A more realistic two-species model than the Lotka-Volterra model (1.2) investigated by Holling [19] and Tanner [41] is the system (1.1) with

$$f(u, v) = r\left\{1 - \frac{u}{K}\right\} - \frac{kv}{u + D}, \qquad g(u, v) = s\left\{1 - h\frac{v}{u}\right\},$$

for positive constants r, K, k, D, s, h, see also Murray [29], Hsu and Hwang [21], Gasull, Kooij, and Torregrosa [15], and Hsu and Hwang [22], [51].

From the viewpoint of nonlinear chemical dynamics, Lotka [25] devised a hypothetical set of chemical reactions

$$
\begin{aligned}
R + X &\xrightarrow{k_1} 2X, \\
X + Y &\xrightarrow{k_2} 2Y, \\
Y &\xrightarrow{k_3} P,
\end{aligned}
$$
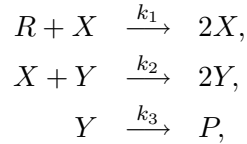
where the reactant R is maintained constant. The law of mass action then yields the rate equations

$$(1.3) \qquad x'(t) = k_1 r x - k_2 xy, \qquad y'(t) = k_2 xy - k_3 y;$$

where $x(t)$, $y(t)$ are the concentrations of intermediates X and Y, respectively, and r is the concentration of R. The system (1.3) is of the type (1.2), and generates sustained temporal oscillations during the net overall reaction $R \to P$. Since then, temporal and spatial oscillations in chemical systems exhibiting limit cycle behavior have been investigated extensively. Some well-known chemical systems include Bray reaction, the Belousov-Zhabotinsky reaction, the Brusselator, and the Oregonator, etc. For more information, see Tyson [42], Nicolis and Prigogine [31], Field and Burger [12], Gray and Scott [18], Scott [38], Schneider and Münster [37], and Epstein and Pojman [9].

In laser dynamics with passively Q-switched microchip lasers, three rate equations in the vicinity of the Hopf bifurcation become

$$
\begin{aligned}
(1.4) \qquad v'(t) &= (1+v)w + \epsilon(1+v)[a_1 v - v'(t)] \\
w'(t) &= -v(a_0 - w) + \{\epsilon(1+v)[a_1 v - v'(t)] - (a_0 + 1)v\}.
\end{aligned}
$$

The dominant term of this system (1.4) for the parameter $0 < \epsilon \ll 1$ is of the form (1.2), see Kozyreff [23, p. 15] and Erneux and Kozyreff [10] for more information. In fact, the shape of a periodic orbit of (1.2) in the phase plane is approximated in four segments in Section 2.5 of Kozyreff [23]. On the other hand, equation (3.1) provides exact expressions of both upper branch and lower branch of such periodic orbit.

The system (1.2) has only one critical point (singular point, or equilibrium point) $(d/c, a/b)$ in the first quadrant of the uv-plane. From the associated Jacobian matrix

$$J(u,v) = \begin{pmatrix} a - bv & -bu \\ cv & cu - d \end{pmatrix},$$

the linearized system of (1.2) at the critical point $(d/c, a/b)$ is

$$(1.5) \qquad u'(t) = -\frac{bd}{c}\{v(t) - \frac{a}{b}\}, \qquad v'(t) = \frac{ac}{b}\{u(t) - \frac{d}{c}\};$$

whose coefficient matrix has eigenpairs

$$(i\sqrt{ad}, (1, -i\frac{c}{b}\sqrt{\frac{a}{d}})^T), \qquad (-i\sqrt{ad}, (1, i\frac{c}{b}\sqrt{\frac{a}{d}})^T).$$

The critical point $(d/c, a/b)$ is then a center for the linearized system (1.5). Moreover, the linearized problem subject to the initial conditions $u(0) = u_0 > 0, v(0) = v_0 > 0$ has the solution

$$u = \frac{d}{c} + (u_0 - \frac{d}{c})\cos(t\sqrt{ad}) - \frac{b}{c}\sqrt{\frac{d}{a}}(v_0 - \frac{a}{b})\sin(t\sqrt{ad}) = \frac{d}{c} + r\cos(t\sqrt{ad} + t_*)$$

$$v = \frac{a}{b} + \frac{c}{b}\sqrt{\frac{a}{d}}\left\{(u_0 - \frac{d}{c})\sin(t\sqrt{ad}) + \frac{b}{c}\sqrt{\frac{d}{a}}(v_0 - \frac{a}{b})\cos(t\sqrt{ad})\right\}$$

$$= \frac{a}{b} + r\frac{c}{b}\sqrt{\frac{a}{d}}\sin(t\sqrt{ad} + t_*);$$

for some t_* satisfying

$$\sin(t_*) = \frac{v_0 - a/b}{r}\frac{b}{c}\sqrt{\frac{d}{a}}, \ \cos(t_*) = \frac{u_0 - d/c}{r}, \ r = \sqrt{(u_0 - \frac{d}{c})^2 + (v_0 - \frac{a}{b})^2\frac{b^2 d}{ac^2}}.$$

Hence the trajectory of the linearized problem is an ellipse with the period $2\pi/\sqrt{ad}$ in the uv-plane. This linear theory, which appeared in both Lotka [25] and Volterra [46], may not predict what happens in the nonlinear system (1.2), but it is closely related to Frame [14] and Waldvogel [48], [49] in some sense. In fact, the period for the nonlinear system (1.2) will be shown to depend on the initial data. This is similar to a simple pendulum of length L, whose equation of motion (3.6) is linearized for small amplitude to be the equation $L\theta''(t) + g\theta = 0$ with the period $2\pi\sqrt{L/g}$, where g is the gravitational acceleration constant.

On the other hand, combining two equations of the system (1.2) yields the separable differential equation

$$(1.6) \qquad\qquad \frac{dv}{du} = \frac{v\{cu - d\}}{u\{a - bv\}}.$$

An elementary integration gives a functional relation between u and v

$$(1.7) \qquad\qquad a\log(v) - bv - cu + d\log(u) + H = 0,$$

where the notation log denotes the natural logarithmic function and H is the constant of integration. Defining $F(u, v) = cu + bv - d\log(u) - a\log(v)$, we then have (1.7) as

$$(1.8) \qquad\qquad F(u, v) = H.$$

Thus the system (1.2) is conservative by virtue of the fact that $F(u, v)$ is a constant as a function of time with $H = F(u_0, v_0)$ on the trajectory passing through the initial data $u(0) = u_0 > 0, v(0) = v_0 > 0$. An elementary technique in calculus further shows that

$$H \geq a + d - a \log(\frac{a}{b}) - d \log(\frac{d}{c}),$$

and the minimum value takes place at $u = d/c, v = a/b$. In the notion of Hamiltonian systems, we write $H = a + d - a \log(a/b) - d \log(d/c) + E$ in (1.8) to obtain

$$(1.9) \qquad cu - d - d \log(\frac{c}{d}u) + bv - a - a \log(\frac{b}{a}v) = E,$$

with the energy $E \geq 0$, and $E = 0$ at $u = d/c, v = a/b$. The system (1.2) has been known to give a one parameter family of periodic solutions (1.9) having $(d/c, a/b)$ as the center point.

The equation (1.7) was obtained from (1.2) in Lotka [25]. Many qualitative properties of the system (1.2) have been obtained from (1.7) since then. We provide a brief survey. Lotka [25] obtained the period of the periodic orbit determined by (1.2) to be about

$$(1.10) \qquad T_l = \frac{2\pi}{\sqrt{ad}}$$

by linearizing the equation (1.7) at $u = d/c, v = a/b$. Volterra [46] separated two variables u, v in (1.8) and then defined each side of the resultant equation to be a new auxiliary variable in his construction of an integral formulation for the period, from which an approximation

$$(1.11) \qquad T_v = \frac{2\pi}{\sqrt{ad}}$$

was followed by approximating the integrand. Frame [14] reduced (1.9) to an equation of the ellipse in a new coordinate system, gave explicit expressions in convergent trigonometric series for the populations of two interacting species, and then derived the following approximation to the exact period

$$
\begin{aligned}
T_f = {} & \frac{2\pi}{\sqrt{ad}} + \frac{\pi(a+d)}{6(ad)^{3/2}}E + \frac{\pi(a+d)^2}{288(ad)^{5/2}}E^2 \\
& - \frac{\pi(a+d)}{155520(ad)^{7/2}}(139a^2 - 154ad + 139d^2)E^3 \\
& - \frac{\pi(a+d)^2}{29859840(ad)^{9/2}}(571a^2 - 586ad + 571d^2)E^4 \\
& - \frac{\pi(a+d)}{62705664000(ad)^{11/2}} \times (829a^4 + 19156a^3d - 9426a^2d^2 + 19156ad^3 + 829d^4)E^5 \\
& + \dots
\end{aligned}
$$

(1.12)

via a lengthy computation, see Equation (4.24), p. 79, of Frame [14]. Dutt [8] used the Hamilton-Jacobi canonical formulation of classical mechanics to obtain nonlinear correction

$$(1.13) \qquad T_d = \frac{2\pi}{\sqrt{ad}} + \frac{\pi(a+d)}{6(ad)^{3/2}}E + \frac{\pi(a+d)^2}{72(ad)^{5/2}}E^2 + \dots,$$

see Equation (24), p. 464, of Dutt [8]. Waldvogel [48], [49] employed a suitable transformation to convert the equation (1.9) into an equation of the circle, from which an integral for the period was obtained. Waldvogel [48] gave the approximation to the period

$$(1.14)\qquad \begin{aligned} T_w = {} & \frac{2\pi}{\sqrt{ad}} + \frac{\pi(a+d)}{6(ad)^{3/2}}E + \frac{\pi(a+d)^2}{288(ad)^{5/2}}E^2 \\ & - \frac{\pi(a+d)}{155520(ad)^{7/2}}(139a^2 + 278ad - 432 + 139d^2)E^3 + \dots, \end{aligned}$$

see Equation (26), p. 1268, of Waldvogel [48]. Based on the theory of Hamiltonian systems and the Laplace transform, Rothe [33], [34] got an integral form for the period, and then gave the following approximation to the period

$$(1.15)\qquad \begin{aligned} T_r = {} & \frac{2\pi}{\sqrt{ad}} + \frac{\pi(a+d)}{6(ad)^{3/2}}E + \frac{\pi(a+d)^2}{288(ad)^{5/2}}E^2 \\ & - \frac{\pi(a+d)}{155520(ad)^{7/2}}(139a^2 - 154ad + 139d^2)E^3 \\ & - \frac{\pi(a+d)^2}{29859840(ad)^{9/2}}(571a^2 - 586ad + 571d^2)E^4 + \dots, \end{aligned}$$

see Equation (23), p. 133, of Rothe [33], and Equation (24), p. 399, of Rothe [34].

Lauwerier [24], Waldvogel [48] and Rothe [34] obtained approximate formulas of the period when the energy E is large. In particular, Lauwerier [24] made the observation that the limiting form of the closed orbit for large E is a triangle. Waldvogel [48] employed a suitable transformation to convert the equation (1.9) into an equation of the circle, from which an integral for the period was obtained by means of inverse Laplace asymptotics, see also [16, pp. 83-86]. Based on the theory of Hamiltonian systems and the Laplace transform, Rothe [34] got an integral form for the period, and then obtained asymptotics of the period for large E, see also [35].

In this paper, our result (4.1) settles the discrepancy of various approximate results (1.10), (1.11), (1.12), (1.13), (1.14), (1.15) in the literature. It is further shown that our power series has an excellent agreement with numerical results of

computing an integral of the period by applying the Gauss-Tschebyscheff integration rule of the first kind. To obtain such result, we first solve (1.9) explicitly for one variable in terms of the other, from which two integral representations of the period are obtained. The basic notations we employ are Lambert's W functions, which are two inverse functions $W(0, x), W(-1, x)$ of $x \exp(x)$ restricted to the intervals $[-1, 0), (-\infty, -1]$, respectively. By comparing methods of Volterra [46], Frame [14], Hsu [20], Waldvogel [48], [49], and Rothe [33], [34], our method of getting integral expressions (3.3), (3.8) for the exact period can be considered to be elementary but elegant. It was shown in Shih [40] that our integral representations (3.3), (3.8) are equivalent to integrals obtained by Volterra [46], Hsu [20], Waldvogel [48], [49], and Rothe [33], [34]. Next, a power series of arbitrary number of terms in small energy is constructed for the period by using expansions of Lambert's functions $W(-k, x)$ at the branch point $x = -\exp(-1)$.

In what follows, two inverse functions of $x \exp(x)$ restricted to $(-\infty, -1]$, $[-1, \infty)$, respectively, are discussed in Section 2 along with some basic properties. We solve the functional relation (1.9) explicitly for one dependent variable in terms of the other, and then give two integral representations of the period in Section 3. We conclude in Section 4 by providing a power series of the period in small energy, which is compared with some approximations to the period (3.8) computed numerically with the use of the Gauss-Tschebyscheff integration rule of the first kind.

2. Inverse Functions of $x \exp(x)$

In order to solve (1.9) explicitly for v in terms of u, one is required to define two auxiliary functions. First of all, the function $x \exp(x)$ has the positive derivative $(x + 1) \exp(x)$ if $x > -1$. Define the inverse function of $x \exp(x)$ restricted on the interval $[-1, \infty)$ to be $W(0, x)$, which is a strictly increasing function mapping from $[-\exp(-1), \infty)$ to $[-1, \infty)$ so that the equivalence relation

$$x \exp(x) = y \Longleftrightarrow W(0, y) = x$$

holds for $x \in [-1, \infty)$, $y \in [-\exp(-1), \infty)$. Similarly, we define the inverse function of $x \exp(x)$ restricted on the interval $(-\infty, -1]$ to be $W(-1, x)$, which is a strictly decreasing function mapping from $[-\exp(-1), 0)$ to $(-\infty, -1]$ so that the equivalence relation

$$x \exp(x) = y \Longleftrightarrow W(-1, y) = x$$

holds for $x \in (-\infty, -1]$, $y \in [-\exp(-1), 0)$. It then follows that

$$(2.1) \qquad W(0, x \exp(x)) = x, \qquad x \geq -1;$$

(2.2) $$W(-1, x \exp(x)) = x, \qquad x \leq -1;$$

(2.3) $$W(0, x) \exp(W(0, x)) = x, \qquad x \geq -\exp(-1);$$

(2.4) $$]W(-1, x) \exp(W(-1, x)) = x, \qquad -\exp(-1) \leq x < 0.$$

For the nature of this study, both $W(0, x)$ and $W(-1, x)$ will be employed only for $x \in [-\exp(-1), 0)$.

Differentiating (2.3), (2.4) with respect to x gives

(2.5) $$W'(0, x) = \frac{\exp(-W(0, x))}{1 + W(0, x)} = \frac{W(0, x)}{x\{1 + W(0, x)\}},$$

(2.6) $$W'(-1, x) = \frac{\exp(-W(-1, x))}{1 + W(-1, x)} = \frac{W(-1, x)}{x\{1 + W(-1, x)\}};$$

for $x \neq -\exp(-1)$.

We conclude this section by providing some remarks on Lambert's W functions. In 1779, Euler obtained a series expansion for the solution of the trinomial equation $x^\alpha - x^\beta = (\alpha - \beta)vx^{\alpha+\beta}$ in the limiting case as $\alpha \to \beta$, which was proposed in 1758 by Lambert. In this case, the equation becomes $\log(x) = vx^\beta$, which has the solution $x = \exp(-W(0, -\beta v)/\beta)$.

In the study of linear differential-difference equations with constant coefficients, one is required to solve some transcendental equations related to $W(-k, x)$ functions. As an illustration, Theorem 3.4 of Bellman and Cooke [3] is given below. The equation $a_0 u'(t) + b_0 u(t) + b_1 u(t - \omega) = 0$ is satisfied by $\sum p_r(t) \exp(s_r t)$, where $\{s_r\}$ is any sequence of zeros of $a_0 s + b_0 + b_1 \exp(-\omega s)$, $p_r(t)$ is a polynomial of degree less than the multiplicity of s_r, and the sum is either finite or is infinite with suitable conditions to ensure convergence. For example, the equation $u'(t) = u(t - 1)$ has a solution of the form $u(t) = \exp(st)$ with $s = W(0, 1)$.

Fritsch, Shafer, and Crowly [13] provided an algorithm of computing $W(0, x)$ for $x > 0$. Shih [39] used $W(0, x)$ with $x > 0$ to describe a slowly moving shock of Burgers' equation in the quarter plane. These inverse functions are also used in constructing the limit cycle of relaxation oscillations of the van der Pol differential equation in the phase plane. A good reference of these functions is Corless, Gonnet, Hare, Jeffrey, and Knuth [5].

The functions $W(-k, x)$ are denoted by Lambert$W(-k, x)$ in the computer algebra system Maple 7, and ProductLog$[-k, x]$ in Mathematica, version 4.1, respectively. Unfortunately, both of them produce erroneous asymptotic behavior for $W(-1, x)$ near $x = -\exp(-1)$.

2.1. Aymptotics of $W(-k, x)$ at the Branch Point $x = -\exp(-1)$

In this investigation, we are interested in the singular behavior of each function $W(-k, x)$ at the branch point $x = -\exp(-1)$. The classical Lagrange inversion formula of complex function theory isn't applicable to the construction of a power series of $W(-k, x)$ at $x = -\exp(-1)$, see for example Fabijonas and Olver [11], because the first derivative of the function $x \exp(x)$ equals zero at $x = -1$.

First of all, for $x = -1$, we have

$$x \exp(x + 1) = \sum_{j=0}^{\infty} \frac{j-1}{j!} (x+1)^j,$$

from which it follows that

$$x \exp(1) = W(-k, x) \exp(W(-k, x) + 1) = \sum_{j=0}^{\infty} \frac{j-1}{j!} \{W(-k, x) + 1\}^j.$$

Thus we obtain the following asymptotic behavior of $W(-k, x)$ as $x \downarrow -\exp(-1)$:

$$(2.7) \qquad W(-k, x) = \begin{cases} -1 + w - \dfrac{1}{3}w^2 + \dfrac{11}{72}w^3 - \dfrac{43}{540}w^4 + \ldots, & k = 0; \\[2mm] -1 - w - \dfrac{1}{3}w^2 - \dfrac{11}{72}w^3 - \dfrac{43}{540}w^4 - \ldots, & k = 1; \end{cases}$$

where $w = \sqrt{2\exp(1)\{x + \exp(-1)\}}$. In other words, we have

$$(2.8) \qquad \frac{1}{1 + W(-k, x)} = \begin{cases} \dfrac{1}{w} + \dfrac{1}{3} - \dfrac{1}{24}w + \dfrac{2}{135}w^2 - \ldots, & k = 0; \\[2mm] \dfrac{-1}{w} + \dfrac{1}{3} + \dfrac{1}{24}w + \dfrac{2}{135}w^2 + \ldots, & k = 1; \end{cases}$$

as $x \downarrow -\exp(-1)$.

Next, instead of deriving a general expression for (2.8)

$$(2.9) \qquad \frac{1}{1 + W(0, x)} = \sum_{k=0}^{\infty} \alpha_k w^{k-1}, \qquad \text{as } x \downarrow -\exp(-1);$$

we proceed the following procedure in order to construct a power series of the period (4.2). Let

$$(2.10) \qquad \psi(\sigma) = \frac{\sigma}{1 + W(0, -\exp(-1 - \frac{1}{2}\sigma^2))}.$$

Then, by using (2.5), $\psi(\sigma)$ satisfies the differential equation

$$(2.11) \qquad\qquad \sigma\psi'(\sigma) = \psi(\sigma) + \sigma\psi(\sigma)^2 - \psi(\sigma)^3.$$

Substituting the series

$$(2.12) \qquad\qquad \psi(\sigma) = \sum_{k=0}^{\infty} \psi_k \sigma^k$$

into (2.11) and equating the like powers of σ give the recursive relation

$$(2.13) \qquad \psi_j = \frac{1}{j+2}\left\{ \sum_{k=0}^{j-1}\psi_k\psi_{j-k-1} - \psi_0\sum_{k=1}^{j-1}\psi_k\psi_{j-k} - \sum_{i=1}^{j-1}\left[\psi_{j-i}\sum_{k=0}^{i}\psi_k\psi_{i-k}\right]\right\},$$

for $j \geq 2$, along with $\psi_0 = 1$, $\psi_1 = 1/3$ obtained from (2.8). Similarly, the function

$$\hat{\psi}(\sigma) = \frac{\sigma}{1 + W(-1, -\exp(-1 - \frac{1}{2}\sigma^2))}$$

has the series expansion $\hat{\psi}(\sigma) = \sum_{k=0}^{\infty}\hat{\psi}_k\sigma^k$ where the coefficients $\hat{\psi}_k$ satisfy (2.13) for $j \geq 2$, along with $\hat{\psi}_0 = -1$, $\hat{\psi}_1 = 1/3$ obtained from (2.8). It can be shown by induction that $\hat{\psi}_j = (-1)^{j+1}\psi_j$ for $j \geq 0$. As an illustration, applying (2.13) recursively gives the following numerical values of ψ_j

$$\psi_2 = \frac{1}{12}, \qquad \psi_3 = \frac{2}{135}, \qquad \psi_4 = \frac{1}{864}, \qquad \psi_5 = \frac{-1}{2835}, \qquad \psi_6 = \frac{-139}{777600}.$$

Thus combining (2.10) and (2.12) gives

$$(2.14) \qquad \frac{1}{1 + W(0, -\exp(-1-s))} = \sum_{k=0}^{\infty} 2^{(k-1)/2}\psi_k s^{(k-1)/2}, \qquad \text{as } s \downarrow 0.$$

Similarly, we have

$$(2.15) \qquad \frac{1}{1 + W(-1, -\exp(-1-s))} = \sum_{k=0}^{\infty} 2^{(k-1)/2}\hat{\psi}_k s^{(k-1)/2}, \qquad \text{as } s \downarrow 0.$$

It then follows from (2.14) and (2.15) that

$$(2.16) \qquad \Phi(s) = \sum_{j=0}^{\infty} 2^{j+1/2}\psi_{2j} s^{j-1/2}, \qquad \text{as } s \downarrow 0;$$

with

$$(2.17) \qquad \Phi(s) = \frac{1}{1 + W(0, -\exp(-1-s))} - \frac{1}{1 + W(-1, -\exp(-1-s))}.$$

3. Two Integral Formulations for the Period

First of all, we solve (1.9) for v in terms of u by using Lambert's functions.

Theorem 3.1. *Equation* (1.9) *is solved explicitly for v in terms of u to give*

$$(3.1) \quad v = g_k(u) \qquad g_k(u) = -\frac{a}{b}W(-k, -(\frac{c}{d}u)^{-d/a}\exp(\frac{c}{a}u - 1 - \frac{d}{a} - \frac{E}{a})),$$

for $k = 0,1$, and $u \in [u_{\min}, u_{\max}]$, where

$$(3.2) \quad u_{\min} = -\frac{d}{c}W(0, -\exp(-1 - \frac{E}{d})), \quad u_{\max} = -\frac{d}{c}W(-1, -\exp(-1 - \frac{E}{d})).$$

Next, the function $v = g_0(u) \in (0, a/b]$ gives the lower branch of the periodic orbit travelling from the point $(u_{\min}, a/b)$ to the point $(u_{\max}, a/b)$ in the counterclockwise direction; while the function $v = g_1(u) \in [a/b, \infty)$ describes the upper branch travelling from the point $(u_{\max}, a/b)$ to the point $(u_{\min}, a/b)$ in the counterclockwise direction. Consequently, the trajectory determined by (1.2) subject to the initial conditions

$$u(0) = u_0 > 0, \qquad v(0) = v_0 > 0$$

is closed and stays in the first quadrant of the phase plane. Furthermore, $v(t)$ oscillates between the minimum value $g_0(d/c)$ and the maximum value $g_1(d/c)$.

Proof. Rewrite (1.9) as

$$\frac{c}{a}u - 1 - \frac{d}{a} - \frac{E}{a} = \frac{d}{a}\log(\frac{c}{d}u) - \frac{b}{a}v + \log(\frac{b}{a}v),$$

or

$$-\frac{b}{a}v\exp(-\frac{b}{a}v) = -(\frac{c}{d}u)^{-d/a}\exp(\frac{c}{a}u - 1 - \frac{d}{a} - \frac{E}{a}),$$

which lies in the interval $[-\exp(-1), 0)$ for positive v. Solving this equation for v gives $g_0(u)$ and $g_1(u)$ defined by (3.1) with $0 < g_0(u) \le a/b \le g_1(u) < \infty$. Next, we determine the range of u. From

$$-(\frac{c}{d}u)^{-d/a}\exp(\frac{c}{a}u - 1 - \frac{d}{a} - \frac{E}{a}) \in [-\exp(-1), 0),$$

we get the inequality

$$\log(\frac{c}{d}u) - \frac{c}{d}u \ge -1 - \frac{E}{d},$$

or

$$-\frac{c}{d}u\exp(-\frac{c}{d}u) \le -\exp(-1 - \frac{E}{d}),$$

which is solved to give $u_{\min}$ and $u_{\max}$ defined by (3.2). The travelling direction in the closed orbit is clearly shown by (1.2) to be in the counterclockwise direction. The function $g_0(u)$, which is strictly decreasing in the interval $(u_{\min}, d/c)$ and strictly increasing in the interval $(d/c, u_{\max})$, has its minimum value at $u = d/c$. On the other hand, the function $g_1(u)$, which is strictly increasing in the interval $(u_{\min}, d/c)$ and strictly decreasing in the interval $(d/c, u_{\max})$, has its maximum value at $u = d/c$. ∎

Now we are ready to obtain integral representations of the period.

Theorem 3.2.   *The closed trajectory determined by* (1.9) *has the period*

$$(3.3) \qquad T = \int_{u_{\min}}^{u_{\max}} \left\{ \frac{1}{u\{a - b\,g_0(u)\}} + \frac{-1}{u\{a - b\,g_1(u)\}} \right\} \, du,$$

where the functions $g_0(u), g_1(u)$ are given by (3.1); *and two endpoints $u_{\min}, u_{\max}$ of the integral are defined by* (3.2). *Furthermore, each term in the integrand of* (3.3) *has a singularity of the square root type at $u = u_{\min}$ and $u = u_{\max}$.*

Proof.   Substituting (3.1) into the first equation of (1.2) gives

$$u'(t) = u\{a - b\,g_k(u)\},$$

or

$$dt = \frac{du}{u\{a - b\,g_k(u)\}}$$

for $k = 0, 1$. Then travelling along the lower branch described by $v = g_0(u)$ from the point $(u_{\min}, a/b)$, with $t = t|_{P_w}$, to the point $(u_{\max}, a/b)$, with $t = t|_{P_e}$, in the counterclockwise direction yields

$$(3.4) \qquad t|_{P_e} - t|_{P_w} = \int_{u_{\min}}^{u_{\max}} \frac{du}{u\{a - b\,g_0(u)\}};$$

while travelling along the upper branch described by $v = g_1(u)$ from the point $(u_{\max}, a/b)$, with $t = t|_{P_e}$, to the point $(u_{\min}, a/b)$, with $t = t|_{P_w}$, in the counterclockwise direction yields

$$(3.5) \qquad t|_{P_w} - t|_{P_e} = \int_{u_{\max}}^{u_{\min}} \frac{du}{u\{a - b\,g_1(u)\}}.$$

It is easy to see from $g_k(u) = a/b$ at $u = u_{\max}$ and $u = u_{\min}$ for $k = 0, 1$ that each integrand in (3.4), (3.5) is singular at each endpoint of the integration. Each integral

has a weak singularity of the square root type at each endpoint of the integration as suggested by (2.8). For example, as $u \downarrow u_{\min}$, we have

$$\frac{1}{a - b\, g_0(u)} = a^{-1}\left\{1 + W(0, -(\frac{c}{d}u)^{-d/a}\exp(\frac{c}{a}u - 1 - \frac{d}{a} - \frac{E}{a}))\right\}^{-1}$$

$$= a^{-1}\left\{\frac{1}{w} + \frac{1}{3} - \frac{w}{24} + \frac{2w^2}{135} + \mathcal{O}(w^3)\right\},$$

with

$$w = \sqrt{2\exp(1)}\sqrt{-(\frac{c}{d}u)^{-d/a}\exp(\frac{c}{a}u - 1 - \frac{d}{a} - \frac{E}{a}) + \exp(-1)}$$

$$= \sqrt{2\exp(1)}\times$$

$$\sqrt{-(\frac{c}{d}u)^{-d/a}\exp(\frac{c}{a}u - 1 - \frac{d}{a} - \frac{E}{a}) + (\frac{c}{d}u_{\min})^{-d/a}\exp(\frac{c}{a}u_{\min} - 1 - \frac{d}{a} - \frac{E}{a})}$$

$$= \sqrt{2\exp(-\frac{d + E}{a})}\sqrt{(\frac{c}{d}u_{\min})^{-d/a}\exp(\frac{c}{a}u_{\min}) - (\frac{c}{d}u)^{-d/a}\exp(\frac{c}{a}u)}$$

$$= \sqrt{2(u - u_{\min})\frac{d/u_{\min} - c}{a}} + \mathcal{O}((u - u_{\min})^2).$$

Thus the period of a complete cycle is determined by the sum of these two convergent improper integrals or (3.3). ■

The period depends on the energy E, and thus on initial data u_0, v_0. This is different from the linearized problem (1.5).

Both Volterra [46] and Hsu [20] used different methods to derive two integrals for the period, which are equivalent to the form (3.3).

The phenomenon of having a weak singularity in the integral for the period takes place even in the linear problems. For instance, consider the linearized problem (1.5), which gives

$$\frac{dv}{du} = \frac{ac^2(u - d/c)}{b^2d(a/b - v)}.$$

Integrating this separable differential equation yields

$$ac^2(u - \frac{d}{c})^2 + b^2d(v - \frac{a}{b})^2 = E$$

for some constant of integration $E \geq 0$. Solving for v in terms of u gives rise to

$$v = \frac{a}{b} \pm \frac{1}{b\sqrt{d}}\sqrt{E - ac^2(u - \frac{d}{c})^2},$$

which represents the upper (with $+$) and lower (with $-$) branches of the ellipse by traveling in the counterclockwise direction for $u \in [d/c - \sqrt{E/(ac^2)}, d/c + \sqrt{E/(ac^2)}]$. Thus, from the first equation of (1.5), we have

$$dt = \frac{c\,du}{bd(a/b - v)},$$

which is integrated to give the period of a complete cycle of ellipse

$$T = \frac{c}{\sqrt{d}} \int_{\frac{d}{c}-\sqrt{E/(ac^2)}}^{\frac{d}{c}+\sqrt{E/(ac^2)}} \frac{du}{\sqrt{E - ac^2(u - d/c)^2}}$$

$$+ \frac{c}{\sqrt{d}} \int_{\frac{d}{c}+\sqrt{E/(ac^2)}}^{\frac{d}{c}-\sqrt{E/(ac^2)}} \frac{du}{-\sqrt{E - ac^2(u - d/c)^2}}$$

$$= \frac{2c}{\sqrt{d}} \int_{\frac{d}{c}-\sqrt{E/(ac^2)}}^{\frac{d}{c}+\sqrt{E/(ac^2)}} \frac{du}{\sqrt{E - ac^2(u - d/c)^2}}.$$

Note that the integrand of the above integral is singular at both endpoints of the integration. An elementary integration technique gives $T = 2\pi/\sqrt{ad}$, which is the period obtained by Lotka [25] and Volterra [46] for a trajectory of (1.9) in a neighborhood of the critical point $(d/c, a/b)$.

A well-known example of nonlinear problems is the mechanical vibration. If the weight of the rod is negligible, the hinge is frictionless, and there is no air resistance, then Newton's second law for rotation and the torque due to the gravitational force give the equation of motion of a simple pendulum of length L

(3.6) $$L\theta''(t) + g\sin(\theta) = 0,$$

where g is the gravitational acceleration constant, and θ is the angle the pendulum makes with the downward vertical. Integration gives

$$L[\theta'(t)]^2 = 2g\left[\cos(\theta) - \cos(a)\right],$$

where the amplitude $a \in (0, \pi/2)$ is defined as the value of θ where $\theta'(t) = 0$. This gives the period of a complete cycle

(3.7) $$T = 2\sqrt{\frac{L}{2g}} \int_{-a}^{a} \frac{d\theta}{\sqrt{\cos(\theta) - \cos(a)}}.$$

Again this integral depends on the initial displacement a, and is an improper integral with a square root singularity at two endpoints of integration.

Next, with a splitting of the integration interval and a simple substitution, one can reduce the integral of the period (3.3) to an alternative form, which is the same as the one appearing in Rothe [33], [34] obtained by using the Laplace transform and theory of Hamiltonian systems.

Theorem 3.3. *The period of the closed trajectory determined by* (1.9) *can be expressed as*

$$(3.8) \qquad T = \frac{1}{ad} \int_0^E \Phi(\frac{s}{d}) \Phi(\frac{E-s}{a}) \, ds,$$

where the function $\Phi(s)$ *is defined by* (2.17).

Proof.   A splitting of the integration interval in (3.3) yields

$$T = \int_{u_{\min}}^{d/c} \left\{ \frac{1}{u\{a - b\,g_0(u)\}} + \frac{-1}{u\{a - b\,g_1(u)\}} \right\} \, du$$

$$+ \int_{d/c}^{u_{\max}} \left\{ \frac{1}{u\{a - b\,g_0(u)\}} + \frac{-1}{u\{a - b\,g_1(u)\}} \right\} \, du.$$

By using (2.3), (2.4), (2.5), and (2.6), the substitution

$$(3.9) \qquad u(s) = -\frac{d}{c} W(0, -\exp(-1 - \frac{s}{d}))$$

gives

$$\int_{u_{\min}}^{d/c} \frac{1}{u\{a - b\,g_0(u)\}} \, du = \frac{1}{ad} \int_0^E \phi_3(s) \, ds,$$

$$\int_{u_{\min}}^{d/c} \frac{-1}{u\{a - b\,g_1(u)\}} \, du = \frac{1}{ad} \int_0^E \phi_2(s) \, ds;$$

and the substitution

$$(3.10) \qquad u(s) = -\frac{d}{c} W(-1, -\exp(-1 - \frac{s}{d}))$$

gives

$$\int_{d/c}^{u_{\max}} \frac{1}{u\{a - b\,g_0(u)\}} \, du = \frac{1}{ad} \int_0^E \phi_4(s) \, ds,$$

$$\int_{d/c}^{u_{\max}} \frac{-1}{u\{a - b\,g_1(u)\}} \, du = \frac{1}{ad} \int_0^E \phi_1(s) \, ds,$$

where $\phi_1(s), \phi_2(s), \phi_3(s), \phi_4(s)$ are defined by

$$\phi_1(\sigma) = \frac{\{1 + W(-1, -\exp(-1 - \sigma/d))\}^{-1}}{1 + W(-1, -\exp(\sigma/a - 1 - E/a))},$$

$$\phi_2(\sigma) = -\frac{\{1 + W(0, -\exp(-1 - \sigma/d))\}^{-1}}{1 + W(-1, -\exp(\sigma/a - 1 - E/a))},$$

$$\phi_3(\sigma) = \frac{\{1 + W(0, -\exp(-1 - \sigma/d))\}^{-1}}{1 + W(0, -\exp(\sigma/a - 1 - E/a))},$$

$$\phi_4(\sigma) = \frac{\{1 + W(-1, -\exp(-1 - \sigma/d))\}^{-1}}{1 + W(0, -\exp(\sigma/a - 1 - E/a))},$$

respectively. It then follows that

$$(3.11) \qquad T = \frac{1}{ad} \int_0^E \{\phi_1(s) + \phi_2(s) + \phi_3(s) + \phi_4(s)\} \, ds,$$

which gives (3.8). ∎

Waldvogel [48], [49] used a suitable transformation to convert the equation (1.9) into an equation of the circle, and then obtains an integral over $[0, 2\pi]$ for the period, which is equivalent to the form (3.8).

4. POWER SERIES AND NUMERICAL INTEGRATION OF THE PERIOD

Theorem 4.1. *The period defined by* (3.8) *has a power series*

$$(4.1) \qquad T = \frac{2\pi}{\sqrt{ad}} \sum_{k=0}^{\infty} \frac{t_k}{k!} E^k, \qquad \text{as } E \downarrow 0;$$

where the coefficients t_k *are of the form*

$$t_k = \sum_{j=0}^{k} \frac{\psi_{2j} \psi_{2k-2j}}{a^{k-j} d^j} \frac{(2j)!(2k-2j)!}{2^k j!(k-j)!},$$

with ψ_j *defined recursively by* (2.13) *for* $j \geq 2$ *along with* $\psi_0 = 1, \psi_1 = 1/3$.

Proof. To obtain a power series of the period T for small energy, a change of variable in (3.8) is used to give

$$(4.2) \qquad T = \frac{E}{ad} \int_0^1 \Phi(E\frac{\sigma}{d}) \Phi(E\frac{1-\sigma}{a}) \, d\sigma.$$

Note that, by using (2.9), we obtain

$$\Phi(s) = \sum_{k=0}^{\infty} a_k [1 - \exp(-s)]^{k-1/2}$$

$$= \frac{\sqrt{2}}{\sqrt{1 - \exp(-s)}} - \frac{\sqrt{2}}{12}\sqrt{1 - \exp(-s)} - \frac{23\sqrt{2}}{864}[1 - \exp(-s)]^{3/2} + \ldots,$$

which, in turn, reduces (4.2) to

$$T = \frac{E}{ad} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} a_{k-j} a_j \int_0^1 [1 - \exp(-E\frac{s}{d})]^{k-j-1/2} [1 - \exp(-E\frac{1-s}{a})]^{j-1/2} \, ds.$$

Such an approximate result seems not to be in the desired form. Instead, substituting (2.16) into (4.2) gives

$$T = \frac{E}{ad} \int_0^1 \sum_{k=0}^{\infty} 2^{k+1/2} \psi_{2k} (E\frac{\sigma}{d})^{k-1/2} \sum_{j=0}^{\infty} 2^{j+1/2} \psi_{2j} (E\frac{1-\sigma}{a})^{j-1/2} \, d\sigma$$

(4.3)
$$= \frac{E}{ad} \int_0^1 \sum_{k=0}^{\infty} \sum_{j=0}^{k} 2^{k+1} \psi_{2j} \psi_{2k-2j} \frac{E^{k-1}}{a^{k-j-1/2}d^{j-1/2}} \sigma^{j-1/2} (1-\sigma)^{k-j-1/2} d\sigma$$

$$= \sum_{k=0}^{\infty} 2^{k+1} \sum_{j=0}^{k} \psi_{2j} \psi_{2k-2j} \frac{E^k}{a^{k-j+1/2}d^{j+1/2}} B(j + \frac{1}{2}, k - j + \frac{1}{2}),$$

where B is the Beta function defined by the Euler integral of the second kind

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt, \qquad x > 0, \quad y > 0;$$

which satisfies the properties

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \quad B(x, y) = B(y, x), \quad B(\frac{1}{2}, \frac{1}{2}) = \pi, \quad B(\frac{1}{2}, \frac{3}{2}) = \frac{\pi}{2}.$$

Here Γ is the Gamma (factorial) function defined by the Euler integral of the first kind

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt, \qquad x > 0;$$

which has the properties

$$\Gamma(\frac{1}{2}) = \sqrt{\pi}, \qquad \Gamma(n+1) = n!, \qquad \Gamma(n + \frac{1}{2}) = \sqrt{\pi} 2^{-n} 1 \cdot 3 \cdot 5 \cdot 7 \cdots (2n-1);$$

for a positive integer n. For more information on these functions, see Abramowitz and Stegun [2] (p. 255 and p. 258). It then follows that

(4.4)
$$B(j + \frac{1}{2}, k - j + \frac{1}{2}) = \frac{\pi (2j)!(2k-2j)!}{2^{2k} k! j! (k-j)!}.$$

Thus we have obtained the desired result by substituting (4.4) into (4.3). ∎

For illustration, we provide a few numerical values of t_k

$$t_0 = 1, \qquad t_1 = \frac{a+d}{12ad}, \qquad t_2 = \frac{(a+d)^2}{288(ad)^2},$$

$$t_3 = -\frac{a+d}{51840(ad)^3}(139a^2 - 154ad + 139d^2),$$

$$t_4 = -\frac{(a+d)^2}{2488320(ad)^4}(571a^2 - 586ad + 571d^2),$$

$$t_5 = \frac{a+d}{209018880(ad)^5}$$
$$\times(163879a^4 - 167876a^3d + 165930a^2d^2 - 167876ad^3 + 163879d^4).$$

For a comparison with some known results of approximating the period in the literature, let

$$(4.5) \qquad\qquad T_j = \frac{2\pi}{\sqrt{ad}} \sum_{k=0}^{j} \frac{t_k}{k!} E^k.$$

Then, from (1.10) and (1.11), both Lotka [25] and Volterra [46] obtained T_0 even though the integral representation of the period derived by Volterra is equivalent to (3.3). The approximate result (1.12) constructed by Frame [14] is identical to T_4 with an error term of the order $\mathcal{O}(E^5)$. The asymptotic formula (1.13) obtained by Dutt [8] is T_1 with an error term of the order $\mathcal{O}(E^2)$. The approximation (1.14) given by Waldvogel [48] is T_2 with an error term of the order $\mathcal{O}(E^3)$ and it was obtained from an integral form of the period, that is equivalent to (3.8). The explicit expansion (1.15) stated in Roth [33], [34] is identical to T_4 and it was based on the integral (3.8).

For the rest of this section, the Gauss-Tschebyscheff integration rule of the first kind is illustrated to be an effective numerical method for computing the integral representation (3.8) of the period.

By using (2.16), the integral (3.8) possesses a weak singularity of the square root type at each endpoint of the integration, which is computed numerically by the Gauss-Tschebyscheff integration rule of the first kind. For more on this numerical quadrature, see for example Davis and Rabinowitz [6]. To proceed further, the integral (3.8) is converted to the form

$$(4.6) \qquad\qquad T = \int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}\, dx,$$

with

$$f(x) = \frac{E}{2ad}\Phi(\frac{E}{2d}(1+x))\Phi(\frac{E}{2a}(1-x))\sqrt{1-x^2}.$$

A numerical approximation to T by using the Gauss-Tschebyscheff integration rule of the first kind is

(4.7) $$T_{\text{sgt}} = \frac{\pi}{n}\sum_{i=1}^{n} f(x_i) \qquad \text{with} \quad x_i = \cos(\frac{(2i-1)\pi}{2n});$$

for $i = 1, \ldots, n$, as well as the error term

$$T - T_{\text{sgt}} = \frac{\pi}{2^{2n-1}(2n)!} f^{(2n)}(\xi) \qquad \text{for some} \quad \xi \in (-1, 1).$$

Computational results T_{sgt} of the period (4.7) given in Table 1 show an excellent agreement with numerical results T_{num} of Grasman and Veling [17], which considered the system (1.2) with $a = b = 1$, $c = d = 2$, $u(0) = u_0$ and $v(0) = 1$. Numerical approximations T_{num} were based on the numerical results of integrating system (1.2) using Zonneveld's Runge-Kutta scheme RK4na, which yield the same results to the required accuracy as a method using the implicit formula (1.8). The polynomial approximations T_j defined by (4.5) show excellent results even for the energy $E > 2.8$. Note that the values of the energy with $u_0 = 1$, $u_0 = 0.25$, $u_0 = 0.1$ are $E = -1 + 2\log(2) \approx 0.386294361$, $E = -3/2 + 2\log(4) \approx 1.272588722$, $E = -9/5 + 2\log(10) \approx 2.805170186$, respectively. The power series (4.1) has a radius of convergence $|E| < 2\pi\min(a, d)$ given by [34].

Table 1. Different approximations of the period

Period	$u_0 = 0.50$ $E = 0.386294361$	$u_0 = 0.25$ $E = 1.272588722$	$u_0 = 0.10$ $E = 2.805170186$
T_0	4.442882938	4.442882938	4.442882938
T_1	4.657415516	5.149628278	6.000763283
T_2	4.660005289	5.177734408	6.137329517
T_3	4.659885795	5.173462174	6.091571362
T_4	4.659884227	5.173277457	6.087210314
T_5	4.659884480	5.173375610	6.092318424
T_{sgt}	4.659884577	5.173375716	6.091989069
T_{num}	4.6599	5.1734	6.0920

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