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A GENERAL THEOREM FOR THE GENERALIZED WEYL FRACTIONAL INTEGRAL OPERATOR INVOLVING THE MULTIVARIABLE H-FUNCTION

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Abstract. In this paper we establish a very general and useful theorem which interconnects the Laplace transform and the generalized Weyl fractional integral operator involving the multivariable H-function of related functions of several variables. Our main theorem involves a multidimensional series with essentially arbitrary sequence of complex numbers. By suitably assigning different values to these sequences, one can easily evaluate the generalized Weyl fractional integral operator of special functions of several variables. We have illustrated it for Srivastava-Daoust multivariable hypergeometric function. On account of general nature of this function a number of results involving special functions of one or more variables can be obtained merely by specializing the parameters.

1. Introduction

(a) The generalized Weyl fractional integral operator occurring in this paper is defined as follows

$$W^{\mu_1,..,\mu_k}\{g(x_1,...,x_k;u_1,...,u_l);\nu_1,...,\nu_k;z_1,...,z_k\}$$

$$(1.1)\quad =\int_{\nu_1}^{\infty}...\int_{\nu_k}^{\infty}\prod_{j=1}^{k}\left\{\frac{(x_j-\nu_j)^{\mu_j-1}}{\Gamma(\mu_j)}\right\}H\begin{matrix}0,0:m_1,n_1;...;m_k,n_k\\ p,q:p_1,q_1;...;p_k,q_k\end{matrix}\left[\begin{matrix}z_1(x_1-\nu_1)^{-\sigma_1}\\ \vdots\\ z_k(x_k-\nu_k)^{-\sigma_k}\end{matrix}\right]$$

$$.g(x_1,...,x_k;u_1,...,u_l)dx_1....dx_k$$

provided that the integral on right-hand side of (1.1) converges absolutely.

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In (1.1) and elsewhere $H[z_1,...,z_k]$ stands for the multivariable H-function introduced by H. M. Srivastava and R. Panda through a series of research papers ([9], [10], [11]). This function is defined and represented in the following manner (see, e.g. [8, p.251. Eq. (C.1)]).

$$H[z_1,...,z_k]=H^{0,n:\{m_k,n_k\}}_{p,q:\{p_k,q_k\}}\left[\begin{matrix}z_1\\ \vdots\\ z_k\end{matrix}\middle|\begin{matrix}(a_j;\alpha'_j,...,\alpha^{(k)}_j)_{1,p}:\left\{(c^{(k)}_j,\gamma^{(k)}_j)_{1,p_k}\right\}\\ (b_j;\beta'_j,...,\beta^{(k)}_j)_{1,q}:\left\{(d^{(k)}_j,\delta^{(k)}_j)_{1,q_k}\right\}\end{matrix}\right]$$

(1.2)

$$=\frac{1}{(2\pi\omega)^k}\int_{L_1}\cdots\int_{L_k}\psi(\xi_1,...,\xi_k)\prod_{i=1}^{k}\left(\phi_i(\xi_i)z_i^{\xi_i}d\xi_i\right)$$

where $\omega=\sqrt{-1}$, and

(1.3) $$\psi(\xi_1,...,\xi_k)=\frac{\prod\limits_{j=1}^{n}\Gamma\left(1-a_j+\sum\limits_{i=1}^{k}\alpha^{(i)}_j\xi_i\right)}{\prod\limits_{j=n+1}^{p}\Gamma\left(a_j-\sum\limits_{i=1}^{k}\alpha^{(i)}_j\xi_i\right)\prod\limits_{j=1}^{q}\Gamma\left(1-b_j+\sum\limits_{i=1}^{k}\beta^{(i)}_j\xi_i\right)}$$

and

(1.4) $$\phi_i(\xi_i)=\frac{\prod\limits_{j=1}^{m_i}\Gamma(d^{(i)}_j-\delta^{(i)}_j\xi_i)\prod\limits_{j=1}^{n_i}\Gamma(1-c^{(i)}_j+\gamma^{(i)}_j\xi_i)}{\prod\limits_{j=m_i+1}^{q_i}\Gamma(1-d^{(i)}_j+\delta^{(i)}_j\xi_i)\prod\limits_{j=n_i+1}^{p_i}\Gamma(c^{(i)}_j-\gamma^{(i)}_j\xi_i)}\quad\forall i\in\{1,2,...,k\}$$

More details of this function can be found in above cited book and research papers. Also $\{m_k,n_k\}$ stands for $m_1,n_1;...;m_k,n_k$ and $\left\{(c^{(k)}_j,\gamma^{(k)}_j)_{1,p_k}\right\}$ stands for the sequence of k ordered pairs $(c^{'}_j,\gamma^{'}_j)_{1,p_1};...;(c^{(k)}_j,\gamma^{(k)}_j)_{1,p_k}$.

(b) The multivariable hypergeometric function introduced by Srivastava and Daoust [6, p. 454] is defined as follows

$$F\left[z_1,...,z_l\right]=F^{\,t\,:\,t_1;\cdots;t_l}_{\,w\,:\,w_1;\cdots;w_l}\left[\begin{matrix}z_1\\ \vdots\\ z_l\end{matrix}\right]$$

(1.5)

$$=F^{\,t\,:\,\{t_l\}}_{\,w\,:\,\{w_l\}}\left[\begin{matrix}\left(g_j;\tau'_j,...,\tau^{(l)}_j\right)_{1,t}:\left\{\left(e^{(l)}_j,E^{(l)}_j\right)_{1,t_l}\right\};\\ \left(h_j;\rho'_j,...,\rho^{(l)}_j\right)_{1,w}:\left\{\left(f^{(l)}_j,F^{(l)}_j\right)_{1,w_l}\right\};\end{matrix}\;z_1,...,z_l\right]$$

$$=\sum_{r_1,...,r_l=0}^{\infty}A_{r_1,...,r_l}\frac{(z_1)^{r_1}}{r_1!}\cdots\frac{(z_l)^{r_l}}{r_l!}$$

where

$$(1.6)\qquad A_{r_1,\ldots,r_l}=\frac{\prod\limits_{j=1}^{t}(g_j)_{\sum\limits_{i=1}^{l} r_i\tau_j^{(i)}}\prod\limits_{j=1}^{t_1}(e'_j)_{r_1E'_j}\cdots\prod\limits_{j=1}^{t_l}(e_j^{(l)})_{r_lE_j^{(l)}}}{\prod\limits_{j=1}^{w}(h_j)_{\sum\limits_{i=1}^{l} r_i\sigma_j^{(i)}}\prod\limits_{j=1}^{w_1}(f'_j)_{r_1F'_j}\cdots\prod\limits_{j=1}^{w_l}(f_j^{(l)})_{r_lF_j^{(l)}}}$$

The multiple series (1.5) converges absolutely (see [7]) for all $z_1,...,z_l$, where $Q_i>0$ or for $Q_i=0$ and $|z_i|<\rho_i$, $(i=1,2,...,l)$ where, ρ_i is defined by equation (5.3) in [7, p. 157] and

$$(1.7)\qquad Q_i=1+\sum_{j=1}^{w}\sigma_j^{(i)}+\sum_{j=1}^{w_i}F_j^{(i)}-\sum_{j=1}^{t}\tau_j^{(i)}-\sum_{j=1}^{t_i}E_j^{(i)},\qquad i=1,2,...,l$$

In this paper we shall establish a theorem which interconnects the well-known multidimensional Laplace transform

$$(1.8)\qquad \begin{aligned}\psi(s_1,...,s_k)&=L\{f(x_1,...,x_k);s_1,...,s_k\}\\&=\int_0^\infty\cdots\int_0^\infty\exp\Big(-\sum_{i=1}^{k}s_ix_i\Big)f(x_1,..,x_k)dx_1...dx_k\\&\quad Re(s_i)>0, i\in\{1,2,...,k\}\end{aligned}$$

and generalized Weyl FIO (defined by (1.1)) of related functions. This theorem is then applied to evaluate generalized Weyl FIO of the Srivastava- Daoust multivariable hypergeometric function defined by (1.5). Several (known and new) special cases of our results are mentioned briefly. Our findings may be useful in handling the problems involving Weyl FIO, as our results involve certain special functions that are highly useful in fractional calculus. It may be remarked here that much more general multidimensional integral transformations than (1.8) were considered in [11].

2. A General Theorem and Its Corollaries

Theorem. *Let* $\{A_{r_1,...,r_l}\}$ *be a sequence of arbitrary complex numbers*, *Re*$(s_i)>0$, *Re*$(\mu_i)>0$, $\nu_i>0$, $\nu_i^{(j)}\in\mathbf{R}$, $i\in\{1,...,k\}$ *and* $j\in\{1,...,l\}$,

$$(2.1)\qquad h(x_1,...,x_k;u_1,...,u_l)=\sum_{r_1,...,r_l=0}^{\infty}A_{r_1,...,r_l}\prod_{j=1}^{l}\frac{(u_j)^{r_j}}{r_j!}\prod_{i=1}^{k}(x_i)^{\sum\limits_{j=1}^{l}\nu_i^{(j)}r_j+\lambda_i-1}$$

and

(2.2) $$g(s_1,...,s_k;u_1,...,u_l)=L\left\{h(x_1,...,x_k;u_1,...,u_l);s_1,...,s_k\right\}$$

then

(2.3)
$$\begin{aligned}&W^{\mu_1,..,\mu_k}\left\{g(x_1,...,x_k;u_1,...,u_l);\nu_1,...,\nu_k;z_1,...,z_k\right\}\\&=\sum_{r_1,...,r_l=0}^{\infty}A_{r_1,...,r_l}\prod_{j=1}^{l}\frac{(u_j)^{r_j}}{r_j!}\prod_{i=1}^{k}\left\{\frac{1}{\Gamma(\mu_i)}(\nu_i)^{-\left(\sum_{j=1}^{l}\nu_i^{(j)}r_j+\lambda_i\right)+\mu_i}\right\}\\&H\begin{array}{l}0,0:\{m_k+1,n_k+1\}\\p,q:\{p_k+1,q_k+1\}\end{array}\\&\left[\begin{array}{c}z_1v_1^{-\sigma_1}\\\vdots\\z_kv_k^{-\sigma_k}\end{array}\middle|\begin{array}{l}(a_j;\alpha_j',...,\alpha_j^{(k)})_{1,p}:\left\{(1-\lambda_k+\mu_k-\sum_{j=1}^{l}\nu_k^{(j)}r_j,\sigma_k),(c_j^{(k)},\gamma_j^{(k)})_{1,p_k}\right\}\\(b_j;\beta_j',...,\beta_j^{(k)})_{1,q}:\left\{(\mu_k,\sigma_k),(d_j^{(k)},\delta_j^{(k)})_{1,q_k}\right\}\end{array}\right]\end{aligned}$$

and provided that

$\sigma_i>0, \mathrm{Re}(\mu_i)-\sigma_i\max_{1\le j\le n_i}\mathrm{Re}\left\{(c_j^{(i)}-1)/\gamma_j^{(i)}\right\}>0, \Omega_i>0$, and $|\arg z_i|<\frac{1}{2}\Omega_i\pi$,

where

(2.4) $$\Omega_i=-\sum_{j=1}^{p}\alpha_j^{(i)}-\sum_{j=1}^{q}\beta_j^{(i)}+\sum_{j=1}^{m_i}\delta_j^{(i)}-\sum_{j=m_i+1}^{q_i}\delta_j^{(i)}+\sum_{j=1}^{n_i}\gamma_j^{(i)}-\sum_{j=n_i+1}^{p_i}\gamma_j^{(i)}\quad\forall i\in\{1,2,...,k\}$$

and the multiple series on the R.H.S. of (*2.1*) *converges absolutely.*

Proof. On substituting the value of $h(x_1,...,x_k;u_1,...,u_l)$ from (2.1) in (2.2) and evaluating the multidimensional Laplace transform, we find that

(2.5)
$$\begin{aligned}&g(\mathrm{s}_1,..,\mathrm{s}_k;u_1,...,u_l)\\&=\sum_{r_{\hat{1}},...,r_{\hat{l}}=0}^{\infty}A_{r_{\hat{1}},...,r_{\hat{l}}}\prod_{j=1}^{l}\frac{(u_j)^{r_j}}{r_j!}\prod_{i=1}^{k}(s_i)^{-\sum_{j=1}^{l}\nu_i^{(j)}r_j-\lambda_i}\Gamma(\lambda_i+\sum_{j=1}^{l}\nu_i^{(j)}r_j)\end{aligned}$$

where $\mathrm{Re}(s_i)>0$, $\mathrm{Re}(\lambda_i)>0$, $\nu_i^{(j)}\in\mathbf{R}$, $i\in\{1,...,k\}, j\in\{1,...,l\}$ and multiple series on R.H.S. of (2.5) is absolutely convergent.

Again taking the multidimensional Laplace transform of the multivariable H-function and replacing $x_{\hat{i}}$ by $(x_{\hat{i}}-\nu_{\hat{i}})$ and using first shift rule therein, we get

(2.6)
$$\begin{aligned}&L\left\{\prod_{i=1}^{k}\left((x_i-\nu_i)^{\mu_i-1}H(x_i-\nu_i)\right)H_1\left[z_1(x_1-\nu_1)^{-\sigma_1},...,z_k(x_k-\nu_k)^{-\sigma_k}\right];s_1,...,s_k\right\}\\&=\exp\left\{-\sum_{i=1}^{k}\nu_is_i\right\}\psi(s_1,...,s_k)\end{aligned}$$

where $H(x-\nu)$ is the well-known Heaviside Unit function and $H_1[z_1,...,z_k] \equiv \{H[z_1,...,z_k]\}_{n=0}$ Also,

$$\psi(s_1,..,s_k)=\prod_{i=1}^{k}(s_i^{-\mu_i})H\begin{matrix}0,0:\{m_k+1,n_k\}\\ p,q:\{p_k,q_k+1\}\end{matrix}\left[\begin{matrix}z_1s_1^{\sigma_1}\\ \vdots\\ z_ks_k^{\sigma_k}\end{matrix}\left|\begin{matrix}(a_j;\alpha'_j,...,\alpha_j^{(k)})_{1,p}:\left\{(c_j^{(k)},\gamma_j^{(k)})_{1,p_k}\right\}\\ (b_j;\beta'_j,...,\beta_j^{(k)})_{1,q}:\left\{(\mu_k,\sigma_k),(d_j^{(k)},\delta_j^{(k)})_{1,q_k}\right\}\end{matrix}\right.\right] \tag{2.7}$$

The sufficient condition of validity for (2.6) are

$$\sigma_i>0,\mathrm{Re}(s_i)>0,|\arg z_i|<\frac{1}{2}\Omega_i\pi,\Omega_i>0,$$

and $\mathrm{Re}(\mu_i)-\sigma_i\max\limits_{1\le j\le n}\mathrm{Re}\left\{(c_j^{(i)}-1)/\gamma_j^{(i)}\right\}>0\ \forall i\in\{1,...,k\}$

Now applying the multidimensional analogue of the Parseval-Goldstein theorem for the Laplace transform for the pairs (2.2) and (2.6), we have

$$\text{L.H.S. of (2.3)}=\int_0^\infty...\int_0^\infty\exp\left\{-\sum_{i=1}^{k}\nu_ix_i\right\}\psi(x_1,...,x_k)\,h(x_1,.x_k;u_1,...,u_l)dx_1...dx_k \tag{2.8}$$

Substituting the series form of $h(x_1,...,x_k;u_1,...,u_1)$ from (2.1), $\psi(x_1,...,x_k)$ from (2.7) in the R.H.S. of (2.8) and interchanging the order of integration and summation therein and then evaluating the x-integral thus obtained, we arrive easily at the required result (2.3).

(i) Taking $l=k$ and choosing, $\nu_i^{(j)}=0,\ i\neq j$ and $\nu_i^{(i)}=\nu_i$ in our main theorem, we immidiately get

Corollary 1. *Let* $\{A_{r_1,...,r_k}\}$ *be a sequence of arbitrary complex numbers*,

$$h(x_1,...,x_k;u_1,...,u_k)=\sum_{r_1,...,r_k=0}^{\infty}A_{r_1,...,r_k}\prod_{i=1}^{k}\left(\frac{(u_i)^{r_i}}{r_i!}(x_i)^{\nu_ir_i+\lambda_i-1}\right) \tag{2.9}$$

and

$$g(s_1,...,s_k;u_1,...,u_k)=L\{h(x_1,..,x_k;u_1,...,u_k);s_1,...,s_k\} \tag{2.10}$$

then

$$W^{\mu_1,..,\mu_k}\{g(x_1,...,x_k;u_1,...,u_k);\nu_1,...,\nu_k;z_1,...,z_k\}$$
$$=\sum_{r_1,...,r_k=0}^{\infty}A_{r_1,...,r_k}\prod_{i=1}^{k}\frac{(u_i)^{r_i}}{r_i!\Gamma(\mu_i)}(\nu_i)^{-\nu_ir_i-\lambda_i+\mu_i}H\begin{matrix}0,0:\{m_k+1,n_k+1\}\\ p,q:\{p_k+1,q_k+1\}\end{matrix}\left[\begin{matrix}z_1v_1^{-\sigma_1}\\ \vdots\\ z_kv_k^{-\sigma_k}\end{matrix}\left|\begin{matrix}(a_j;\alpha'_j,...,\alpha_j^{(k)})_{1,p}:\left\{(1-\lambda_k+\mu_k-\nu_kr_k,\sigma_k),(c_j^{(k)},\gamma_j^{(k)})_{1,p_k}\right\}\\ (b_j;\beta'_j,...,\beta_j^{(k)})_{1,q}:\left\{(\mu_k,\sigma_k),(d_j^{(k)},\delta_j^{(k)})_{1,q_k}\right\}\end{matrix}\right.\right] \tag{2.11}$$

provided that the conditions (modified appropriately) given with the main theorem are satisfied.

(ii) Letting $p = q = 0,\ m_i = 1,\ n_i = p_i,\ q_i \to q_i + 1,\ \sigma_i = 1\ \forall i = 1, 2, .., k$, all γ 's and δ's equal to unity in (2.3) and using the known result [8, p.18, Eq. (2.6.3)] therein, we have after a little simplification

Corollary 2.

$$W_1^{\mu_1,..,\mu_k}\{g(x_1,..,x_k; u_1, ..., u_l); \nu_1, ..., \nu_k; z_1, ..., z_k\}$$

$$= \int_{\nu_1}^{\infty} \cdots \int_{\nu_k}^{\infty} \prod_{i=1}^{k} \left\{ \frac{(x_i - \nu_i)^{\mu_i - 1}}{\Gamma(\mu_i)} {}_{p_i}F_{q_i}\left[(c_{p_i}); (d_{q_i}); \frac{-z_i}{x_i - v_i}\right] \right\} .g(x_1,..,x_k; u_1, ..., u_l) dx_1 ... dx_k$$

$$= \prod_{i=1}^{k} \left\{ \frac{v_i^{\mu_i - \lambda_i}}{\Gamma(\mu_i)} \prod_{j=1}^{q_i} \Gamma(d_j^{(i)}) \left\{ \prod_{j=1}^{p_i} \Gamma(c_j^{(i)}) \right\}^{-1} \right\} \sum_{r_1,...,r_l=0}^{\infty} A_{r_1,...,r_l} \prod_{j=1}^{l} \left(\frac{(u_j)^{r_j}}{r_j!} \right)$$
$$\cdot \prod_{i=1}^{k} (\nu_i)^{-\sum_{j=1}^{l} \nu_i^{(j)} r_j} G^{\,2, p_i+1}_{p_i+1, q_i+2}\left[\frac{z_i}{\nu_i} \middle| \begin{matrix} 1 - \lambda_i + \mu_i - \sum_{j=1}^{l} \nu_i^{(j)} r_j, (1 - c_{p_i}^{(i)}) \\ 0, \mu_i, (1 - d_{q_i}^{(i)}) \end{matrix} \right] \tag{2.12}$$

where $h(x_1, ...x_k; u_1, , ..., u_l)$ *and* $g(x_1, ...x_k; u_1, , ..., u_l)$ *are given by* (*2.1*) *and* (*2.2*) *respectively. The conditions of validity for* (*2.12*) *are*

(*a*) $\nu_i > 0$, *Re*$(\mu_i) > 0$ *and* $\nu_i^{(j)} \in \mathrm{R}$, (*j =1, 2,..., l*)

(*b*) $p_i \le q_i$ *or* $p_i = q_i + 1$ *with* $|z_i| < 1, i \in \{1, ..., k\}$

(*c*) *the multiple series on the R.H.S. of* (*2.12*) *converges absolutely.*

(iii) Further, taking $p_i = 1, q_i = 0, z_i = v_i$ and replacing $\mu_i + c_i$ by μ_i (i=1,...,k) in Corollary 2 and using known results [5, p. 74, Eq. (1); 8, p. 91, Eq. (6.4.19)] we arrive at the following interesting result contained in

Corollary 3.

$$\mathrm{W}_2^{\mu_1,...,\mu_k}[g(x_1,..,x_k; u_1, ..., u_l); \nu_1, ..., \nu_k]$$

$$= \int_{v_1}^{\infty} \cdots \int_{v_k}^{\infty} \prod_{i=1}^{k} \left\{ \frac{(x_i - \nu_i)^{\mu_i - 1}}{\Gamma(\mu_i)} x_i^{-c_i} \right\} g(x_1,..,x_k\,;\, u_1, ..., u_l) dx_1 .... dx_k$$

$$= \sum_{r_1,...,r_l=0}^{\infty} A_{r_1,...,r_l} \prod_{j=1}^{l} \left(\frac{(u_j)^{r_j}}{r_j!} \right) \prod_{i=1}^{k} \left[(\nu_i)^{-\sum_{j=1}^{l} \nu_i^{(j)} r_j + \mu_i - c_i - \lambda_i} \Gamma(\lambda_i + \sum_{j=1}^{l} \nu_i^{(j)} r_j) \right.$$
$$\left. .\ \Gamma(\lambda_i - \mu_i + c_i + \sum_{j=1}^{l} \nu_i^{(j)} r_j) \left(\Gamma(\lambda_i + c_i + \sum_{j=1}^{l} \nu_i^{(j)} r_j) \right)^{-1} \right] \tag{2.13}$$

If we take $k = l$ and $\nu_i^{(j)} = 0$ for $i \neq j$ and $\nu_i^{(i)} = \nu_i$ in Corollary 3, we get result due to Goyal and Garg [3, p.274, Theorem 2]. Again taking $l = 2$, $k = 1$, $\nu_1 = \nu_2 = 1$ in (2.13), we arrive at the known Theorem 3 due to Jain and Pathan [4, p.53, Eq. (2.9)] and for $l = 2$, $k = 1$, $\nu_1 = 0$, $\nu_2 = 1$, the Corollary 3 reduces to Theorems 1 and 2 due to them. Also for k = 1, the main Theorem reduces to a result recently given by the authors [2].

3. Further Consequences and Applications

If we choose $\{A_{r_1,\ldots,r_l}\}$ as given by (1.6), then from (1.5), (2.1) and (2.2) we have,

$$\begin{aligned} &g(s_1,..,s_k; u_1, ..., u_l) \\ &= \prod_{i=1}^{k}\left(\Gamma(\lambda_i)s_i^{-\lambda_i}\right)F_1\left[u_1\prod_{i=1}^{k}\left(s_i^{-\nu_i'}\right), ..., u_l\prod_{i=1}^{k}\left(s_i^{-\nu_i^{(l)}}\right)\right] \end{aligned} \tag{3.1}$$

where

$$\begin{aligned} &F_1[u_1, ..., u_l] = F^{\,t+k:\{t_l\}}_{\,w:\{w_l\}} \\ &\left[\begin{array}{ll} \left(\lambda_i;\nu_1^{(i)},...,\nu_l^{(i)}\right)_{1,k},\left(g_j;\tau_j',...,\tau_j^{(l)}\right)_{1,t} & :\left\{\left(e_j^{(l)},E_j^{(l)}\right)_{1,t_l}\right\}; u_1,...,u_l \\ \left(h_j;\rho_j',...,\rho_j^{(l)}\right)_{1,w} & :\left\{\left(f_j^{(l)},F_j^{(l)}\right)_{1,w_l}\right\}; \end{array}\right] \end{aligned} \tag{3.2}$$

provided that Re$(s_i) > 0$, Re$(\lambda_i) > 0$, $\nu_i^{(j)} \in \mathbf{R}^+$, $(i = 1, ..., k; j = 1, ..., l)$ and the conditions mentioned with (1.6) are satisfied.

Substituting the value of $\{A_{r_1,\ldots,r_l}\}$ and $g(x_1, ...x_k; u_1, , ..., u_l)$ in (2.3) of the main theorem, then using (1.2), (1.5) and [8, p. 254, Eq. (C.9)] therein and reinterpreting the expression so obtained in terms of H-Function of (k+*l*) variables, we finally obtain

$$\begin{aligned} &W^{\mu_1,\ldots,\mu_k}\left\{\prod_{i=1}^{k}\left(x_i^{-\lambda_i}\right)F_1\left[u_1\prod_{i=1}^{k}\left(x_i^{-\nu_i'}\right),...,u_l\prod_{i=1}^{k}\left(x_i^{-\nu_i^{(l)}}\right)\right];\nu_1,...,\nu_k;z_1,..,z_k\right\} \\ &= \prod_{i=1}^{k}\left(\frac{\nu_i^{\mu_i-\lambda_i}}{\Gamma(\mu_i)\Gamma(\lambda_i)}\right)\prod_{j=1}^{w}\Gamma(h_j)\left\{\prod_{j=1}^{t}\Gamma(g_j)\right\}^{-1}\prod_{i=1}^{l}\left[\prod_{j=1}^{w_i}\Gamma\left(f_j^{(i)}\right)\left\{\prod_{j=1}^{t_i}\Gamma\left(e_j^{(i)}\right)\right\}^{-1}\right] \end{aligned} \tag{3.3}$$

$$.H\begin{array}{l}0,t+k:\{m_k+1,n_k\};\{1,t_l\}\\ p+t+k,w+q:\{p_k,q_k+1\};\{t_l,w_l+1\}\end{array}\left[\begin{array}{c}z_1\nu_1^{-\sigma_1}\\ \vdots\\ z_k\nu_k^{-\sigma_k}\\ -u_1\prod_{i=1}^{k}\left(\nu_i^{-\nu_i'}\right)\\ \vdots\\ -u_l\prod_{i=1}^{k}\left(\nu_i^{-\nu_i^{(l)}}\right)\end{array}\right|$$

$$\cdot\left(1-\lambda_j+\mu_j;0,..,\sigma_j,...0,\nu_j',...,\nu_j^{(l)}\right)_{1,k},$$

$$\overline{\qquad},$$

$$\left(1-g_j;\underbrace{0,..,0}_{k-times},\tau_j',...,\tau_j^{(l)}\right)_{1,t},(a_j;\alpha_j',...,\alpha_j^{(k)},\underbrace{0,..,0}_{l-times})_{1,p}:\left\{(c_j^{(k)},\gamma_j^{(k)})_{1,p_k}\right\};$$

$$\left(1–h_j;\underbrace{0,..,0}_{k–times},\rho\prime_j,...,\rho_j^{(l)}\right)_{1,w},\left(b_j;\beta_j',...,\beta_j^{(k)},\underbrace{0,..,0}_{l-times}\right)_{1,\mathrm{q}}:\left\{(\mu_k,\sigma_k),(d_j^{(k)},\delta_j^{(k)})_{1,q_k}\right\};$$

$$\left.\begin{array}{l}\left\{\left(1-e_j^{(l)},E_j^{(l)}\right)_{1,t_l}\right\}\\ \left\{(0,1),\left(1-f_j^{(l)},F_j^{(l)}\right)_{1,w_l}\right\}\end{array}\right]$$

The conditions of validity of (3.3) are

(i) $\nu_i>0,\ \sigma_i>0,\nu_i^{(j)}\in\mathbf{R}^+,\ i\in\{1,2,...,k\},\ j\in\{1,2,...,l\}$

$\mathrm{Re}(\lambda_i)+\sigma_i\min_{1\le j\le m_i}\mathrm{Re}\,(d_j/\delta_j)>1$ and

$\mathrm{Re}(\mu_i)-\sigma_i\max_{1\le j\le n_i}\mathrm{Re}\,\{(c_j-1)/\gamma_j\}>0,$

(ii) $\Omega_i>0,\ |\arg z_i|<\frac{1}{2}\Omega_i\pi,$ *where* Ω_i *is defined by* (2.4)

$\forall i\in\{1,2,...,k\}$

(iii) *the conditions (modified appropriately) which are given just below* (1.6).

Next we mention certain interesting special cases of the aforementioned result by considering following examples:

(i) If we let $l=1$ and $t=w=0$ in (3.3) and use a known result (c.f., e.g. [8, p. 19, Eq. (2.6.11)]) therein, we obtain the following result involving the Fox-Wright Psi- function.

$$W^{\mu_1,\dots,\mu_k}\left\{\prod_{i=1}^{k} x_i^{-\lambda_i}\,{}_{t+k}\psi_w\left[\begin{matrix}(\lambda_j,\nu_j)_{1,k},(e_j,E_j)_{1,t}; & \prod_{i=1}^{k} x_i^{-\nu_i}\\ (f_j,F_j)_{1,w}; & \end{matrix}\right];\nu_1,\dots,\nu_k;z_1,\dots,z_k\right\}$$

(3.4)

$$=\prod_{i=1}^{k}\left\{\frac{\nu_i^{\mu_i-\lambda_i}}{\Gamma(\mu_i)}\right\} H\begin{matrix}0,k:\{m_k+1,n_k\};1,t\\ p+k,q:\{p_k,q_k+1\};t,w+1\end{matrix}\left[\begin{matrix}z_1\nu_1^{-\sigma_1}\\ \vdots\\ z_k\nu_k^{-\sigma_k}\\ -u\prod_{i=1}^{k}\left(\nu_i^{-\nu_i}\right)\end{matrix}\right|$$

$$(1-\lambda_j+\mu_j;0,..,\sigma_j,...0,\nu_j)_{1,k},$$

$$\text{——},$$

$$\left.\begin{matrix}\left(a_j;\alpha_j',\dots,\alpha_j^{(k)},\underbrace{0,..,0}_{l-times}\right)_{1,p}:\left\{(c_j^{(k)},\gamma_j^{(k)})_{1,p_k}\right\};(1-e_j,E_j)_{1,t}\\ \left(b_j;\beta_j',\dots,\beta_j^{(k)},\underbrace{0,..,0}_{l-\ times}\right)_{1,q}:\left\{(\mu_k,\sigma_k),(d_j^{(k)},\delta_j^{(k)})_{1,q_k}\right\};(0,1),(1-f_j,F_j)_{1,w}\end{matrix}\right]$$

(ii) If we substitute $\{A_{r_1,\dots,r_l}\}$ from (1.6) in (2.13) , we obtain

(3.5)

$$W_2^{\mu_1,\dots,\mu_k}\left\{\prod_{i=1}^{k}\left(x_i^{-\lambda_i}\right)F_1\left[u_1\prod_{i=1}^{k}\left(x_i^{-\nu_i'}\right),\dots,u_l\prod_{i=1}^{k}\left(x_i^{-\nu_i^{(l)}}\right)\right];\nu_1,\dots,\nu_k\right\}$$

$$=\prod_{i=1}^{k}\left\{\frac{\nu_i^{\mu_i-c_i-\lambda_i}\Gamma(\lambda_i-\mu_i+c_i)}{\Gamma(\lambda_i+c_i)}\right\}F\begin{matrix}t+2k:\{t_k\}\\ \text{w}+k:\{w_k\}\end{matrix}\left[\begin{matrix}\left(\lambda_j-\mu_j+c_j;\nu_j',\dots,\nu_j^{(l)}\right)_{1,k},\\ \left(\lambda_j+c_j;\nu_j',\dots,\nu_j^{(l)}\right)_{1,k},\end{matrix}\right.$$

$$\begin{matrix}\left(\lambda_j;\nu_j',\dots,\nu_j^{(l)}\right)_{1,k},\left(g_j;\tau_j',\dots,\tau_j^{(l)}\right)_{1,t}:\left\{\left(e_j^{(l)},E_j^{(l)}\right)_{1,t_l}\right\};\\ \left(h_j;\rho_j',\dots,\rho_j^{(l)}\right)_{1,w}:\left\{\left(f_j^{(l)},F_j^{(l)}\right)_{1,w_l}\right\};\end{matrix}$$

$$\left.u_1\prod_{i=1}^{k}\left(\nu_i^{-\nu_i'}\right),\dots,u_k\prod_{i=1}^{k}\left(\nu_i^{-\nu_i^{(l)}}\right)\right]$$

The conditions of validity for results (3.4) and (3.5) are easily obtainable from the main result (3.3), however we prefer to omit them.

(ii) Also on taking $k=1$, $l=2$ in (3.5) we arrive at a known result due to Jain and Pathan [4, p.54, Eq. (3.4)], which also contains another two results (3.6) and (3.8) due to them. Finally on taking $l=k$, $\nu_i^{(j)}=0$ for $i\neq j$ and $\nu_i^{(i)}=\nu_i$, we get the result (4.5) due to Goyal and Garg [3] as a special case of our result (3.5).

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