TAIWANESE JOURNAL OF MATHEMATICS
Vol. 8, No. 3, pp. 503-513, September 2004
This paper is available online at http://www.math.nthu.edu.tw/tjm/

SOLVABILITY OF NONLINEAR ORDINARY DIFFERENTIAL EQUATION WHEN ITS ASSOCIATED LINEAR EQUATION HAS NO NONTRIVIAL OR SIGN-CHANGING SOLUTION

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Abstract. In this paper we investigate the existence of nontrivial solutions of a two-point boundary value problem. Under the condition that the associated linear boundary value problem has no nontrivial solutions or no sign-changing solutions and some other additional conditions, we prove some existence theorems of (nontrivial) solutions.

1. Introduction

We consider the two point boundary value problem:

$$u'' + u + g(x,u) = h(x) \quad \text{in} \quad (0,\pi), \quad u(0) = u(\pi) = 0, \tag{1}$$

where $g : (0,\pi) \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory function, that is, $g(x,u)$ is measurable in $x \in (0,\pi)$ for each $u \in \mathbb{R}$, continuous in $u \in \mathbb{R}$ for a.e. $x \in (0,\pi)$ and satisfies for each $r > 0$, there exists $\alpha_r(x) \in L^1(0,\pi)$ such that $\mid g(x,u) \mid \ \leq \alpha_r(x)$ for a.e. $x \in (0,\pi)$ and $\mid u \mid \ \leq r$. And $h(x) \in L^1(0,\pi)$ is a given function. We prove some existence theorems of (nontrivial) solutions to (1) under some conditions on $g(x,u)$ by coincidence degree method.

The following assumption is used in this paper.

(C1) $\limsup_{|u|\to\infty} g(x,u)/u = \Gamma(x)$, where $\Gamma(x) \in L^1(0,\pi)$ and the convergence is uniform for a.e. $x \in \Omega$.

If $\Gamma(x) \leq const < 0$, the problem (1) has been studied by topological degree method, sub- and supersolution method and critical point theory (boundedness from below for $\Gamma(x)$ is needed sometimes). We refer to [8] for refererce. Instead of the assumption of boundedness from above for $\Gamma(x)$, we use the condition (H1)(see

Received March 28, 2003; Accepted September 12, 2003.
Communicated by Hal Smith.
2000 *Mathematics Subject Classification*: 34B15.
Key words and phrases: Caratheodory conditions, boundary value problems, coincidence degree.

section 2) similar to the one used in [3], which is satisfied under a type of Lyapunov condition.

When $\Gamma(x) \geq 0,\ \Gamma(x) \leq 3$ and the strict inequality holds in a subset of $(0, \pi)$ with positive measure, the problem has been widely investigated in the literature under some further conditions, e.g. Landesman-Lazer condition. We refer to [2], [4], [7], [8], [9] and the references therein. Of particular interest to our work is the paper of Dancer and Gupta ([1]) where they proposed a condition on an initial value problem related to $\Gamma(x)$. This makes it possible to investigate the problem (1) with unbounded $\Gamma(x)$. In this paper, we propose a different kind of condition (H2) on $\Gamma(x)$ involving a boundary value problem to deal with (1) in this case. Some situations satisfying the above condition on $\Gamma(x)$ are also presented. Finally we prove a theorem about the existence of nontrivial solutions for (1). It seems that there are not many results in this respect.

In this paper, $C_0^1[0, \pi]$ denotes the subspace of $C^1[0, \pi]$ of functions satisfying $u(0) = u(\pi) = 0$. The usual norm of $C[0, \pi]$ is denoted by $\|u\|_\infty$. For any $v(x) \in H_0^1(0, \pi)$ with Fourier series $\sum_{n=1}^{n=\infty} b_n \sin nx$, set $v_0(x) = b_1 \sin x,\ \widetilde{v}(x) = \sum_{n=2}^{n=\infty} b_n \sin nx$.

2. Existence Theorem Under Condition (H1)

We assume the following condition on $\Gamma(x)$ in (C1).

(H1) For every $a(x) \in L^1(0, \pi)$ with $-1 \leq a(x) \leq \Gamma(x)$, the boundary value problem

$$u'' + u + a(x)u = 0 \quad \text{in} \quad (0, \pi), \quad u(0) = u(\pi) = 0 \tag{2}$$

has only the trivial solution in $H_0^1(0, \pi)$.

(C2) There exist $r > 0$ and $\alpha(x),\ \beta(x)$ in $L^1(0, \pi)$ such that for a.e. $x \in (0, \pi)$

$$u + g(x, u) \geq \beta(x) \quad \text{if} \quad u \geq r; \quad u + g(x, u) \leq \alpha(x) \quad \text{if} \quad u \leq -r.$$

Theorem 1 *Assume* (*C1*), (*C2*) *and* (*H1*). *Then the boundary value problem* (*1*) *has at least one solution in* $C[0, \pi]$.

Proof. By (H1) and the variational characterization ([1]) of the least eigenvalue of

$$u'' + u + \Gamma(x)u = -\lambda u \quad \text{in} \quad (0, \pi), \quad u(0) = u(\pi) = 0$$

we can prove that $\Gamma(x) + \delta$ also satisfies the condition (H1) for some $\delta > 0$.

For the given $\delta > 0$, by (C1), we can get $R > 0$ such that $g(x, u)/u \leq \Gamma(x) + \delta$ for $|u| \geq R$. Without loss of generality, we assume $R = r$, where r is in (C2). It is a standard argument (e.g. see [1], [5], [7]) that we can decompose $g(x,u)+u = \gamma(x,u)u + \widetilde{g}(x,u)$, where $\gamma(x,u)$ and $\widetilde{g}(x,u)$ are Caratheodory functions and, moreover, satisfy

$$|\widetilde{g}(x,u)| \le \alpha(x) \in L^1(0,\pi), \tag{3}$$

$$0 \le \gamma(x,u) \le 1 + \Gamma(x) + \delta \tag{4}$$

for a.e. $x \in (0,\pi)$ and every $u \in \mathbb{R}$.

Let $X = C[0,\pi],\ Z = L^1(0,\pi)$, dom$L = \{u \in X\colon\ u'$ is absolutely continuous in $[0,\pi]\}$,

$$L : \text{dom}L \subset X \longrightarrow Z,\ u \longmapsto u''$$

$$N : X \longrightarrow Z,\ u \longmapsto \gamma(\cdot,u)u + \widetilde{g}(\cdot,u) - h$$

$$A : X \longrightarrow Z,\ u \longmapsto (\Gamma(\cdot)+\delta+1)u.$$

It is standard to check that N and A are L-compact in bounded subsets of X, and that A is a linear Fredholm operator of index zero. In order to apply Theorem iv.5 in [8]. It suffices to prove the solutions of $Lu+(1-\lambda)Au+\lambda Nu = 0$ are bounded uniformly in $\lambda \in [0,1]$ in X. We shall prove by contradiction. Suppose that there exist a sequence $\{u_n\}$ in dom L and a corresponding sequence $\{\lambda_n\}$ in $[0,1]$ such that $\|u_n\|_\infty \to \infty$ and

$$u_n''+(1-\lambda_n)(\Gamma(x)+\delta+1)u_n+\lambda_n\gamma(x,u_n)u_n+\lambda_n\widetilde{g}(x,u_n) = \lambda_n h(x) \quad \text{in} \quad (0,\pi),$$

$$u_n(0) = u_n(\pi) = 0.$$

Let $v_n(x) = u_n(x)/\|u_n\|_\infty$. Dividing the above equation by $\|u_n\|_\infty$, we have

$$v_n'' + p_n(x)v_n = h_n(x) \quad \text{in} \quad (0,\pi), \quad v_n(0) = v_n(\pi) = 0, \tag{5}$$

where

$$p_n(x) = (1-\lambda_n)(\Gamma(x)+\delta+1) + \lambda_n\gamma(x,u_n),$$

$$h_n(x) = (\lambda_n h(x) - \lambda_n\widetilde{g}(x,u_n))/\|u_n\|_\infty.$$

Hence, we can easily obtain $\alpha(x) \in L^1(0,\pi)$ such that $\mid v_n''(x) \mid\ \le \alpha(x)$ for a.e. $x \in (0,\pi)$. From $v_n(0) = v_n(\pi) = 0$, one can choose $\xi_n \in (0,\pi)$ satisfying $v_n'(\xi_n) = 0$. So,

$$\mid v_n'(x) \mid\ \le\ \mid \int_{\xi_n}^{x} v''(s)ds \mid\ \le\ \int_0^\pi \alpha(s)ds. \tag{6}$$

Thus $\{v_n\}$ is a bounded subset of equicontinuous functions in $C[0,\pi]$. Hence by the Arzela-Ascoli theorem, without loss of generality, there is a $v \in C[0,\pi]$ such that $v_n \to v$ in $C[0,\pi]$ (for a subsequence, all convergent series in the following are understood like this and the $n \to \infty$ is also omitted). Furthermore, it follows from (5), (6) that $\{v_n'\}$ is also compact in $C[0,\pi]$. Hence $v_n \to v$ in $C^1[0,\pi]$.

On the other hand,

$$0 \le p_n(x) \le 1 + \Gamma(x) + \delta \in L^1(0,\pi).$$

Hence from the Dunford-Pettis theorem and Mazur theorem we assume that $\{p_n\}$ converges weakly to p in $L^1(0,\pi)$ and $0 \le p(x) \le 1 + \Gamma(x) + \delta$.

Now, multiplying (5) by $\phi \in C_0^\infty(0,\pi)$ and letting $n \to \infty$, yield that $v(x)$ is a weak solution of the following problem

$$v'' + p(x)v = 0 \quad \text{in} \quad (0,\pi), \quad v(0) = v(\pi) = 0.$$

Since $\Gamma(x)+\delta$ satisfies condition (H1), we conclude that $v = 0$. It is a contradiction to the fact $\|v_n\|_\infty = 1$. Hence the proof is completed.

If $a(t),\ b(t)$ are measurable functions in $(0,\pi)$, $a(t) \le b(t)$ and the strict inequality holds on a subset of $(0,\pi)$ with positive measure, then we denote $a(t) \preceq b(t)$ or $b(t) \succeq a(t)$.

Now, we give some remarks on the condition (H1).

Remark 1. If $a(t) \preceq 0 = \lambda_1 - 1$, where $\lambda_n = n^2 (n = 1, 2, \cdots)$ are the eigenvalues of $-u'' = \lambda u$ together with the boundary condition $u(0) = u(\pi) = 0$, $a(t)$ satisfies condition (H1). This is well-known in the literature, see [7].

Remark 2. We claim without proof that:

$$\max_{x\in[0,\pi]} |u(x)| \le \frac{\sqrt{\pi}}{2} \|u'\|_{L^2}$$

for $u \in C_0^1[0,\pi]$. Multiplying the equation (2) by u' integrating over $(0,\pi)$ and noticing the above inequality yield condition (H_1) provided $a(x)$ satisfies the Lyapunov type inequality $\|1 + a(x)\|_{L^1} < 4/\pi$. Hence functions $a(x) \in L^1(0,\pi)$ satisfying the Lyapunov type inequality have the property (H1). This fact can be used to deal with the resonance problem (1) which may cross infinitely many eigenvalues.

Remark 3. Assume that $\Gamma(x) \in L^1(0,\pi)$ and the boundary value problem

$$u'' + u + \Gamma(x)u = 0 \quad \text{in} \quad (0,\pi), \quad u(0) = u(\pi) = 0$$

has a positive solution in $(0,\pi)$. Then any measurable function $a(x) \in L^1(0,\pi)$ with $a(x) \preceq \Gamma(x)$ satisfies the condition (H1). Particularly, $\Gamma(x) \equiv 0$ is such a function. In fact, we first note that in this case $\lambda_1 = 1$ is the least eigenvalue of the problem

$$u'' + \Gamma(x)u = -\lambda u \quad \text{in} \quad (0,\pi), \quad u(0) = u(\pi) = 0,$$

and the assertion follows immediately from the variational characterization of λ_1.

Remark 4. Theorem 1 is quite similar to Theorem 2.2 in [3] which can be applied to more general nonlinearities $g(x,u)$ than (C1). But we use a weaker condition (H1) than the corresponding condition (ii) in Theorem 3.1 there, at the expense of an extra condition (C2). By Theorem 3.1, it seems that the condition (H1) can not guarantee the positive definiteness of the quadratic form in (B1) in [3] and hence Theorem 1 is not a direct corollary of Theorem 2.2 there and the proof there can not be directly applied to our Theorem 1.

3. Existence Theorem Under Condition (H2)

In this section, we let $\Gamma(x)$ in (C1) be nonnegative. We introduce the following condition.

(H2) The boundary value problem (2) has no sign-changing solution for all $a(x) \in L^1(0,\pi)$ satisfying $0 \le a(x) \le \Gamma(x)$.

Theorem 2 *Let $\Gamma(x) \in L^1(0,\pi)$, $\Gamma(x) \ge 0$ for a.e. $x \in (0,\pi)$, satisfy conditions (C1) and (H2). Let $g : (0,\pi) \times \mathbb{R} \to \mathbb{R}$ be a given Caratheodory function and there exists $\rho > 0$ such that $g(x,u(x)) \ge 0$ for all $u(x) \in C_0^1[0,\pi]$ with $u(x) \ge \rho \sin x$ and $g(x,u(x)) \le 0$ for all $u(x) \in C_0^1[0,\pi]$ with $u(x) \le -\rho \sin x$. Then for each $h(x) \in L^1(0,\pi)$ with*

$$\int_0^\pi h(x) \sin x dx = 0$$

the boundary value problem (1) has at least one solution $u(x) \in C[0,\pi]$.

Proof. In order to apply Theorem iv.5 in [8], we need to prove

$$u'' + u + \epsilon u + \Gamma(x)u = 0 \quad \text{in} \quad (0,\pi), \quad u(0) = u(\pi) = 0$$

has only the trivial solution for ϵ sufficiently small. If the boundary value problem (2) with $a(x) = \Gamma(x)$ has only the trivial solution, as we have pointed out in Theorem 1, it is true for small positive ϵ. So we can suppose that it has one-sign solutions on $(0,\pi)$. If the above assertion were not true, we can find a sequence of nontrivial solutions $\{u_n\}$ for

$$u_n'' + u_n + \epsilon_n u_n + \Gamma(x)u_n = 0 \quad \text{in} \quad (0,\pi), \quad u_n(0) = u_n(\pi) = 0,$$

where $\epsilon_n > 0$ and $\epsilon_n \to 0$. First multiplying the above equation by $\sin x$ and integrating over $[0, \pi]$, yield that $u_n(x)(n = 1, 2, \cdots)$ must be sign-changing solutions. Define $v_n(x) = u_n(x)/\|u_n\|_\infty$. Then $v_n(x)(n = 1, 2, \cdots)$ are solutions of

$$v_n'' + v_n + \epsilon_n v_n + \Gamma(x) v_n = 0 \quad \text{in} \quad (0, \pi), \quad v_n(0) = v_n(\pi) = 0.$$

As in the proof of Theorem 1, we assume that $v_n \to v_0$ in $C_0^1[0, \pi]$, where v_0 is a solution of the following problem

$$v_0'' + v_0 + \Gamma(x) v_0 = 0 \quad \text{in} \quad (0, \pi), \quad v_0(0) = v_0(\pi) = 0.$$

By condition (H2), $v_0(x)$ is not sign-changing in $(0, \pi)$. Without loss of generality, we may suppose that $v_0(x) > 0$ on $(0, \pi)$. Also, by the basic existence result about initial value problems, we have $v_0'(0) \neq 0$ and $v_0'(1) \neq 0$. Hence v_0 is in the interior of the usual positive cone in $C_0^1[0, \pi]$. This is a contradiction, since $v_n(x)$ is sign-changing and $v_n \to v_0$ in $C_0^1[0, \pi]$.

For the $\Gamma(x)$ in this theorem, the same argument can be used to prove that for $\delta = \delta(\Gamma) > 0$ sufficiently small, $\Gamma(x) + \delta$ still satisfies condition (H2). We also, as in Theorem 1, decompose $g(x, u) = \gamma(x, u)u + \widetilde{g}(x, u)$, where $\gamma(x, u)$ and $\widetilde{g}(x, u)$ are Caratheodory functions satisfying (3) and (4).

Let $X = C[0, \pi]$, $Z = L^1(0, \pi)$, dom L=$\{u \in X\colon\ u'$ is absolutely continuous in $[0, \pi]\}$,

$$L : \text{dom}L \subset X \longrightarrow Z,\ u \longmapsto u'' + u$$

$$N : X \longrightarrow Z,\ u \longmapsto \gamma(\cdot, u)u + \widetilde{g}(\cdot, u) - h$$

$$A : X \longrightarrow Z,\ u \longmapsto (\Gamma(\cdot) + \delta)u.$$

It is routine to check that N and A are L-compact in bounded subsets of X and A is a linear Fredholm operator of index zero. In order to apply Theorem iv.5 in [8], we first prove that all solutions of $Lu + \lambda Au + (1 - \lambda)Nu = 0$ are bounded in X, uniformly in $\lambda \in [0, 1]$. Otherwise, there exist a sequence $\{\lambda_n\}$ in $[0, 1]$ and solutions $\{u_n\}$ satisfying $\|u_n\|_\infty \to \infty$ and

$$\text{(7)} \quad u_n'' + u_n + \lambda_n \delta u_n + \lambda_n \Gamma(x) u_n + (1 - \lambda_n) g(x, u_n) = (1 - \lambda_n) h(x) \quad \text{in} \quad (0, \pi)$$

$$u_n(0) = u_n(\pi) = 0.$$

Let $v_n(x) = u_n(x)/\|u_n\|_\infty$. As in the proof of Theorem 1, we can prove that $\{v_n\}$ is compact in $C_0^1[0, \pi]$ and $v(x)$ is a weak solution of the equation

$$\text{(8)} \quad v'' + v + \overline{\lambda}\delta v + \overline{\lambda}\Gamma(x) v + (1 - \overline{\lambda}) z(x) v = 0 \quad \text{in} \quad (0, \pi), \quad v(0) = v(\pi) = 0,$$

where $v_n \to v$ in $C_0^1[0, \pi]$ and $\lambda_n \to \overline{\lambda}$ and $0 \leq z(x) \leq \Gamma(x) + \delta$ for some $z(x) \in L^1(0, \pi)$. By the preceding argument, if $\delta > 0$ is sufficiently small, (8) has no sign-changing solutions. Since $v(x) \not\equiv 0$, without loss of generality we assume that $v(x)$ is positive in $(0,\pi)$. Multiplying (2) by $\sin x$ and integrating over $[0,\pi]$, yield that $v(x) = \sin x$. Obviously, $\sin x$ is in the interior of the usual positive cone in $C_0^1[0,\pi]$. Hence $u_n(x) > 0$ in $(0,\pi)$ for n sufficiently large. Set $\widetilde{v}_n(x) = \widetilde{u}_n(x)/\|u_n\|_\infty$. We now prove that $\widetilde{v}_n \to 0$ in $C_0^1[0,\pi]$. In fact, since $\{u_n\}$ are solutions of (8), $\{\widetilde{v}_n\}$ satisfy

$$\widetilde{v}_n'' + \widetilde{v}_n = \beta_n(x,u) \quad \text{in} \quad (0,\pi), \quad \widetilde{v}_n(0) = \widetilde{v}_n(\pi) = 0,$$

$$\int_0^\pi \widetilde{v}_n(x) \sin x dx = 0,$$

where $|\ \beta_n(x,u)\ | \ \leq \beta_0(x) \in L^1(0,\pi)$, $\beta_n(x,u) = h(x)(\|u_n\|_\infty)^{-1} - \lambda_n \delta v_n - \lambda_n \Gamma(x) v_n - (1-\lambda_n)\gamma(x,u_n(x))v_n(x) - (1-\lambda_n)\widetilde{g}(x,u_n(x))(\|u_n\|_\infty)^{-1}$. Hence $\{\widetilde{v}_n\}$ is uniformly bounded in $[0,\pi]$. Furthermore, $|\ (\widetilde{v}_n)''(x))\ | \ \leq \beta(x)$ for some $\beta(x) \in L^1(0,\pi)$. As in the proof of Theorem 1, we can prove that $\{\widetilde{v}_n\}$ is compact in $C_0^1[0,\pi]$. Since $v_n \to \sin(\cdot)$ in $C_0^1[0,\pi]$, we may assume that $\widetilde{v}_n \to 0$ in $C_0^1[0,\pi]$. Hence if we set $v_n^0(x) = u_n^0(x)/\|u_n\|_\infty = k_n \sin x$, then $k_n \to 1$. Therefore

$$\begin{aligned} u_n(x) &= \|u_n\|_\infty (u_n^0(x)/\|u_n\|_\infty + \widetilde{u}_n(x)/\|u_n\|_\infty) \\ &\geq \|u_n\|_\infty (\tfrac{1}{2}\sin x - \tfrac{1}{4}\sin x) \geq \rho \sin x \end{aligned} \tag{9}$$

for $n > N$ (N is only dependent on ρ), where we have used the known inequality

$$|\ u(x)\ | \ \leq \frac{\pi}{2} \ |\ u'(x)\ |\ \sin x$$

for all $u(x) \in C_0^1[0,\pi]$ and the boundedness of $\{\widetilde{v}_n\}$ in $C_0^1[0,\pi]$.

Finally, from the equation (7) we have

$$\int_0^\pi [(\lambda_n \delta + \lambda_n \Gamma(x))u_n + (1-\lambda_n)g(x,u_n)] \sin x dx = 0.$$

This is a contradiction, since $g(x,u_n(x))\sin x \geq 0$, $\Gamma(x) + \delta \geq \delta$ and $u_n(x)$ is positive in $(0,\pi)$ for n sufficiently large. This completes the proof.

Now, we give some remarks on the condition (H2).

Remark 5. It is known in the literature that a measurable function $a(x)$ such that $0 \leq a(x) \preceq \Gamma(x) \equiv 3 = \lambda_2 - \lambda_1$ satisfies condition (H2).

Remark 6. If

$$u'' + u + \Gamma(x)u = 0 \quad \text{in} \quad (0,\pi), \quad u(0) = u(\pi) = 0$$

has a solution $u(x)$ possessing only one zero in $(0,\pi)$, where $\Gamma(x) \in L^\infty(0,\pi)$, then any $a(x) \in L^\infty(\Omega)$ with $a(x) \preceq \Gamma(x)$ satisfies the property (H2). In fact, by choosing A such that $\Gamma(x) - A + 1 < 0$ and considering the eigenvalue problem

$$u'' + u + (\Gamma(x) - A)u = -\lambda u - Au \quad \text{in} \quad (0,\pi), \quad u(0) = u(\pi) = 0,$$

the assertion follows from the Sturm-Liouville theory and variational characterization of the eigenvalues. Obviously, $\Gamma(x) \equiv 3 = \lambda_2 - \lambda_1$ is such a particular function.

Remark 7. As is proved in [1], $\Gamma(x) \in L^1(\Omega)$ with $\|\Gamma\|_{L^1} \leq 4$ satisfies the condition (H2).

4. Existence Theorem for Nontrivial Solutions

In this section, we investigate the existence of nontrivial solutions to (1). We need the following result about the computation of coincidence degree proved in [6] (see also [2]). For the terminology and more information about the degree, see [8].

Theorem 3. *Let $X,\ Z$ be Banach spaces and X be infinite dimensional, and let $L : domL \subset X \longrightarrow Z$ be a linear Fredholm operator of index 0. Furthermore we let $\Omega \subset X$ be a bounded open subset, N and N_1: $\overline{\Omega} \longrightarrow Z$ be two L-compact operators. If we assume the following two conditions*

(i) $Lu - Nu \neq \lambda N_1 u$ *for all* $\lambda \geq 0,\ u \in \mathrm{dom}L \cap \partial\Omega$,

(ii) $\inf_{u\in\partial\Omega} \|K_{P,Q}N_1u + JQN_1u\| > 0$,

then the coincidence degree $D((L,N),\Omega) = 0$.

Theorem 4. *Assume* ($C1$), ($H2$) *and the following condition*

($C3$) *There exist* $R > r > 0$ *and* $a > 0$ *such that*

(i) $g(x,u) \geq 0$ *for* $u \geq -r$, *a.e.* $x \in \Omega$,

(ii) $g(x,u) \leq -a$ *for* $u \leq -R$, *a.e.* $x \in \Omega$.

Then for every $h(x) \in L^1(0,\pi)$ *with* $\int_0^\pi h(x)\sin x dx = 0$, ($1$) *has at least one nontrivial solution in* $C[0,\pi]$.

Proof. Let $\delta > 0$ be a constant such that $\Gamma(x) + \delta$ satisfies the condition (H2). We shall use the decomposition $g(x,u) = \gamma(x,u)u + \widetilde{g}(x,u)$ as in Theorem 2.

Let the Banach spaces $X,\ Z$ and the operators $L,\ N$ and A be defined as in the proof of Theorem 2. It is immediate that

$$\mathrm{Ker}L = \{u \in X : u = a\sin(\cdot), a \in R\},$$

$$\mathrm{Im}L = \{u \in Z : \int_0^\pi u(x)\sin(x)dx = 0\}.$$

We define

$$P : X \to X,\ Pu(\cdot) = \frac{1}{\pi}(\int_0^\pi u(x)\sin(x)dx)\sin(\cdot)$$

$$Q : Z \to Z,\ Qu(\cdot) = \frac{1}{\pi}(\int_0^\pi u(x)\sin(x)dx)\sin(\cdot).$$

Hence $K_P : \mathrm{Im}L \to \mathrm{dom}L \cap \mathrm{Ker}P$ (the inverse of L restricted to $\mathrm{dom}L \cap \mathrm{Ker}P$) is given by

$$(K_P z)(x) = \int_0^x \sin(x-s)z(s)ds - \frac{1}{\pi}(\int_0^\pi (\pi - s)\cos s + \sin s)z(s)ds)\sin x.$$

We recall that $K_{P,Q} = K_P(I - Q)$. It follows that $K_P : \mathrm{Im}L \to \mathrm{dom}L \cap \mathrm{Ker}P$ is completely continuous and hence $A,\ N : X \to Z$ are L-compact.

We first prove that the solutions of $Lu + \lambda Au + (1-\lambda)Nu = 0$ are bounded in X uniformly in $\lambda \in [0,1]$. Otherwise, there exists a sequence $\{\lambda_n\}$ in $[0,1]$ and solutions $\{u_n\}$ satisfying $\|u_n\|_\infty \to \infty$ and

$$u'' + u + \lambda\delta u + \lambda\Gamma(x)u + (1-\lambda)g(x,u) = (1-\lambda)h(x) \quad \text{in} \quad (0,\pi),$$
$$u(0) = u(\pi) = 0. \tag{10}$$

Let $v_n(x) = u_n(x)/\|u_n\|_\infty$. Similar to the proof of Theorem 1, we can prove that $\{v_n\}$ is compact in $C_0^1[0,\pi]$ and $v(x)$ is a weak solution of the equation

$$v'' + v + \overline{\lambda}\delta v + \overline{\lambda}\Gamma(x)v + (1-\overline{\lambda})z(x)v = 0 \quad \text{in} \quad (0,\pi),$$
$$v(0) = v(\pi) = 0, \tag{11}$$

where $v_n \to v$ in $C_0^1[0,\pi]$ and $\lambda_n \to \overline{\lambda}$ and $0 \le z(x) \le \Gamma(x) + \delta$ for some $z(x) \in L^1(0,\pi)$. By the preceding argument, if $\delta > 0$ is sufficiently small, (11) has no sign-changing solutions. If we suppose that $v(x)$ is positive in $(0,\pi)$, we will obtain a contradiction following the proof of Theorem 2. Hence we may assume that $v(x)$ is negative in $(0,\pi)$. Also, following the proof of (9), we obtain

$$u_n(x) = \|u_n\|_\infty(u_n^0(x)/\|u_n\|_\infty + \widetilde{u}_n(x)/\|u_n\|_\infty)$$

$$\le \|u_n\|_\infty(-\frac{1}{2}\sin x + \frac{1}{4}\sin x) \le -\frac{1}{2}\|u_n\|_\infty \sin x, \quad \text{for } n \text{ suffiently large.}$$

Set $I'_n = \{x \in [0,\pi] : u_n(x) \le -R\}$, $I''_n = [0,\pi] \setminus I'_n = \{x \in [0,\pi] : -R \le u_n(x) \le 0\} \subset \{x \in [0,\pi] : \sin x \le \frac{2R}{\|u_n\|_\infty}\}$. Multiplying the equation (10) by $\sin x$, we have

$$\left(\int_{I_{n'}} + \int_{I_{n''}}\right)[(\lambda_n\delta + \lambda_n\Gamma(x))u_n + (1-\lambda_n)g(x,u_n)]\sin x dx = 0.$$

Noticing that the integrand can be estimated by a function $\alpha(x) \in L^1(0,\pi)$ and measure $(I_n'') \to 0$, we have

$$\int_{I_n''}[(\lambda_n\delta + \lambda_n\Gamma(x))u_n + (1-\lambda_n)g(x,u_n)]\sin x dx \to 0.$$

On other hand, by the condition (C3) of the Theorem, it follows that

$$\begin{aligned}&\int_{I_{n'}}[(\lambda_n\delta + \lambda_n\Gamma(x))u_n + (1-\lambda_n)g(x,u_n)]\sin x dx\\ \le\ &\int_{I_{n'}}[-R\lambda_n(\delta+\Gamma(x)) + (1-\lambda_n)(-a)]\sin x dx\\ \to &-\int_0^\pi [R\overline{\lambda}(\delta+\Gamma(x)) + (1-\overline{\lambda})a]\sin x dx < 0.\end{aligned}$$

This is a contradiction. Hence we can choose $R_0 > r$ sufficiently large such that $|D((L,-N), B_{R_0})| = 1$.

Without loss of generality, we assume that $Lu + Nu \neq 0, \forall u \in \partial B_r$. Now we prove

$$Lu + Nu \neq \lambda(-\frac{\pi}{4}),\ \forall \lambda > 0, u \in \partial B_r.$$

In fact if there are $\lambda_0 > 0$ and $u_0 \in \partial B_r$ such that

$$Lu_0 + Nu_0 = \lambda_0(-\frac{\pi}{4}),$$

taking the inner product with $\sin x$ in $L^2(0,\pi)$, we have

$$\int_0^\pi g(x,u_0(x))\sin x dx + \lambda_0\frac{\pi}{2} = 0.$$

Hence, by condition (C3), we get $\lambda_0 \le 0$. Therefore, the condition (i) in Theorem 3 is satisfied with $N_1 = -\frac{\pi}{4}$. If we notice $JQ(-\frac{\pi}{4}) = -\sin x$ and $K_{P,Q}(-\frac{\pi}{4}) \in \{u : \int_0^\pi u(x)\sin(x)dx = 0\}$, we obtain $JQ(-\frac{\pi}{4}) + K_{P,Q}(-\frac{\pi}{4}) \neq \theta$. By Theorem 3, we have $D((L,-N), B_r) = 0$. Hence by the additivity of the coincidence degree, equation (1) has at least one nontrivial solution in dom$L \cap (B_{R_0} \backslash \overline{B}_r)$.

Acknowledgement

The author thanks the support of Academy of Mathematics and System Sciences, Chinese Academy of Sciences and the support of CSC for studying at TU Wien, Austria where the work was done. The author also would like to thank the anonymous referee for pointing out to me the reference [3] which is closely related to this paper.

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