TAIWANESE JOURNAL OF MATHEMATICS
Vol. 8, No. 3, pp. 477-487, September 2004
This paper is available online at http://www.math.nthu.edu.tw/tjm/

VECTOR SUPERIOR AND INFERIOR

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Abstract. Let $(\mathcal{Z}\,,\,\mathcal{C})$ be an ordered Hausdorff real topological vector space. Some conditions for assuring that a nonempty set $K\subset\mathcal{Z}$ has a nonempty superior or inferior are established. Ordering-conically compact ordered Hausdorff real topological vector spaces are introduced so that in such a space every nonempty bounded below (respectively, bounded above) set has a nonempty inferior (respectively, superior).

1. INTRODUCTION

In this paper, all topological vector spaces are assumed to be real topological vector spaces. Let $\mathcal{Z}$ be a topological vector space. A cone C in $\mathcal{Z}$ is called proper if it is not the whole space $\mathcal{Z}$. Note that a closed convex cone $C\subset\mathcal{Z}$ is proper if and only if Int$C$ does not contain the zero vector, where Int$C$ denotes the interior of C.

Let C be a proper closed convex cone in $\mathcal{Z}$ with Int$C\neq\emptyset$. Then C induces a reflexive and transitive order $\preceq_C$ on $\mathcal{Z}$ defined by $x\preceq_C y$ whenever $x,\,y\in\mathcal{Z}$ with $y-x\in C$. When there is no confusion, we shall simply write $x\preceq y$ or $y\succeq x$ if $x\preceq_C y$. We shall also write $x\prec y$ or $y\succ x$ whenever $y-x\in$ Int$C$. Let $(\mathcal{Z}\,,\,C)$ denote the space $\mathcal{Z}$ together with the order induced by C, called an ordered topological vector space. The cone C is called an ordering cone in $\mathcal{Z}$.

Let X be a Hausdorff topological space, and let K be a nonempty subset of X. A real valued bifunction f on $K\times K$ is called topologically pseudomonotone [2, p. 410] if for any net $\{x_\alpha\}$ staying in a compact subset of K and converging to $\widehat{x}$ with

$$\liminf_{\alpha} f(x_\alpha\,,\,\widehat{x})\geq 0\,,$$

its limit $\widehat{x}$ satisfies

Received February 3, 2002; resived June 12, 2003.
Communicated by Mau-Hsiang Shih.
2000 *Mathematics Subject Classification*: 49J53.
Key words and phrases: Ordered real topological vector spaces, vector superior and inferior. .
The work was partially supported by grants from the National Science Council of the Republic of China.

$$\limsup_{\alpha} f(x_\alpha\,,\, y) \le f(\widehat{x}\,,\, y) \quad \text{for all } y \in K.$$

The notion of topological pseudomonotonicity has been generalized in [3] to bifunctions f of $K \times K$ into an ordered topological vector space.

In a very recent paper, Chadli, Schaible and Yao established an existence result for regularized equilibrium problems with the corresponding real functions topologically pseudomonotone. See Theorem 3.1 in [4]. The motivation for this work was an attempt to prove a vector version of the theorem. Topological pseudomonotonicity, in both scalar and vector cases, is essentially based on the notions of superior and inferior. This paper is devoted to studying vector superior and inferior. The work for properties of vector topologically pseudomonotone functions will appear elsewhere.

Let K be a subset of an ordered topological vector space $(\mathcal{Z}\,,\, C)$. Several kinds of vector superior Sup (K, C) and vector inferior Inf (K, C) of K with respect to C have been defined in literature. See [1, 6, 7, 10, 11] and references therein.

In [1, 6, 10, 11], Sup (K, C) and Inf (K, C) are defined to be subsets of the closure $\overline{K}$ of K. While in [7], Sup (K, C) and Inf (K, C) could be disjoint from $\overline{K}$ [7, Example 1.7]. The definitions of Sup (K, C) and Inf (K, C) given in [1, 6, 11] are essentially the same. In [10], the points of vector inferior of K are called strictly C-infimal points of K, and the set of strictly C-infimal points of K is written by Inf$_s(K, C)$.

In this paper, we shall consider vector superiors and inferiors defined in [1]. From now on, Sup (K, C) and Inf (K, C) are respectively used for the vector superior and inferior of K defined in [1]. In [8], elements of the inferior of K are called *weakly efficient points* of $\overline{K}$ with respect to C. See [8] Chapter 2, Proposition 2.3. We shall prove in Remark 2.1 that Inf$_s(K, C) \subset$ Inf (K, C) and Inf$_s(K, C)$ may not coincide with Inf (K, C).

Let K be a subset of an ordered topological vector space $(\mathcal{Z}\,,\, C)$.

(i) K is called *bounded below with respect to* C if there is a $z \in \mathcal{Z}$ such that $K \subset z + C$.

(ii) K is called *bounded above with respect to* C if there is a $z \in \mathcal{Z}$ such that $K \subset z - C$.

If K is nonempty, then K is bounded below with respect to C if and only if there is a $z \in \mathcal{Z}$ such that $z \preceq x$ for all $x \in K$, and K is bounded above with respect to C if and only if there is a $z \in \mathcal{Z}$ such that $x \preceq z$ for all $x \in K$.

It is well known that any nonempty bounded below (respectively, bounded above) subset of $\mathbb{R}$ has an infimum (respectively, supremum), where $\mathbb{R}$ is the set of all real numbers. The main work of the paper is to find conditions on a nonempty bounded above (respectively, bounded below) subset K of an ordered Hausdorff topological vector space for assuring that K has a nonempty superior (respectively, inferior).

The rest of the paper is organized as follows. In Section 2, we recall the definitions of vector superior and inferior, and establish some preliminary results. In Section 3, we state the main theorems (Theorems 3.3, 3.4 and 3.7). Also, in Section 3, we define a family of ordered Hausdorff topological vector spaces, called ordering-conically compact, so that in such a space every nonempty bounded below (respectively, bounded above) set has a nonempty inferior (respectively, superior), (see Theorem 3.7). The proofs of the main theorems are given in Section 4.

In the sequel, for any subset A of a topological space X, we shall denote by A^c the complement of A in X, $\overline{A}$ the closure of A in X, Int$A$ the interior of A in X, and ∂A the boundary of A.

2. Vector Superior and Inferior

Throughout this section, let $(\mathcal{Z}\,,\,C)$ denote an ordered topological vector space. For a subset K of $\mathcal{Z}$, the superior of K with respect to C is defined by

$$\text{Sup}\,(K\,,\,C) = \{x \in \overline{K}\,:\,(x + \text{Int}C) \cap K = \emptyset\}\,,$$

and the inferior of K with respect to C is defined by

$$\text{Inf}\,(K\,,\,C) = \{x \in \overline{K}\,:\,(x - \text{Int}C) \cap K = \emptyset\}\,.$$

Remark 2.1. For any $K \subset \mathcal{Z}$, according to Tanaka, a point $x \in \mathcal{Z}$ is a strictly C-infimal point of K if and only if $x \in \overline{K}$ and $(x - C) \cap \overline{K} = \{x\}$. Since x is not in $x - \text{Int}C$, we have $x \in \text{Inf}\,(K, C)$ if x is a strictly C-infimal point of K. Therefore, $\text{Inf}_s(K, C) \subset \text{Inf}\,(K, C)$.

The following example illustrates that $\text{Inf}_s(K, C)$ may not equal to $\text{Inf}\,(K, C)$. Let $\mathcal{Z} = \mathbb{R}^2$, let $C = \{(x, y) \in \mathbb{R}^2 : x \geq 0 \text{ and } y \geq 0\}$ and $K = \{(x, y) \in \mathbb{R}^2 : x > 0 \text{ or } y > 0\}$. Note that the origin lies in $\text{Inf}\,(K, C)$ but not in $\text{Inf}_s(K, C)$.

In the rest of this section, we shall consider the superior and inferior of subsets of $\mathcal{Z}$ with respect to a fixed ordering cone C. Therefore, we shall simply write

$$\text{Sup}\,(K\,,\,C) = \text{Sup}\,K \quad \text{and} \quad \text{Inf}\,(K\,,\,C) = \text{Inf}\,K\,.$$

Remark 2.2. For any subset K of $\mathcal{Z}$, one proves easily that

(i) $\text{Inf}\,(-K) = -\text{Sup}\,K$ and $\text{Sup}\,(-K) = -\text{Inf}\,K$;

(ii) $\text{Inf}\,K = \text{Inf}\,\overline{K}$ and $\text{Sup}\,K = \text{Sup}\,\overline{K}$.

For a proof of (ii), see [3, Proposition 2.1].

Proposition 2.3. *Assume that $\mathcal{Z}$ is Hausdorff. If $K \subset \mathcal{Z}$, then* Sup K *and* Inf K *are closed.*

Proof. By Remark 2.2 (i), it suffices to show that Inf K is closed. Assume that Inf K is nonempty. Let x be any point in the closure of Inf K, and let $\{x_\alpha\}$ be a net in Inf K converging to x. Note that $x \in \overline{K}$ since $x_\alpha \in \overline{K}$ for all α. To complete the proof, we have to show that $(x - \text{Int}C) \cap K = \emptyset$.

Suppose on the contrary that there is a point $v \in \text{Int}C$ such that $x - v \in K$. Since $\lim_\alpha (x - v - x_\alpha) = -v \in -\text{Int}C$, there is an α such that $x - v - x_\alpha \in -\text{Int}C$. Thus

$$x - v = x_\alpha + (x - v - x_\alpha) \in (x_\alpha - \text{Int}C) \cap K\,.$$

This is a contradiction to the definition of x_α. Hence, $(x - \text{Int}C) \cap K = \emptyset$.

In the following proposition, we give a necessary condition for a point lying in the inferior or superior of a nonempty subset of an ordered Hausdorff topological vector space.

Proposition 2.4. *Assume that $\mathcal{Z}$ is Hausdorff, and let K be a nonempty subset of $\mathcal{Z}$.*

(i) *If $x \in$ Inf K, and if $v \in$ Int$C$, then $(x + v - \text{Int}C) \cap K \neq \emptyset$, i.e., there is a $x_v \in K$ such that $x_v \prec x + v$.*

(ii) *If $x \in$ Sup K, and if $v \in$ Int$C$, then $(x - v + \text{Int}C) \cap K \neq \emptyset$, i.e., there is a $x_v \in K$ such that $x_v \succ x - v$.*

Proof. We shall prove (i). By Remark 2.2 (i), the statement (ii) will follow. Note that $x + v - \text{Int}C$ is open in $\mathcal{Z}$, and that

$$x = x + v - v \in (x + v - \text{Int}C)\,.$$

Since $x \in \overline{K}$, there is a net $\{x_\alpha\}$ in K converging to x. Thus, $x + v - \text{Int}C$ must contain some x_α. This completes the proof.

The following theorem is important to our later discussion.

Theorem 2.5. *Let $\mathcal{Z}$ be Hausdorff, and let K be a nonempty subset of $\mathcal{Z}$. If $x \in K$, then*

$$\text{Inf}\,(\overline{K} \cap (x - C)) \subset \text{Inf}\,K \quad \text{and} \quad \text{Sup}\,(\overline{K} \cap (x + C)) \subset \text{Sup}\,K\,.$$

Proof. For every $x \in K$, let $L_K(x) = \overline{K} \cap (x - C)$ and $U_K(x) = \overline{K} \cap (x + C)$. Note that $L_K(x)$ and $U_K(x)$ are closed and satisfy :

$$x \in L_K(x) \cap U_K(x) \quad \text{and} \quad L_K(x) \cup U_K(x) \subset \overline{K}\,.$$

We shall prove Inf $L_K(x) \subset$ Inf K. By a similar argument, one proves Sup $U_K(x) \subset$ Sup K.

There is nothing to prove if Inf $L_K(x) = \emptyset$. Let $z \in$ Inf $L_K(x)$ be arbitrary. By definition, $z \in L_K(x) \subset \overline{K}$ and $(z - \text{Int}C) \cap L_K(x) = \emptyset$. We claim that

$$(z - \text{Int}C) \cap \overline{K} = \emptyset\,.$$

This implies $z \in$ Inf $\overline{K} =$ Inf K, and completes the proof.

Suppose on the contrary that there is a $v_1 \in$ Int$C$ such that $y = z - v_1 \in \overline{K}$. Since $z \in L_K(x) \subset x - C$, then $z = x - v_2$ for some $v_2 \in C$, and

$$y = x - (v_1 + v_2) \in x - C\,.$$

Thus, $y \in L_K(x) \cap (z - \text{Int}C)$. This is a contradiction.

3. The Main Theorems

To state our main results, we first relate an ordered Hausdorff topological vector space $(\mathcal{Z}\,,\,C)$ to a continuous (real) linear functional on $\mathcal{Z}$.

From [5] Theorem IV.3.3 and its proof, we obtain :

Proposition 3.1. *If $(\mathcal{Z}\,,\,C)$ is an ordered Hausdorff topological vector space, there is a continuous linear functional $\varphi : \mathcal{Z} \longrightarrow \mathbb{R}$ such that Int$C \subset \{z \in \mathcal{Z} : \varphi(z) > 0\}$. Consequently,*

$$-\text{Int}C \subset \{z \in \mathcal{Z} : \varphi(z) < 0\},\ C \subset \{z \in \mathcal{Z} : \varphi(z) \geq 0\}\ and\ -C \subset \{z \in \mathcal{Z} : \varphi(z) \leq 0\}.$$

Let $C_\varphi = \{z \in \mathcal{Z} : \varphi(z) \geq 0\}$. The set C_φ is a closed half space bounded by the hyperplane $\ker\varphi$. Thus $\partial C_\varphi = \ker\varphi$. Note that $(\mathcal{Z}\,,\,C_\varphi)$ is also an ordered Hausdorff topological vector space.

Proposition 3.2. *Let $(\mathcal{Z}, C)$ be an ordered Hausdorff topological vector space, and let φ and C_φ be given above. Then $C = C_\varphi$ if and only if $(\text{Int}C)^c$ is a convex cone in $\mathcal{Z}$.*

Proof. I f $C = C_\varphi$, then $(\text{Int}C)^c = -C$ is a convex cone. Since $C = C_\varphi$ if and only if

$$\text{Int}C = \{z \in \mathcal{Z} : \varphi(z) > 0\},\ -\text{Int}C = \{z \in \mathcal{Z} : \varphi(z) < 0\}\ and\ \partial C = \ker\varphi = \partial(-C)\,,$$

it remains to show that if $(\text{Int}C)^c$ is a convex cone, then the above assertions hold.

First, we prove that $\mathcal{Z} = C \cup (-C)$. This implies that $\ker\varphi \subset \partial C \cup \partial(-C)$ by Proposition 3.1. Since $(\mathrm{Int}C)^c$ is a proper closed convex cone, we have

$$\mathrm{Int}(\mathrm{Int}C)^c \cap [-\mathrm{Int}(\mathrm{Int}C)^c] = \emptyset\,.$$

By definition, $C^c = \mathrm{Int}(\mathrm{Int}C)^c$ and $-C^c = -\mathrm{Int}(\mathrm{Int}C)^c$. Therefore,

$$(C \cup (-C))^c = C^c \cap (-C^c) = \emptyset \quad \text{and} \quad \mathcal{Z} = C \cup (-C)\,.$$

Next, we prove that $\partial C \subset \ker\varphi$. This implies that $\partial C \cup \partial(-C) = \ker\varphi$, and that

$$\mathrm{Int}C = \{z \in \mathcal{Z} \,:\, \varphi(z) > 0\} \quad \text{and} \quad -\mathrm{Int}C = \{z \in \mathcal{Z} \,:\, \varphi(z) < 0\}\,.$$

Suppose on the contrary that $\varphi(v_0) \neq 0$ for some $v_0 \in \partial C$. If $\varphi(v_0) > 0$, then by the continuity of φ there is a neighborhood U of v_0 such that $\varphi(v) > 0$ for all $v \in U$. Thus $U \cap (-\mathrm{Int}C) = \emptyset$. Since $\mathcal{Z} = C \cup (-C)$, C^c is an open subset of $-C$. Thus

$$C^c = -\mathrm{Int}C \quad \text{and} \quad U \cap C^c = \emptyset\,.$$

This is a contradiction to the definition of v_0. By a similar argument, one is led to a contradiction if $\varphi(v_0) < 0$.

Finally, we have to show that $\ker\varphi \subset \partial C$. Let $v \in \ker\varphi$ be arbitrary, and let U be an arbitrary open neighborhood of v. Since a non-constant linear functional on a Hausdorff topological vector space is an open map [9, (8.3.2), p.153], $\varphi(U)$ is an open neighborhood of $0 \in \mathbb{R}$. There is an $\varepsilon > 0$ such that $(-\varepsilon\,,\,\varepsilon) \subset \varphi(U)$. This implies that

$$U \cap \mathrm{Int}C \neq \emptyset \quad \text{and} \quad U \cap (-\mathrm{Int}C) \neq \emptyset\,.$$

Since $-\mathrm{Int}C = C^c$, we obtain $v \in \partial C$. Therefore, $\ker\varphi \subset \partial C$.

For C and C_φ given above, let $\boldsymbol{\mathcal{N}}$ be the vector in $\mathrm{Int}C_\varphi$ such that $\varphi(\boldsymbol{\mathcal{N}}) = 1$. It is well known that $\mathcal{Z} = \langle\boldsymbol{\mathcal{N}}\rangle \oplus \partial C_\varphi$, where $\langle\boldsymbol{\mathcal{N}}\rangle$ is the vector subspace of $\mathcal{Z}$ spanned by $\boldsymbol{\mathcal{N}}$. Let $\Pi \,:\, \mathcal{Z} \longrightarrow \langle\boldsymbol{\mathcal{N}}\rangle$ and $\Pi_0 \,:\, \mathcal{Z} \longrightarrow \partial C_\varphi$ be the canonical projections.

Now, we state our main theorems as follows.

Theorem 3.3. *Assume that $(\mathcal{Z}\,,\,C)$ is an ordered Hausdorff topological vector space. Let $\varphi \,:\, \mathcal{Z} \longrightarrow \mathbb{R}$ be the linear functional given in Proposition 3.1, and let $K \subset \mathcal{Z}$ be nonempty.*

(i) *If K is bounded below with respect to C, and if there is an $x_0 \in K$ such that the set $\varphi(\overline{K} \cap (x_0 - C))$ is closed in $\mathbb{R}$, then Inf$(K\,,\,C)$ is nonempty.*

(ii) *If* K *is bounded above with respect to* C*, and if there is an* $x_0 \in K$ *such that the set* $\varphi(\overline{K} \cap (x_0 + C))$ *is closed in* $\mathbb{R}$*, then Sup*(K, C) *is nonempty.*

Theorem 3.4. *Let* $(\mathcal{Z}, C)$ *be an ordered Hausdorff topological vector space, let* $\varphi : \mathcal{Z} \longrightarrow \mathbb{R}$ *be the linear functional given in Proposition 3.1, let* Π_0 *be the canonical projection of* $\mathcal{Z}$ *onto* $\ker \varphi$*, and let* $K \subset \mathcal{Z}$ *be nonempty. Assume that* $\mathcal{Z}$ *is locally convex.*

(i) *If* K *is bounded below with respect to* C*, and if there is an* $x_0 \in K$ *such that the set* $\Pi_0(\overline{K} \cap (x_0 - C))$ *is compact, then Inf*$(K, C) \neq \emptyset$.

(ii) *If* K *is bounded above with respect to* C*, and if there is an* $x_0 \in K$ *such that the set* $\Pi_0(\overline{K} \cap (x_0 + C))$ *is compact, then Sup*$(K, C) \neq \emptyset$.

The following two corollaries are immediate consequences of Theorem 3.3.

Corollary 3.5. *Assume that* $(\mathcal{Z}, C)$ *is an ordered Hausdorff topological vector space, and let* $K \subset \mathcal{Z}$ *be nonempty.*

(i) *If* K *is bounded below with respect to* C*, and if there is an* $x_0 \in K$ *such that* $\overline{K} \cap (x_0 - C)$ *is compact, then Inf*(K, C) *is nonempty.*

(ii) *If* K *is bounded above with respect to* C*, and if there is an* $x_0 \in K$ *such that* $\overline{K} \cap (x_0 + C)$ *is compact, then Sup*(K, C) *is nonempty.*

Corollary 3.6. *Assume that* $(\mathcal{Z}, C)$ *is an ordered Hausdorff topological vector space. If* K *is a nonempty compact subset of* $\mathcal{Z}$*, then Inf*(K, C) *and Sup*(K, C) *are nonempty.*

We end this section by introducing *ordering-conically compact* spaces. Let K be a nonempty subset of an ordered Hausdorff topological vector space $(\mathcal{Z}, C)$, and assume that K is bounded below with respect to C. Let $z \in \mathcal{Z}$ be such that $K \subset z + C$. Clearly,

$$\overline{K} \cap (x - C) \subset (z + C) \cap (x - C) \quad \text{for any } x \in K.$$

If $(z + C) \cap (x - C)$ is compact, then so is $\overline{K} \cap (x - C)$. It follows from Corollary 3.5 that Inf(K, C) is nonempty. This leads to the following definition.

Definition of ordering-conically compact space An ordering-conically compact space is an ordered Hausdorff topological vector space $(\mathcal{Z}, C)$ with the property that for any two x, $y \in \mathcal{Z}$, the set $(x - C) \cap (y + C)$ is compact.

For instance, $(\mathbb{R}, \mathbb{R}^+)$ is ordering-conically compact, where $\mathbb{R}^+$ is the set of non-negative real numbers. For one more example, let C be any proper closed convex cone in $\mathbb{R}^2$ with nonempty interior. If C is pointed, i.e., $C \cap (-C) = \{0\}$, then $(\mathbb{R}^2\,,\, C)$ is ordering-conically compact.

Theorem 3.7. *Let $(\mathcal{Z}\,,\, C)$ be an ordering-conically compact space, and let $K \subset \mathcal{Z}$ be nonemtpty.*

(i) *If K is bounded below with respect to C, then Inf(K, C) is nonempty.*

(ii) *If K is bounded above with respect to C, then Sup(K, C) is nonempty.*

4. Proofs of Main Theorems

This section is devoted to proving Theorems 3.3 and 3.4. Throughout this section, $(\mathcal{Z}, C)$ is an ordered Hausdorff topological vector space.

We shall use the notation given in Section 3. In the rest of the paper, we simply write $x \preceq y$ for $x \preceq_{C_\varphi} y$ whenever $x,\ y \in \mathcal{Z}$. Note that $x \preceq y$ or $y \preceq x$ for x, $y \in \mathcal{Z}$ since $\mathcal{Z} = C_\varphi \cup (-C_\varphi)$.

Lemma 4.1. *Let $K \subset \mathcal{Z}$ be nonempty, and let $\overline{x} \in$ Inf(K, C_φ) and $\widehat{x} \in$ Sup$(K\,,\, C_\varphi)$. Then the following statements hold.*

(i) *$\overline{x} \preceq x \preceq \widehat{x}$ for all $x \in K$.*

(ii) *If $x' \preceq x$ for all $x \in K$, then $x' \preceq \overline{x}$.*

(iii) *If $x \preceq x'$ for all $x \in K$, then $\widehat{x} \preceq x'$.*

Proof. By Remark 2.2 (i), we only have to show that $\overline{x} \preceq x$ for all $x \in K$, and that $x' \preceq \overline{x}$ whenever $x' \preceq x$ for all $x \in K$.

Suppose that there is an $x_0 \in K$ such that $x_0 - \overline{x} = -v$ for some $v \in \text{Int}C_\varphi$. Then we are led the contradiction $x_0 = \overline{x} - v \in (\overline{x} - \text{Int}C_\varphi) \cap K$. Therefore, $\overline{x} \preceq x$ for all $x \in K$.

By assumption, $-x' + K \subset C_\varphi$. Since the map $z \longmapsto -x' + z$ is a homeomorphism of $\mathcal{Z}$ onto itself, $-x' + \overline{K} = \overline{(-x' + K)} \subset C_\varphi$. This implies that $-x' + \overline{x} \in C_\varphi$ and $x' \preceq \overline{x}$.

Remark 4.2. By Lemma 4.1, if Inf$\,(K, C_\varphi)$ (respectively, Sup$\,(K, C_\varphi)$) is nonempty, then K is bounded below (respectively, bounded above) with respect to C_φ.

This is not true for any ordered Hausdorff topological vector space. For instance, let $\mathcal{Z} = \mathbb{R}^2$, let $C = \{(x, y) \in \mathbb{R}^2 : x \geq 0 \text{ and } y \geq 0\}$, and let $K = \{(0, y) : y \in \mathbb{R}\}$. Then $K = \text{Inf}\,(K, C) = \text{Sup}\,(K, C)$. But K is neither bounded above nor bounded below.

Corollary 4.3. *Let $K \subset \mathcal{Z}$ be nonempty, and let $\widehat{x} \in \overline{K}$.*

(i) $\widehat{x} \in$ *Inf* $(K\,,\,C_\varphi)$ *if and only if* $\widehat{x} \preceq x$ *for all* $x \in \overline{K}$.

(ii) $\widehat{x} \in$ *Sup* $(K\,,\,C_\varphi)$ *if and only if* $\widehat{x} \succeq x$ *for all* $x \in \overline{K}$.

Proof. We only prove (i). By Lemma 4.1, it remains to show that $\widehat{x} \in \text{Inf}\,(K\,,\,C_\varphi)$ if $\widehat{x} \preceq x$ for all $x \in \overline{K}$. Since $\overline{K} \subset \widehat{x} + C_\varphi$,

$$(\widehat{x} - \text{Int}C_\varphi) \cap K \subset (\widehat{x} - \text{Int}C_\varphi) \cap (\widehat{x} + C_\varphi) = \emptyset\,.$$

Therefore, $\widehat{x} \in \text{Inf}\,(K\,,\,C_\varphi)$.

Corollary 4.4. *Let* $K \subset \mathcal{Z}$ *be nonempty. If* $x \in K$*, then*

$$\textit{Inf}\,(K, C_\varphi) = \textit{Inf}\,(\overline{K} \cap (x - C_\varphi), C_\varphi) \ \ \textit{and} \ \ \textit{Sup}\,(K\,,\,C_\varphi) = \textit{Sup}\,(\overline{K} \cap (x + C_\varphi)\,,\,C_\varphi)\,.$$

Proof. It follows from Corollary 4.3 that

$$\text{Inf}\,(K\,,\,C_\varphi) \subset \text{Inf}\,(\overline{K} \cap (x - C_\varphi)\,,\,C_\varphi) \ \ \text{and} \ \ \text{Sup}\,(K, C_\varphi) \subset \text{Sup}\,(\overline{K} \cap (x + C_\varphi), C_\varphi).$$

Now, the corollary follows immediately from Theorem 2.5.

Lemma 4.5. *If* $z,\, z' \in \mathcal{Z}$*, then*

(i) $z \preceq z'$ *if and only if* $\varphi(z) \le \varphi(z')$*, and*

(ii) $z \prec z'$ *if and only if* $\varphi(z) < \varphi(z')$.

Proof. Since $\varphi(z' - z) = \varphi(z') - \varphi(z)$,

$$\begin{aligned} z \preceq z' &\iff z' - z \in C_\varphi \iff \varphi(z') - \varphi(z) \ge 0\,;\\ z \prec z' &\iff z' - z \in \text{Int}C_\varphi \iff \varphi(z') - \varphi(z) > 0\,. \end{aligned}$$

From Corollary 4.3 and Lemma 4.5, we obtain :

Corollary 4.6. *Let* $K \subset \mathcal{Z}$ *be nonempty.*

(i) *Inf* $(K\,,\,C_\varphi) \ne \emptyset$ *if and only if there is an* $\widehat{x} \in \overline{K}$ *such that* $\varphi(\widehat{x}) \le \varphi(x)$ *for all* $x \in \overline{K}$.

(ii) *Sup* $(K\,,\,C_\varphi) \ne \emptyset$ *if and only if there is an* $\widehat{x} \in \overline{K}$ *such that* $\varphi(\widehat{x}) \ge \varphi(x)$ *for all* $x \in \overline{K}$.

Lemma 4.7. *If* $K \subset \mathcal{Z}$*, then* *Inf* $(K, C_\varphi) \subset$ *Inf* $(K\,,\,C)$ *and* *Sup* $(K, C_\varphi) \subset$ *Sup* $(K\,,\,C)$.

Proof. We assume that K is closed, and prove that Inf $(K, C_\varphi) \subset$ Inf (K, C) as follows. Let $x \in K$. If $x \in$ Inf $(K\,,\, C_\varphi)$, then $(x - \text{Int}C) \cap K \subset (x - \text{Int}C_\varphi) \cap K = \emptyset$. Therefore, $x \in$ Inf (K, C).

Now, we are ready to prove Theorems 3.3 and 3.4.

Proof of Theorem 3.3 (i). We assume that K is closed. By Theorem 2.5, it suffices to show that Inf $(K_0\,,\, C) \neq \emptyset$, where $K_0 = K \cap (x_0 - C)$. By assumption, there is a $z_0 \in \mathcal{Z}$ such that $K \subset z_0 + C$. Since

$$K_0 \subset (z_0 + C) \cap (x_0 - C) \subset (z_0 + C_\varphi) \cap (x_0 - C_\varphi)\,,$$

we have $\varphi(z_0) \leq \varphi(x) \leq \varphi(x_0)$ for all $x \in K_0$ by Lemma 4.5. This proves that $\varphi(K_0)$ is compact in $\mathbb{R}$ since it is closed in $\mathbb{R}$. There is an $\widehat{x} \in K_0$ such that

$$\varphi(\widehat{x}) = \min_{x \in K_0} \varphi(x)\,.$$

Thus, $\widehat{x} \in$ Inf $(K_0\,,\, C_\varphi)$ by Corollary 4.6. The proof is complete by Lemma 4.7.

Proof of Theorem 3.4 (i). We assume that K is closed. Let $K_0 = K \cap (x_0 - C)$ and let $A = \Pi_0(K_0)$. There is a $z_0 \in \mathcal{Z}$ such that $K \subset z_0 + C$. Recall that Π is the orthogonal projection of $\mathcal{Z}$ onto $\langle \boldsymbol{\mathcal{N}} \rangle$, and that $\varphi\Pi(z) = \varphi(z)$ for all $z \in \mathcal{Z}$. By writing $\varphi(x_0) = t$ and $\varphi(z_0) = r$, we have

$$\Pi(K_0) \subset \{\lambda \boldsymbol{\mathcal{N}} \,:\, \lambda \in \mathbb{R} \text{ with } r \leq \lambda \leq t\}\,.$$

This implies that $K_0 \subset \mathbb{B} = \{\lambda \boldsymbol{\mathcal{N}} + z \,:\, r \leq \lambda \leq t \text{ and } z \in A\}$. Note that $\mathbb{B}$ is the image of set $[r\,,\, t] \times A$ mapped by the continuous function $\Psi \,:\, \mathbb{R} \times \partial C_\varphi \longrightarrow \mathcal{Z}$ defined by

$$\Psi(\lambda\,,\, z) = \lambda \boldsymbol{\mathcal{N}} + z \quad \text{ for } \lambda \in \mathbb{R} \text{ and for } z \in \partial C_\varphi.$$

Since $[r\,,\, t] \times A$ is compact, $\mathbb{B}$ is compact, and so is K_0. This implies that $\varphi(K_0)$ is closed. Now, the theorem follows from Theorem 3.3.

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