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WILLMORE SURFACES IN THE UNIT N-SPHERE

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Abstract. Let M^2 be a compact Willmore surface in the n-dimensional unit sphere. Denote by ϕ_{ij}^{α} the tracefree part of the second fundamental form h_{ij}^{α} of M^2, and by $\mathbb{H}$ the mean curvature vector of M^2. Let Φ be the square of the length of ϕ_{ij}^{α} and $H = |\mathbb{H}|$. We prove that if $0 \le \Phi \le C(1 + \frac{H^2}{8})$, where $C = 2$ when $n = 3$ and $C = \frac{4}{3}$ when $n \ge 4$, then either $\Phi = 0$ and M^2 is totally umbilic or $\Phi = C(1 + \frac{H^2}{8})$. In the latter case, either $n = 3$ and M^2 is the Clifford torus or $n = 4$ and M^2 is the Veronese surface.

1. Introduction

Let M^2 be a compact surface in the n-dimensional unit sphere S^n. Denote by $[h_{ij}^{\alpha}]$ the second fundamental form of M^2, by $H^{\alpha} = \sum h_{ii}^{\alpha}$ the component of the mean curvature vector $\mathbb{H}$, and by ϕ_{ij}^{α} the tensor $h_{ij}^{\alpha} - \frac{H^{\alpha}}{2}\delta_{ij}$ of the tracefree part of the second fundamental form $[h_{ij}^{\alpha}]$. Let Φ denote the square of the length of $[\phi_{ij}^{\alpha}]$ and H the length of $\mathbb{H}$.

The Willmore functional is defined by

$$W(x) = \int_M \Phi$$

where the integration is with respect to the area measure of M^2. This functional is invariant under conformal transformations of S^n. The critical surfaces of W are called Willmore surface (see [2]). More precisely, M^2 is a Willmore surface if and only if

$$\triangle^{\perp} H^{\alpha} + \sum h_{ij}^{\alpha} h_{ij}^{\beta} H^{\beta} - \frac{H^2}{2} H^{\alpha} = 0, \qquad 3 \le \alpha \le n,$$

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where $\triangle^{\perp}$ is the Laplacian in the normal bundle NM (see [1] and [8]). In other words, M^2 is a Willmore surface if and only if

$$\triangle^{\perp}H^{\alpha}+\sum\phi_{ij}^{\alpha}\phi_{ij}^{\beta}H^{\beta}=0,\qquad 3\leq\alpha\leq n.$$

Any minimal surface in S^n is a Willmore surface. The class of all Willmore surfaces turns out to be rather large.

It is well known that if $H=0$ and $0\leq\Phi\leq\frac{2n-4}{2n-5}$, then M^2 is the equatorial sphere, or the Clifford torus, or the Veronese surface (see [3]). Recently, Li extended the above result to Willmore surfaces (see [5] and [6]). He proved that if $0\leq\Phi\leq C$, where $C=2$ when $n=3$ and $C=\frac{4}{3}$ when $n\geq 4$, then M^2 is the totally umbilical sphere, or the Clifford torus, or the Veronese surface. In this paper we shall give a sharper estimate which improves Li's results. The crucial point is that we replace his estimate of $\triangle S$ by the estimate of $\triangle\Phi$. We will prove the following theorem:

Theorem 1.1. *Let M^2 be a compact Willmore surface in the n-dimensional unit sphere S^n. Then*

$$\int_M\Phi\,(\,C(1+\frac{H^2}{8})-\Phi\,)\leq 0$$

where $C=2$ for $n=3$ and $C=\frac{4}{3}$ for $n\geq 4$. In particular, if

$$0\leq\Phi\leq C(1+\frac{H^2}{8}),$$

then either $\Phi=0$ and M is totally umbilic, or $\Phi=C(1+\frac{H^2}{8})$. In the latter case, either $n=3$ and M^2 is the Clifford torus; or $n=4$ and M^2 is the Veronese surface.

Because of the fact that the Willmore functional is conformal invariant, we construct two examples in the last section. Both of examples show that our result is no longer working if we make a slight change in the pinching condition.

2. Basic Lemmas

We begin by introducing some terminologies. Let $x:M^2\to S^n$ be a surface in the n-dimensional unit sphere S^n. We choose a local orthonormal frame field $\{e_1,\cdots,e_n\}$ in S^n, restricted to $x(M)$, so that the vectors e_1,e_2 are tangent to $x(M)$ and $\{e_3,\cdots,e_n\}$ is a local frame field in the normal bundle NM of M^2. Let $\{\omega_1,\cdots,\omega_n\}$ denote the dual coframe field in S^n. We shall use the following ranges of indices

$$1 \le i, j, k, \cdots \le 2; \qquad 3 \le \alpha, \beta, \gamma, \cdots \le n.$$

Then the structure equations are given by

$$\begin{aligned} dx &= \sum \omega_i e_i, \\ de_i &= \sum \omega_{ij} e_j + \sum h_{ij}^\alpha \omega_j e_\alpha - \omega_i x, \\ de_\alpha &= -\sum h_{ij}^\alpha \omega_j e_i + \sum \omega_{\alpha\beta} e_\beta, \qquad h_{ij}^\alpha = h_{ji}^\alpha, \end{aligned}$$

where $[h_{ij}^\alpha]$ is the second fundamental form of M^2. From the structure equations of M^2, the Gauss equations are

$$R_{ijkl} = (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + \sum (h_{ik}^\alpha h_{jl}^\alpha - h_{il}^\alpha h_{jk}^\alpha), \tag{2.1}$$

$$R_{ik} = \delta_{ik} + \sum H^\alpha h_{ik}^\alpha - \sum h_{ij}^\alpha h_{jk}^\alpha, \tag{2.2}$$

$$2K = 2 + H^2 - S, \tag{2.3}$$

$$R_{\alpha\beta kl} = \sum (h_{ki}^\alpha h_{il}^\beta - h_{li}^\alpha h_{ik}^\beta), \tag{2.4}$$

were K is the Gaussian curvature of M^2, $S = \sum (h_{ij}^\alpha)^2$ is the square of the length of the second fundamental form, $\mathbb{H} = \sum H^\alpha e_\alpha = \sum h_{ii}^\alpha e_\alpha$ is the mean curvature vector, $H = \sum h_{ii}$ if $n = 3$ and $H = |\mathbb{H}|$ is the length of mean curvature vector of M^2 if $n \ge 4$.

The covariant derivative ∇h_{ij}^α of the second fundamental form h_{ij}^α of M^2 with components h_{ijk}^α is defined by

$$\sum h_{ijk}^\alpha \omega_k = dh_{ij}^\alpha + \sum h_{kj}^\alpha \omega_{ki} + \sum h_{ik}^\alpha \omega_{kj} + \sum h_{ij}^\beta \omega_{\beta\alpha},$$

and the covariant derivative $\nabla^2 h_{ij}^\alpha$ of ∇h_{ij}^α with components h_{ijkl}^α is defined by

$$\sum h_{ijkl}^\alpha \omega_l = dh_{ijk}^\alpha + \sum h_{ljk}^\alpha \omega_{li} + \sum h_{ilk}^\alpha \omega_{lj} + \sum h_{ijl}^\alpha \omega_{lk} + \sum h_{ijk}^\beta \omega_{\beta\alpha}.$$

Then the Codazzi equation and the Ricci formula are given by

$$h_{ijk}^\alpha - h_{ikj}^\alpha = 0, \tag{2.5}$$

$$h_{ijkl}^\alpha - h_{ijlk}^\alpha = \sum h_{mj}^\alpha R_{mikl} + \sum h_{im}^\alpha R_{mjkl} + \sum h_{ij}^\beta R_{\beta\alpha kl}. \tag{2.6}$$

Let ϕ_{ij}^α denote the tensor $h_{ij}^\alpha - \frac{H^\alpha}{2}\delta_{ij}$ and $\Phi = \sum (\phi_{ij}^\alpha)^2$ the square of the length of the tracefree tensor ϕ_{ij}^α.

Lemma 2.1. $\frac{1}{2}\triangle\Phi = \sum (\phi_{ijk}^\alpha)^2 + \sum \phi_{ij}^\alpha H_{ij}^\alpha + \Phi(2 - \Phi + \frac{H^2}{2}) - \sum R_{\alpha\beta 12}^2.$

Proof. Using (2.1) (2.2) (2.3) (2.5) and (2.6), we have

$$\begin{aligned}\triangle\phi^\alpha_{ij} &= \sum \phi^\alpha_{ijkk}\\ &= \sum \phi^\alpha_{kkij} + H^\alpha_{ij} - \frac{\triangle^\perp H^\alpha}{2}\delta_{ij} + (2-\Phi+\frac{H^2}{2})\phi^\alpha_{ij} + \sum \phi^\beta_{ki}R_{\beta\alpha jk}\\ &\quad + \sum_\beta \frac{H^\beta}{2}R_{\beta\alpha ji},\end{aligned}$$

where $\triangle^\perp$ is the Laplacian in the normal bundle. Since $\sum\phi^\alpha_{kk}=0$, it follows that

$$\begin{aligned}\frac{1}{2}\triangle\Phi &= \sum(\phi^\alpha_{ijk})^2 + \sum\phi^\alpha_{ij}\phi^\alpha_{ijkk}\\ &= \sum(\phi^\alpha_{ijk})^2 + \sum\phi^\alpha_{ij}H^\alpha_{ij} + (2-\Phi+\frac{H^2}{2})\Phi\\ &\quad + \sum\phi^\alpha_{ij}\phi^\beta_{ki}R_{\beta\alpha jk} + \sum\frac{H^\beta}{2}\phi^\alpha_{ij}R_{\beta\alpha ji}\\ &= \sum(\phi^\alpha_{ijk})^2 + \sum\phi^\alpha_{ij}H^\alpha_{ij} + (2-\Phi+\frac{H^2}{2})\Phi\\ &\quad + \sum(\phi^\alpha_{i1}\phi^\beta_{2i} - \phi^\alpha_{i2}\phi^\beta_{1i})R_{\beta\alpha 12} + \sum\frac{H^\beta}{2}(\phi^\alpha_{12}R_{\beta\alpha 21} + \phi^\alpha_{21}R_{\beta\alpha 12})\\ &= \sum(\phi^\alpha_{ijk})^2 + \sum\phi^\alpha_{ij}H^\alpha_{ij} + \Phi(2-\Phi+\frac{H^2}{2}) - \sum R^2_{\alpha\beta 12}.\end{aligned}$$

■

Lemma 2.2. $\sum\phi^\alpha_{ijj}H^\alpha_i = \frac{1}{2}|\nabla^\perp\mathbb{H}|^2$, *where* $|\nabla^\perp\mathbb{H}|^2 = \sum(H^\alpha_i)^2$.

Proof. It is an immediate consequence of the fact $\sum\phi^\alpha_{ijj} = \frac{H^\alpha_i}{2}$. ■

Lemma 2.3. $\sum(\phi^\alpha_{ijk})^2 \geq \frac{1}{4}|\nabla^\perp\mathbb{H}|^2$.

Proof. Since $0=\phi^\alpha_{11}+\phi^\alpha_{22}$, we therefore have $\phi^\alpha_{111} = -\phi^\alpha_{221}$ and $\phi^\alpha_{112} = -\phi^\alpha_{222}$, which implies

$$\begin{aligned}\sum(\phi^\alpha_{ijk})^2 &= \sum[\,(\phi^\alpha_{111})^2 + (\phi^\alpha_{112})^2\\ &\quad +2(\phi^\alpha_{121})^2 + 2(\phi^\alpha_{122})^2 + (\phi^\alpha_{221})^2 + (\phi^\alpha_{222})^2\,]\\ &= \sum 2\,[\,(\phi^\alpha_{111})^2 + (\phi^\alpha_{222})^2 + (\phi^\alpha_{211})^2 + (\phi^\alpha_{122})^2)\,]\\ &\geq \sum[\,(\phi^\alpha_{111}+\phi^\alpha_{122})^2 + (\phi^\alpha_{222}+\phi^\alpha_{211})^2\,]\\ &= \sum[\,(\frac{H^\alpha_1}{2})^2 + (\frac{H^\alpha_2}{2})^2)\,]\\ &= \frac{1}{4}\,|\,\nabla^\perp\mathbb{H}\,|^2\,.\end{aligned}$$

■

Lemma 2.4. ([5] *and* [6].) *Let M^2 be a compact surface in the n-dimensional unit sphere S^n. Then M^2 is a Willmore surface if and only if*

$$\triangle^{\perp} H^{\alpha} + \sum \phi^{\alpha}_{ij}\phi^{\beta}_{ij}H^{\beta} = 0, \qquad \text{for all } \ 3 \leq \alpha \leq n.$$

Lemma 2.5. *Let M^2 be a compact Willmore surface in the n-dimensional unit sphere S^n. Then*

$$\int_M |\nabla^{\perp}\mathbb{H}|^2 \leq \int_M \Phi H^2.$$

Equality holds if and only if either $n = 3$ or $n \geq 4$ and $\phi^{\ \alpha}_{ij} = C_{ij}H^{\alpha}$ for some functions C_{ij} at the points where $\Phi \neq 0$ and $H \neq 0$.

Proof. By use of Lemma 2.4 and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
\int_M |\nabla^{\perp}\mathbb{H}|^2 &= -\int_M \sum H^{\alpha}\triangle^{\perp}H^{\alpha} \\
&= \int_M \sum \phi^{\alpha}_{ij}\phi^{\beta}_{ij}H^{\alpha}H^{\beta} \\
&= \int_M \sum_{i,j}(\sum_{\alpha}\phi^{\alpha}_{ij}H^{\alpha})^2 \\
&\leq \int_M (\sum(\phi^{\alpha}_{ij})^2)(\sum(H^{\alpha})^2) \\
&= \int_M \Phi H^2.
\end{aligned}$$

∎

Lemma 2.6. $\sum(R_{\alpha\beta12})^2 \leq \frac{2-C}{C}\Phi^2$, *where* $C = 2$ *if* $n = 3$, *and* $C = \frac{4}{3}$ *if* $n \geq 4$. *Equality holds if and only if either $n = 3$ or $n \geq 4$,* $\sum(\phi^{\alpha}_{11})^2 = \sum(\phi^{\alpha}_{12})^2$ *and* $\sum \phi^{\alpha}_{11}\phi^{\alpha}_{12} = 0$.

Proof. In the case of $n = 3$, $\sum R^2_{\alpha\beta12} = 0$. For $n \geq 4$, according to (2.4), we have

$$\begin{aligned}
\sum(R_{\alpha\beta12})^2 &= \sum \left[\, \sum(\phi^{\beta}_{1i}\phi^{\alpha}_{i2} - \phi^{\alpha}_{1i}\phi^{\beta}_{i2})\, \right]^2 \\
&= 4\sum(\phi^{\alpha}_{11}\phi^{\beta}_{12} - \phi^{\beta}_{11}\phi^{\alpha}_{12})^2 \\
&= 4\sum \left[\, (\phi^{\alpha}_{11})^2(\phi^{\beta}_{12})^2 + (\phi^{\beta}_{11})^2(\phi^{\alpha}_{12})^2 - 2\phi^{\alpha}_{11}\phi^{\beta}_{11}\phi^{\alpha}_{12}\phi^{\beta}_{12}\, \right] \\
&= 8\sum(\phi^{\alpha}_{11})^2\sum(\phi^{\alpha}_{12})^2 - 8(\sum \phi^{\alpha}_{11}\phi^{\alpha}_{12})^2.
\end{aligned}$$

On the other hand,

$$\begin{aligned}\Phi &= \sum(\phi_{ij}^{\alpha})^2 = \sum(\phi_{11}^{\alpha})^2 + 2\sum(\phi_{12}^{\alpha})^2 + \sum(\phi_{22}^{\alpha})^2 \\ &= 2\,[\,\sum(\phi_{11}^{\alpha})^2 + \sum(\phi_{12}^{\alpha})^2\,].\end{aligned}$$

It follows that

$$\begin{aligned}\sum(R_{\alpha\beta 12})^2 &= 8\,[\,\sum(\phi_{11}^{\alpha})^2\sum(\phi_{12}^{\alpha})^2 - (\sum\phi_{11}^{\alpha}\phi_{12}^{\alpha})^2\,] \\ &\leq 8\sum(\phi_{11}^{\alpha})^2\sum(\phi_{12}^{\alpha})^2 \\ &\leq 2\,[\,\sum(\phi_{11}^{\alpha})^2 + \sum(\phi_{12}^{\alpha})^2\,]^2 \\ &= \frac{\Phi^2}{2}\,.\end{aligned}$$

∎

Lemma 2.7. ([4]) *Let M^2 be a compact minimal surface in the n-dimensional unit sphere S^n. If $0 \leq S \leq \frac{4}{3}$ then either $S = 0$ and M^2 is totally geodesic, or $S = \frac{4}{3}$, $n = 4$ and M^2 is the Veronese surface.*

Lemma 2.8. *For* $n = 3,\ \Phi\sum\phi_{ijk}^2 = \frac{|\nabla\Phi|^2}{2} + \Phi\frac{|\nabla H|^2}{2} - \sum\phi_{ij}H_i\Phi_j.$

Proof. One compute

$$\begin{aligned}\sum\phi_{ijk}^2 &= 4\phi_{111}^2 + 4\phi_{222}^2 - 2\phi_{111}H_1 - 2\phi_{222}H_2 + \frac{|\nabla H|^2}{2}, \\ \frac{|\nabla\Phi|^2}{2} &= 4\Phi\phi_{111}^2 + 4\Phi\phi_{222}^2 - 8\phi_{12}^2\phi_{111}H_1 - 8\phi_{12}^2\phi_{222}H_2 \\ &\quad +8\phi_{12}(\phi_{11}\phi_{111}H_2 + \phi_{22}\phi_{222}H_1) + 2\phi_{12}^2|\nabla H|^2,\end{aligned}$$

and

$$\begin{aligned}\sum\phi_{ij}H_i\Phi_j &= -4\phi_{12}^2\phi_{111}H_1 - 4\phi_{12}^2\phi_{222}H_2 + 4\phi_{11}^2\phi_{111}H_1 + 4\phi_{22}^2\phi_{222}H_2 \\ &\quad +8\phi_{12}(\phi_{11}\phi_{111}H_2 + \phi_{22}\phi_{222}H_1) + 2\phi_{12}^2|\nabla H|^2.\end{aligned}$$

The proof is then straightforward. ∎

3. Proof of Theorem

Now, we prove the Theorem. Integrating both sides of the Lemma 2.1 over M^2, we have

$$\begin{aligned}0 &= \int_M[\,\sum(\phi_{ijk}^{\alpha})^2 + \sum\phi_{ij}^{\alpha}H_{ij}^{\alpha} + \Phi(2 - \Phi + \frac{H^2}{2}) - \sum R_{\alpha\beta 12}^2\,] \\ &= \int_M[\,\sum(\phi_{ijk}^{\alpha})^2 - \sum\phi_{ijj}^{\alpha}H_i^{\alpha} + \Phi(2 - \Phi + \frac{H^2}{2}) - \sum R_{\alpha\beta 12}^2\,].\end{aligned}$$

By using Lemmas 2.2 and 2.3,

$$\begin{aligned} 0 &\geq \int_M [\,\frac{1}{4}|\nabla^\perp \mathbb{H}|^2 - \frac{1}{2}|\nabla^\perp \mathbb{H}|^2 + \Phi(2 - \Phi + \frac{H^2}{2}) - \sum R^2_{\alpha\beta 12}\,] \\ &= \int_M [\,-\frac{1}{4}|\nabla^\perp \mathbb{H}|^2 + \Phi(2 - \Phi + \frac{H^2}{2}) - \sum R^2_{\alpha\beta 12}\,]. \end{aligned}$$

From Lemma 2.5, we get

$$\begin{aligned} 0 &\geq \int_M [\,-\frac{1}{4}\Phi H^2 + \Phi(2 - \Phi + \frac{H^2}{2}) - \sum R^2_{\alpha\beta 12}\,] \\ &= \int_M \Phi(2 - \Phi + \frac{H^2}{4}) - \sum R^2_{\alpha\beta 12}. \end{aligned}$$

By Lemma 2.6,

$$\begin{aligned} 0 &\geq \int_M \Phi(2 - \Phi + \frac{H^2}{4}) - \frac{2-C}{C}\Phi^2 \\ &= \int_M \Phi(2 - \frac{2}{C}\Phi + \frac{H^2}{4}) \qquad (3.1) \\ &= \frac{2}{C}\int_M \Phi(C(1 + \frac{H^2}{8}) - \Phi). \end{aligned}$$

If $0 \leq \Phi \leq C(1 + \frac{H^2}{8})$, from (3.1) we can conclude that either $\Phi = 0$ and M is totally umbilic or $\Phi = C(1 + \frac{H^2}{8})$. In the latter case, all the integral inequalities become equalities.

For $n = 3$ and $\Phi = 2(1 + \frac{H^2}{8}) > 0$, it follows from Lemmas 2.1 and 2.8 that

$$\begin{aligned} \int_M (2 + \frac{H^2}{2} - \Phi) &= \int_M (\frac{1}{2}\frac{\triangle\Phi}{\Phi} - \frac{\sum \phi^2_{ijk}}{\Phi} - \frac{\sum \phi_{ij}H_{ij}}{\Phi}) \\ &= \int_M (\frac{1}{2}\frac{\triangle\Phi}{\Phi} - \frac{|\nabla\Phi|^2}{2\Phi^2} - \frac{|\nabla H|^2}{2\Phi} + \frac{\sum \phi_{ij}H_i\Phi_j}{\Phi^2} - \frac{\sum \phi_{ij}H_{ij}}{\Phi}) \\ &= \int_M (\frac{1}{2}\triangle \log\Phi - \frac{|\nabla H|^2}{2\Phi} + \frac{\sum \phi_{ij}H_i\Phi_j}{\Phi^2} - \frac{\sum \phi_{ij}H_{ij}}{\Phi}) \\ &= \int_M (-\frac{|\nabla H|^2}{2\Phi} + \frac{\sum \phi_{ij}H_i\Phi_j}{\Phi^2} + \sum (\frac{\phi_{ij}}{\Phi})_j H_i) \\ &= \int_M (-\frac{|\nabla H|^2}{2\Phi} + \frac{\sum \phi_{ij}H_i\Phi_j}{\Phi^2} + \sum \frac{\Phi\phi_{ijj} - \phi_{ij}\Phi_j}{\Phi^2}H_i) \\ &= \int_M (-\frac{|\nabla H|^2}{2\Phi} + \frac{\sum H_i^2}{2\Phi}) \\ &= 0. \end{aligned}$$

This implies that

$$0 = \int_M (2 + \frac{H^2}{2} - \Phi) = \int_M \frac{H^2}{4}.$$

Thus M^2 is a minimal surface of S^3 with $S = 2$, we conclude that M^2 is the Clifford torus (see [3]).

Now we consider the case $n \geq 4$ and $\Phi = \frac{4}{3} + \frac{1}{6}H^2$. Assuming that $H(p) \neq 0$ at some point $p \in M^2$, we shall derive a contradiction. Since by assumption $H(p) \neq 0$, $\Phi(p) > 0$, Lemma 2.5 gives $\phi_{ij}^\alpha = C_{ij}H^\alpha$ for some functions C_{ij}. Furthermore, Lemma 2.6 implies $C_{11}(p) = C_{12}(p) = 0$, and hence $C_{21}(p) = C_{22}(p) = 0$. This means that $\Phi(p) = 0$, a contradiction. This contradiction shows that M^2 is a minimal surface with $S = \frac{4}{3}$ and M^2 is the Veronese surface (see [4]). This completes the proof of the Theorem. ■

4. EXAMPLES

Let D^{n+1} be open unit ball in $\mathbb{R}^{n+1}$ and G the conformal group of S^n. For each $g \in D^{n+1}$, let $g : S^n \to S^n$ be the mapping

$$g(x) = \begin{cases} \dfrac{x + (\mu < x, g > +\lambda)g}{\lambda(< x, g > +1)} & \text{if} \quad g \neq 0, \\ x & \text{if} \quad g = 0, \end{cases} \tag{4.1}$$

where $\lambda = \frac{1}{\sqrt{1-|g|^2}}$ and $\mu = \frac{\lambda-1}{|g|^2}$ when $g \neq 0$. Then each conformal transformation of S^n can be expressed by $T \circ g$, where T is an orthogonal transformation of S^n and g is given by (4.1) (see [7]).

Let $x : M^2 \to S^n$ be a compact Willmore surface. Since the Willmore functional W is invariant under conformal transformations of S^n, it follows that $\bar{x} = g \circ x$ is also a compact Willmore surface. The second fundamental form $\bar{h}_{ij}^\alpha$ and h_{ij}^α of $\bar{x}$ and x, respectively, are related by

$$\bar{h}_{ij}^\alpha = \lambda(< x, g > +1)h_{ij}^\alpha + \lambda < e_\alpha, g > \delta_{ij}.$$

The point of the following examples is that it shows our result may fail to be true if we make a slight change in the pinching condition.

Example 4.1. Let $x : S^1 \times S^1 \to S^3$ be the Clifford torus,

$$x(\theta, \varphi) = \frac{1}{\sqrt{2}}(\cos\theta,\ \sin\theta,\ \cos\varphi,\ \sin\varphi).$$

Consider the Willmore surface $x_\epsilon = g \circ x$, where $g = (a, 0, a, 0)$ with $a = \frac{\epsilon}{\sqrt{2}(4+\epsilon)}$. Since the Clifford torus is a minimal surface with $\Phi = 2$, we have

$$\Phi_\epsilon - \frac{1}{4}H_\epsilon^2 = \frac{2}{1-2a^2}[\,(\frac{a}{\sqrt{2}}\cos\theta + \frac{a}{\sqrt{2}}\cos\varphi + 1)^2 - \frac{1}{2}(-\frac{a}{\sqrt{2}}\cos\theta + \frac{a}{\sqrt{2}}\cos\varphi)^2\,].$$

The maximal value of $\Phi_\epsilon - \frac{1}{4}H_\epsilon^2$ over $S^1 \times S^1$ is

$$2\frac{1+\sqrt{2}a}{1-\sqrt{2}a} = 2 + \epsilon.$$

Thus for every $\epsilon > 0$, there is a compact Willmore surface M^2 in S^3, it is not the Clifford torus, with $0 < \Phi \le 2 + \frac{H^2}{4} + \epsilon$.

Example 4.2. Let $x : S^2(\sqrt{3}) \to S^4$ be the Veronese surface,

$$\begin{aligned} x(\theta,\varphi) \;=\; & (\sqrt{3}\cos\theta\sin\theta\sin\varphi,\ \sqrt{3}\cos\theta\sin\theta\cos\varphi,\ \sqrt{3}\cos^2\theta\cos\varphi\sin\varphi, \\ & \frac{\sqrt{3}}{2}\cos^2\theta(\cos^2\varphi - \sin^2\varphi),\ \frac{1}{2}\cos^2\theta - \sin^2\theta). \end{aligned}$$

Consider the Willmore surface $x_\epsilon = g \circ x$, where $g = (a, a, 0, 0, 0)$ with $a = \frac{-\sqrt{6}+\sqrt{6+3\epsilon(\frac{7}{2}+\frac{3\epsilon}{2})}}{7+3\epsilon}$. Since the Veronese surface is a minimal surface with $\Phi = \frac{4}{3}$, we must have

$$\Phi_\epsilon - \frac{1}{6}H_\epsilon^2 = \frac{1}{1-2a^2}\{\,[\,a(\sin\varphi + \cos\varphi)\sin 2\theta + \frac{2}{\sqrt{3}}\,]^2 - \frac{2a^2}{3}(\cos\varphi - \sin\varphi)^2\cos^2\theta\,\}.$$

The maximal value of $\Phi_\epsilon - \frac{1}{6}H_\epsilon^2$ over $S^2(\sqrt{3})$ is

$$\frac{1}{1-2a^2}(a + \frac{2}{\sqrt{3}})^2 = \frac{4}{3} + \epsilon.$$

Thus for every $\epsilon > 0$, there is a compact Willmore surface M^2 in S^4, it is not the Veronese surface, with $0 < \Phi \le \frac{4}{3} + \frac{H^2}{6} + \epsilon$.

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