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ON CYCLICITY IN THE SPACE $H^p(\beta)$

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Abstract. Let $\{\beta(n)\}$ be a sequence of positive numbers with $\beta(0) = 1$ and let $p > 0$. By the space $H^p(\beta)$, we mean the set of all formal power series $\sum_{n=0}^{\infty} \hat{f}(n) z^n$ for which $\sum_{n=0}^{\infty} |\hat{f}(n)|^p \beta(n)^p < \infty$. In this paper, we study cyclic vectors for the forward shift operator and supercyclic vectors for the backward shift operator on the space $H^p(\beta)$.

1. Introduction

Let x be a vector in a Banach space X, and T be an operator on X. The *orbit* of x under T is defined by

$$orb(T, x) = \{T^n x : n = 0, 1, 2, \cdots\}.$$

We recall that a vector x in a separable Banach space X is *cyclic* for an operator T on X if the closed linear span of $orb(T, x)$ is equal to X; it is *supercyclic* if the set of all scalar multiples of the elements of $orb(T, x)$ is dense in X; also it is said to be *hypercyclic* if $orb(T, x)$ is dense in X. An operator T is called a *cyclic*, *hypercyclic*, or *supercyclic operator*, respectively, if it has a cyclic, hypercyclic, or supercyclic vector. Nowadays, the study of these vectors for operators is in progress. For instance, one can see [4, 5, 6, 9, 10, 11, 12]. Suppose that $p > 0$ and $\{\beta(n)\}$ denotes a sequence of positive numbers such that $\beta(0) = 1$. For a sequence $f = \{\hat{f}(n)\}$, we define

$$||f||_p^p = \sum_{n=0}^{\infty} |\hat{f}(n)|^p \beta(n)^p < \infty.$$

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Furthermore, we shall use the formal notation $f(z) = \sum_{n=0}^{\infty} \hat{f}(n)z^n$ regardless whether the series converges for any complex value of z. Throughout this article, by the space $H^p(\beta)$ we mean

$$H^p(\beta) = \{f : f(z) = \sum_{n=0}^{\infty} \hat{f}(n)z^n, ||f||_p < \infty\}.$$

This notation is taken from [14] where $p = 2$.

From now on, p' denstes the complex conjugate of $p > 1$, i.e., $1/p + 1/p' = 1$. Define the finite measure μ on the set of nonnegative integers N_0 by $\mu(K) = \sum_{n \in K} \beta(n)^p, K \subseteq N_0$. Since $H^p(\beta) \cong l^p(\mu)$, we conclude that $H^p(\beta)$ is, indeed, a Banach space. Moreover, it is known that the dual of $l^p(\mu)$, is $(l^p(\mu))^* = l^{p'}(\mu)$, which implies that $(H^p(\beta))^*$, the dual of $H^p(\beta)$, is $H^{p'}(\gamma)$, where $\gamma = \beta^{p/p'}$. For more information on the space $H^p(\beta)$ see [8, 13, 15, 16]. For the sake of completeness, we first recall the following definition.

Definition 1.1. The operator M_z on $H^p(\beta)$ given by $(M_z f)(\xi) = \xi f(\xi)$ is called the *forward shift*; furthermore, the *backward shift* is the operator B on $H^p(\beta)$ given by $(Bf)(z) = f(z) - f(0)/z$.

The conditions for the boundedness of the forward shift and backward shift are given in the following two elementary lemmas.

Lemma 1.2. *If* $\sup_n \beta(n+1)/\beta(n) < \infty$, *then the operator* M_z *is bounded on* $H^p(\beta)$. *Indeed,* $||M_z|| = \sup_n \beta(n+1)/\beta(n)$.

Proof. For $f \in H^p(\beta)$ it is seen that

$$\begin{aligned} ||zf||_p^p &= \sum_{n=0}^{\infty} |(zf)\hat{}(n)|^p \beta(n)^p \\ &= \sum_{n=1}^{\infty} |\hat{f}(n-1)|^p \beta(n)^p \\ &= \sum_{n=0}^{\infty} |\hat{f}(n)|^p \beta(n+1)^p \\ &\leq (\sup_n \frac{\beta(n+1)}{\beta(n)})^p \sum_{n=0}^{\infty} |\hat{f}(n)|^p \beta(n)^p \\ &= (\sup_n \frac{\beta(n+1)}{\beta(n)})^p ||f||_p^p, \end{aligned}$$

and thus $||M_z|| \leq \sup_n \beta(n+1)/\beta(n)$. On the other hand, $||z^{n+1}||_p \leq ||M_z||\ ||z^n||_p$ and so $\beta(n+1) \leq ||M_z||\beta(n)$; hence $\sup_n \beta(n+1)/\beta(n) \leq ||M_z||$ and the result holds. ∎

Lemma 1.3. *If* $\sup_{n\geq 1}\beta(n-1)/\beta(n)<\infty$*, then the operator* B *is bounded on* $H^p(\beta)$*. In fact,* $||B||=\sup_{n\geq 1}\beta(n-1)/\beta(n)$.

Proof. The proof is similar to the previous lemma and so is omitted.

2. Forward Shift on $H^p(\beta)$

Assume that β can be chosen so that $H^p(\beta)$ consists of all analytic functions on the open unit disc $\mathbb{D}$, and the function f in $H^p(\beta)$ is noncyclic if and only if f has a zero in $\mathbb{D}$. In this case, the set of all noncyclic vectors is an open subset of $H^p(\beta)\backslash\{0\}$. The reason is that if $f\in H^p(\beta)\backslash\{0\}$ is noncyclic, then $f(w)=0$ for some w in $\mathbb{D}$. Now, if f is not an interior point of the set of noncyclic vectors in $H^p(\beta)\backslash\{0\}$, then for each n, one can find a cyclic function f_n such that $||f-f_n||<1/n$. Since $f_n\to f$ as $n\to+\infty$ on compact subsets of $\mathbb{D}$, a corollary to Hurwitz Theorem [3] indicates that there exists a positive integer N so that for every $n>N$, f_n has a zero in $\mathbb{D}$. This contradicts the cyclicity of f_n's.

Before stating the next two theorems, we first bring a lemma, useful in their proofs.

Lemma 2.1. *If lim inf* $\beta(n)^{1/n}=||M_z||=1$*, then every function in* $H^p(\beta)$ *is analytic on the open unit disc* $\mathbb{D}$*. Furthermore, the convergence in* $H^p(\beta)$ *implies the uniform convergence on compact subsets of* $\mathbb{D}$.

Proof. Since $1=||M_z||=\sup_n\beta(n+1)/\beta(n)$, we see that

$$\beta(n)\leq\beta(0)=1 \quad \text{for all } n\geq 0. \tag{2.1}$$

Thus,

$$1=\liminf\sqrt[n]{\beta(n)}\leq\limsup\sqrt[n]{\beta(n)}\leq 1,$$

which implies that $\sqrt[n]{\beta(n)}$ converges to 1 as $n\to+\infty$. Now, if $f(z)=\sum_{n=0}^{\infty}\hat{f}(n)z^n$ is in $H^p(\beta)$, then

$$\limsup\sqrt[n]{|\hat{f}(n)|^p}=\limsup\sqrt[n]{|\hat{f}(n)|^p\beta(n)^p}\leq 1.$$

Therefore, $\limsup\sqrt[n]{|\hat{f}(n)|}\leq 1$, which means that the radius of convergence of $f(z)$ is at least 1. Hence, $f(z)$ is analytic on $\mathbb{D}$.

Furthermore, If $f(z)\in H^p(\beta)$, then

$$\begin{aligned}|f(z)|=|\sum_{n=0}^{\infty}\hat{f}(n)z^n| &\leq \left(\sum_{n=0}^{\infty}|\hat{f}(n)|^p\beta(n)^p\right)^{1/p}\left(\sum_{n=0}^{\infty}\frac{|z|^{np'}}{\beta(n)^{p'}}\right)^{1/p'}\\ &= ||f||_p\left(\sum_{n=0}^{\infty}\frac{|z|^{np'}}{\beta(n)^{p'}}\right)^{1/p'}.\end{aligned}$$

The convergence of the series $\sum_{n=0}^{\infty}|z|^{np'}/\beta(n)^{p'}$ for every z with $|z|<1$ completes the proof of the second part of the lemma. ■

Theorem 2.2. *Suppose that* $\liminf \beta(n)^{1/n}=||M_z||=1$. *Then a polynomial* $m(z)$ *is cyclic for* M_z *if and only if* $m(z)$ *has no zero in the open unit disc* $\mathbb{D}$.

Proof. Let $m(z)$ be cyclic. There exists a sequence of polynomials $\{m_n\}$ such that $m_n m \to 1$ in $H^p(\beta)$ and so $m_n(z)m(z)\to 1$ for every $z\in\mathbb{D}$. It follows that $m(z)$ has no zero in $\mathbb{D}$.
For the converse, suppose that $m(z)$ is a polynomial with no zero in $\mathbb{D}$. Without loss of generality, assume that $m(z)=(z-\alpha_1)\cdots(z-\alpha_k)$. Using induction on k, we are going to show that $m(z)$ is cyclic. Let $m(z)=z-\alpha$, and define the isometric isomorphism U from ℓ^p onto $H^p(\beta)$ by

$$U(\{a_j\})=\sum_{j=0}^{\infty}\frac{a_j}{\beta(j)}z^j.$$

Suppose that L is a complex bounded linear functional on $H^p(\beta)$ such that $L(z^n m(z))=0$ for $n=0,1,2,3,\cdots$. Since LU is a bounded linear functional on ℓ^p, there exists a sequence $\{b_j\}_j$ in $\ell^{p'}$ such that

$$(LU)(\{a_j\})=L\left(\sum_{j=0}^{\infty}\frac{a_j}{\beta(j)}z^j\right)=\sum_{j=0}^{\infty}a_j\overline{b}_j. \tag{2.2}$$

Fix n, and choose a sequence $\{a_j\}_{j=0}^{\infty}$ so that $a_n=-\alpha\beta(n), a_{n+1}=\beta(n+1)$, and $a_j=0$ for $j\neq n,n+1$. Then $(LU)(\{a_j\})=L(z^{n+1}-\alpha z^n)=L(z^n m(z))=0$; moreover, (2.2) implies that

$$(LU)(\{a_j\})=\beta(n+1)\overline{b_{n+1}}-\alpha\beta(n)\overline{b_n}.$$

Thus

$$\beta(n+1)\overline{b}_{n+1}-\alpha\beta(n)\overline{b}_n=0,\quad n=0,1,2,\cdots,$$

and consequently,

$$|b_{n+1}|=\frac{\beta(n)}{\beta(n+1)}|\alpha|\,|b_n|,\quad n=0,1,2,\cdots.$$

It follows that $|b_n|=\beta(0)/\beta(n)|\alpha|^n|b_0|$, for every positive integer n.

If $b_0\neq=0$, knowing the fact that $\{b_n\}_{n=0}^{\infty}$ is in $\ell^{p'}$, the above equality says that $\{|\alpha|^n/\beta(n)\}_{n=0}^{\infty}$ is also in $\ell^{p'}$. But it is impossible; because by (2.1), $\beta(n)\leq 1$ for all n and α with $|\alpha|\geq 1$. Hence, $b_n=0$ for all n, which implies that $L=0$.

Using the Hahn–Banach theorem we observe that the polynomial multiples of $m(z)$ are dense in $H^p(\beta)$, and so $m(z)$ is cyclic. Now, by the induction hypothesis, $s(z) = (z-\alpha_1)\cdots(z-\alpha_k)$ is cyclic. Thus, there exists a sequence of polynomials $\{s_n(z)\}_{n=0}^{\infty}$ such that $s_n s \to 1$ in $H^p(\beta)$. Therefore, $s_n(z)m(z) \to z-\alpha_{k+1}$, where $m(z) = (z-\alpha_1)\cdots(z-\alpha_{k+1})$. But $z-\alpha_{k+1}$ is cyclic, and so is $m(z)$. This completes the proof of the assertion of the theorem. ∎

The natural question which now arises is whether, under the hypotheses of Theorem 2.2, every function with no zero in the open unit disc is cyclic for M_z; or, equivalently, is there a noncyclic function in $H^p(\beta)$ so that it never vanishes on $\mathbb{D}$?

In the rest of this section, we are going to discuss this problem and give some sufficient conditions for the existence of these kinds of functions.

Theorem 2.3. *Let $\mathcal{P}_N$ be the set of all polynomials with no zeros in the open unit disc $\mathbb{D}$. Then the closure of $\mathcal{P}_N$ in $H^p(\beta)$ contains many functions other than polynomials which never vanish on $\mathbb{D}$.*

Proof. Choosing the sequence $\{a_n\}$ so that $|a_n| > 2^{n+1}/\beta(n)$ for $n > 0$ and $a_0 = 1$, we observe that for every complex number c with $|c| = 1$,

$$\left|\sum_{k=1}^{n} c^k/a_k\beta(k)\right| < 1.$$

Applying Rouché's theorem [2] to the analytic functions $f(z) = 1$ and $g_n(z) = \sum_{k=1}^{n} z^k/a_k\beta(k)$, we conclude that $h_n(z) = 1 + g_n(z) \in \mathcal{P}_N$.

It is easily seen that the sequence $\{a_n\}$ can be chosen so that $\sum_{n=0}^{\infty}(1/a_n^p) < \infty$. Thus, the function $h(z) = \sum_{n=0}^{\infty}(z^n/a_n\beta(n))$ is in $H^p(\beta)$ and the sequence $\{h_n(z)\}_n$ converges to $h(z)$ in $H^p(\beta)$. To show that $h(z)$ does not have any zero in the open unit disc, let w be any complex number with $|w| < 1$ and $B(w, r)$ be the open disc with center w and radius r whose closure lies in $\mathbb{D}$. Considering the fact that

$$\left|\sum_{n=1}^{\infty} \frac{z^n}{a_n\beta(n)}\right| < 1, \quad |z| < 1\,,$$

and applying Rouché's theorem to the constant function 1 and the function $\sum_{n=1}^{\infty}(z^n/a_n\beta(n))$, we see that $h(z)$ never vanishes on $B(w, r)$. Since this holds for every disc with the closure in $\mathbb{D}$, the result follows. ∎

To obtain the next results, we need to introduce the concept of *bounded point evaluation* for the space $H^p(\beta)$. Recall that for a complex number w, the functional e_w defined on polynomials by $e_w(m(z)) = m(w)$ is called *evaluation at* w. A point w is said to be a *bounded point evaluation* on $H^p(\beta)$ if the functional e_w

can be extended to a bounded linear functional on $H^p(\beta)$. In this case, we denote $e_w(f)$ by $f(w)$ for f in $H^p(\beta)$. Density of polynomials in $H^p(\beta)$ implies the equivalency of the above definition to the existence of a constant $c > 0$ such that $|e_w(m(z))| \leq c||m(z)||_p$ for all polynomials $m(z)$. Since the spaces $H^p(\beta)$ and $l^p(\mu)$ are isometrically isomorphic for a measure μ on nonnegative integers, we conclude that there is a unique element k_w in $H^{p'}(\gamma)$ where $\gamma = \beta^{p/p'}$, such that for all $f \in H^p(\beta)$ we have

$$f(w) = e_w(f) = \sum_{n=0}^{\infty} \hat{f}(n)\overline{\hat{k}_w(n)}\beta(n)^p, \quad ||e_w|| = ||k_w||_{p'}.$$

The element k_w is called the *reproducing kernel at the point* w.

By taking for f the monomial $f_n(z) = z^n$ we obtain

$$\hat{k}_w(n) = \frac{\overline{w}^n}{\beta(n)^p}.$$

Hence w is a bounded point evaluation if and only if

$$||k_w||_{p'}^{p'} = \sum_{n=0}^{\infty} \frac{|w|^{np'}}{\beta(n)^{p'}} < \infty.$$

Theorem 2.4. *Suppose that lim inf $\beta(n)^{1/n} = ||M_z|| = 1$ and G is an open disc in $\mathbb{D}$ which is tangent to $\partial\mathbb{D}$ at 1. If $p \geq 2$, and there exists a positive constant c such that for every w in G,*

$$\sum_{n=0}^{\infty} \frac{|w|^{np'}}{\beta(n)^{p'}} \leq c|\sum_{n=0}^{\infty} \frac{w^n}{\beta(n)^{p'}}|^{p'},$$

then there is a noncyclic function in $H^p(\beta)$ which never vanishes on the open unit disc $\mathbb{D}$.

Proof. First note that the Hardy space H^2 is, indeed, the space $H^2(\beta)$ with $\beta(n) = 1$, for all n. Considering this fact along with (2.1), we observe that H^2 is a subset of $H^2(\beta)$ for every β satisfying the hypothesis of the theorem. Thus $H^\infty \subseteq H^2(\beta)$. Suppose that $f = \sum_{n=0}^{\infty} \hat{f}(n)z^n$ is in $H^p(\beta)$. Then there exists a positive integer N such that $|\hat{f}(n)|^p\beta(n)^p < 1$ for all $n \geq N$. Moreover,

$$|\hat{f}(n)|^p\beta(n)^p \leq |\hat{f}(n)|^2\beta(n)^2$$

for all $n \geq N$, and consequently, $H^\infty \subseteq H^2(\beta) \subseteq H^p(\beta)$.

Now, define $s(z) = exp(z+1)/(z-1)$. Obviously, s is in H^∞ and so is in $H^p(\beta)$. Furthermore, s never vanishes on $\mathbb{D}$. It remains to show that s is noncyclic. On the contrary, assume that it is cyclic. So there exists a sequence of polynomials $\{p_n\}$ such that $p_n s \to 1$ in $H^p(\beta)$ as $n \to +\infty$.

Now, considering the proof of Lemma 2.1, and applying the ratio test, it is easily seen that if $|w| < 1$, then the series $\sum_{n=0}^{\infty} |w|^{np'}/\beta(n)^{p'}$ is convergent; so w is a bounded point evaluation. In fact, $k_w(z) = \sum_{n=0}^{\infty}(\overline{w}^n/\beta(n)^p)z^n$ is the reproducing kernel for $H^p(\beta)$ at w. Consequently, if $f \in H^p(\beta)$, then

$$|f(w)| \leq ||f||_p||k_w||_{p'} = ||f||_p(\sum_{n=0}^{\infty}\frac{|w|^{np'}}{\beta(n)^{p'}})^{1/p'}.$$

Replacing $f(w)$ by $p_n(w)s(w)$, the boundedness of the sequence $\{p_n s\}$ implies the existence of a constant M such that

$$|p_n(w)s(w)| \leq M(\sum_{n=0}^{\infty}\frac{|w|^{np'}}{\beta(n)^{p'}})^{1/p'}. \tag{2.3}$$

For $\delta > 0$, let C_δ be the circle with center $\delta/(1+\delta)$ and radius $1/(1+\delta)$ which is tangent to $\partial\mathbb{D}$ at 1. Choose δ so large that if $w \in C_\delta$ and $w \neq 1$ then $w \in G$. If $w \neq 1$ ranges over the circle C_δ, then $|s(w)| = e^{-\delta}$; thus by (2.3)

$$\begin{aligned} |p_n(w)s(w)| &= e^{-\delta}|p_n(w)| \\ &\leq M(\sum_{j=0}^{\infty}\frac{|w|^{jp'}}{\beta(j)^{p'}})^{1/p'} \\ &\leq Mc^{\frac{1}{p'}}|\sum_{j=0}^{\infty}\frac{w^j}{\beta(j)^{p'}}|. \end{aligned}$$

On the other hand,

$$1 < (\sum_{n=0}^{\infty}\frac{|w|^{np'}}{\beta(n)^{p'}})^{1/p'} \leq c^{\frac{1}{p'}}|\sum_{n=0}^{\infty}\frac{w^n}{\beta(n)^{p'}}|, \quad w \in G.$$

This implies that

$$|p_n(w)(\sum_{j=0}^{\infty}\frac{w^j}{\beta(j)^{p'}})^{-1}| \leq c^{\frac{1}{p'}}Me^{\delta}, \quad n = 1, 2, 3, \cdots.$$

Let G_δ consist of all points inside the circle C_δ, and define

$$f_n(w) = \begin{cases} p_n(w)(\sum_{j=0}^{\infty}\frac{w^j}{\beta(j)^{p'}})^{-1} & \text{if } w \in \overline{G_\delta}\backslash\{1\}, \\ 0 & \text{if } w = 1. \end{cases}$$

It is apparent that f_n is analytic on G_δ and continuous on $\overline{G}_\delta$, and so

$$\sup_{w\in G_\delta}|f_n(w)|\le c^{\frac{1}{p'}}Me^{\delta},\ \ n=1,2,3,\cdots.$$

Since $p_n(w)$ converges to $1/s(w)$ for every w in the open unit disc, we have

$$1\le c^{\frac{1}{p'}}Me^{\delta}|s(w)|\,|\sum_{j=0}^{\infty}\frac{w^j}{\beta(j)^{p'}}|,\quad w\in G_\delta. \tag{2.4}$$

Now, let w range over the set $G_\delta\cap[0,1)$. Putting $h(w)=(w^j/\ \beta(j)^{p'})$ exp$((w+1)/(w-1))$, a straightforward computation shows that

$$\sup_{w\in[0,1)}h(w)=h(\frac{j+1-\sqrt{2j+1}}{j})\le\frac{1}{\beta(j)^{p'}}e^{\frac{2j}{1-\sqrt{2j+1}}+1}.$$

Using the ratio test and considering the fact that $\lim\sqrt[j]{\beta(j)}=1$, it is easily observed that the series $\sum_{j=0}^{\infty}(1/\beta(j)^{p'})$ exp $((2j/(1-\sqrt{2j+1}))+1)$ is convergent, and so $\sum_{j=0}^{\infty}(w^j/\beta(j)^{p'})$ exp $((w+1)/(w-1))$ converges uniformly on [0,1). Therefore, by using Lebesgue's dominated convergence theorem, it follows that

$$\lim_{w\to1^-}s(w)\sum_{j=0}^{\infty}\frac{w^j}{\beta(j)^{p'}}=\sum_{j=0}^{\infty}\lim_{w\to1^-}e^{\frac{w+1}{w-1}}\frac{w^j}{\beta(j)^{p'}}=0,$$

which contradicts (2.4) ■

Example 2.5. Let $\beta(n)=1$ for all $n>0$, and $p=2$. Clearly $\lim\beta(n)^{1/n}=||M_z||=1$. Put

$$f(w)=\sum_{n=0}^{\infty}|\frac{w^{np'}}{\beta(n)^{p'}}|=\sum_{n=0}^{\infty}|w|^{2n}=\frac{1}{1-|w|^2}$$

and

$$g(w)=|\sum_{n=0}^{\infty}\frac{w^n}{\beta(n)^{p'}}|^{p'}=|\sum_{n=0}^{\infty}w^n|^2=\frac{1}{|1-w|^2}$$

for $w\in\mathbb{D}$. Suppose $w=x+iy\ne1$ ranges over the circle C_δ, with center $\delta/(1+\delta)$ and radius $1/(1+\delta),\delta>0$. Then $|w|^2=(1-\delta+2x\delta)/(1+\delta)$ and $|1-w|^2/(1-|w|^2)=1/\delta$; thus $f(w)=g(w)/\delta$. For a fixed $\delta_1>0$ we see that $f(w)\le g(w)/\delta_1$, for all w on C_δ where $\delta\ge\delta_1$. Thus the inequality in the theorem holds for G consisting of all points inside the circle C_{δ_1}, and $c=1/\delta_1$.

3. Backward Shift on $H^p(\beta)$

In this section, we first present necessary and sufficient conditions for a vector in $H^p(\beta)$ to be supercyclic for certain weighted backward shift that we will denote by $\tilde{B}$. Next, we discuss the hypercyclicity of the operator B.

The operator $\tilde{B}$ is defined on $H^p(\beta)$ by

$$\tilde{B}(\sum_{n=0}^{\infty}\hat{f}(n)z^n)=\sum_{n=0}^{\infty}\hat{f}(n+1)\frac{\beta(n+1)^2}{\beta(n)^2}z^n.$$

Similar to the proof of Lemma 1.2 it can be shown that

$$||\tilde{B}||=\sup_{n\geq 1}(\frac{\beta(n)}{\beta(n-1)})^p.$$

Theorem 3.1. *Suppose that $\beta(i+1)\beta(i-1)\leq\beta(i)^2\leq 1$ for all $i\geq 1$, and $\{\beta(i)/\beta(i-1)\}_{i=1}^{\infty}\in\ell^p$. Then $f(z)$ in $H^p(\beta)$ is supercyclic for $\tilde{B}$ if and only if $f(z)$ is not a polynomial.*

Proof. Fix $\varepsilon>0$, and let $f(z)=\sum_{i=0}^{\infty}\hat{f}(i)z^i$ be in $H^p(\beta)$. Choose the integer k so that $\sum_{i=k}^{\infty}\beta(i)^p/\beta(i-1)^p<\varepsilon$. Since $\lim_{i\to+\infty}|\hat{f}(i)|^p\beta(i)^p=0$ there exists an integer n such that $n\geq k$ and $|\hat{f}(n)|^p\beta(n)^p=\max\{|\hat{f}(i)|^p\beta(i)^p$: $i\geq k\}$. Suppose that $f(z)$ is not a polynomial. Then $\hat{f}(n)\neq 0$. Moreover,

$$|\frac{\hat{f}(i)}{\hat{f}(n)}|^p\frac{\beta(i)^p}{\beta(n)^p}\leq 1 \quad \text{for} \quad i\geq k. \tag{3.1}$$

Now, an easy computation shows that

$$((\tilde{B})^n f)(z)=\sum_{i=0}^{\infty}\hat{f}(i+n)\frac{\beta(i+n)^2}{\beta(i)^2}z^i,$$

and so

$$\frac{((\tilde{B})^n f)(z)}{\beta(n)^2\hat{f}(n)}=\sum_{i=1}^{\infty}\frac{\hat{f}(i+n)}{\hat{f}(n)}\frac{\beta(i+n)^2}{\beta(i)^2\beta(n)^2}z^i+1.$$

Put

$$h_n(z)=\sum_{i=1}^{\infty}\frac{\hat{f}(i+n)}{\hat{f}(n)}\frac{\beta(i+n)^2}{\beta(i)^2\beta(n)^2}z^i.$$

Let $Q(i)$ denote the statement $\beta(i-1)\beta(n-1)\leq\beta(i-2)\beta(n)$. Using induction on i, we show that $Q(i)$ holds for every $i\geq n+1$. Clearly, $Q(n+1)$ holds. Suppose that $Q(i)$ holds. Then

$$\begin{aligned}\beta(i)\beta(n-1) &\leq \frac{\beta(i)}{\beta(i-1)}\beta(i-1)\beta(n-1) \leq \frac{\beta(i)}{\beta(i-1)}\beta(i-2)\beta(n) \\ &\leq \frac{\beta(i-1)^2}{\beta(i-1)}\beta(n) \text{ (by hypothesis of the theorem)} \\ &= \beta(i-1)\beta(n).\end{aligned}$$

Thus $Q(i+1)$ holds. Similarly, applying induction it can be shown that

$$\beta(i-j-1)\beta(n-1-j) \leq \beta(n-j)\beta(i-j-2)$$

for all $i \geq n+1$ and $0 \leq j \leq n-2$. Considering these preliminaries all together, we see that

$$\begin{aligned}||h_n||_p^p &= ||\sum_{i=n+1}^{\infty} \tfrac{\hat{f}(i)}{\hat{f}(n)} \tfrac{\beta(i)^2}{\beta(i-n)^2} \tfrac{1}{\beta(n)^2} z^{i-n}||_p^p \\ &= \sum_{i=n+1}^{\infty} |\frac{\hat{f}(i)}{\hat{f}(n)}|^p \frac{\beta(i)^{2p}}{\beta(i-n)^{2p}} \frac{\beta(i-n)^p}{\beta(n)^{2p}} \\ &= \sum_{i=n+1}^{\infty} |\frac{\hat{f}(i)}{\hat{f}(n)}|^p \frac{\beta(i)^p}{\beta(n)^p} \frac{\beta(i)^p}{\beta(i-n)^p} \frac{1}{\beta(n)^p} \\ &\leq \sum_{i=n+1}^{\infty} \frac{\beta(i)^p}{\beta(n)^p \beta(i-n)^p} \qquad \text{(by (3.1))} \\ &= \sum_{i=n+1}^{\infty} (\frac{\beta(i)}{\beta(i-1)})^p (\prod_{j=0}^{n-2} \frac{\beta(i-j-1)\beta(n-1-j)}{\beta(i-j-2)\beta(n-j)})^p \frac{1}{\beta(1)^p} \\ &\leq \sum_{i=n+1}^{\infty} (\frac{\beta(i)}{\beta(i-1)})^p \frac{1}{\beta(1)^p} < \frac{\varepsilon}{\beta(1)^p}.\end{aligned}$$

It follows that $(\tilde{B})^n f/(\beta(n)^2\hat{f}(n))$ converges to 1 in $H^p(\beta)$. Now, let $M_j = \vee_{i=j}^{\infty}\{z^i\}$, $j \geq 1$, and $P_j : H^p(\beta) \to M_j$ be the mapping defined by $P_j(\sum_{i=0}^{\infty} \hat{f}(i)z^i)$ = $\sum_{i=j}^{\infty} \hat{f}(i)z^i$. If B_j is the operator defined on M_j by $B_j f = P_j \tilde{B} f$, then for a fixed $j \geq 1$,

$$B_j z^k = \begin{cases} 0 & \text{if } k = j, \\ \tilde{B} z^k & \text{if } k > j, \end{cases}$$

and so there exists a sequence $\{\gamma_n\}$ of scalars such that $\gamma_n B_j^n f$ converges to z^j as $n \to +\infty$. But $(\tilde{B})^n f = B_j^n f$ for a sufficient large n; hence $\gamma_n(\tilde{B})^n f$ converges to z^j. Now let $\sum_{k=1}^m c_{j_k} z^{j_k}$ be a finite combination of z^j's, $j \geq 0$. So for every $1 \leq k \leq m$ there exists a sequence $\{\gamma_{k,n}\}_n$ such that $\gamma_{k,n}(\tilde{B})^n f$ converges to z^{j_k} as $n \to +\infty$. Thus $(\sum_{k=1}^m c_{j_k}\gamma_{k,n})(\tilde{B})^n f$ converges to $\sum_{k=1}^m c_{j_k} z^{j_k}$. It follows that f is supercyclic for $\tilde{B}$. To prove the converse, if $f(z)$ is a polynomial then $(\tilde{B})^n f = 0$ for a sufficient large n; hence $f(z)$ is not supercyclic for $\tilde{B}$. ∎

Example 3.2. Let $0 < \beta(1) < 1$ be fixed and $\beta(i) = \beta(1)/(i-1)!, i > 1$. If $p = 2$ then it is easily seen that all conditions of the theorem are satisfied.

The following theorem can be considered, in some way, as a generalization of Theorem 3.1 of [10]. It is shown in [7] that when the operator satisfies the Hypercyclicity Criterion and the essential spectrum meets the unit disc, then it has an infinite dimensional Banach space of hypercyclic vectors. That a backward shift satisfies the Hypercyclicity Criterion is shown in [10]. So we give the following result.

Theorem 3.3. *If* $\lim_{n\to+\infty} \beta(n) = 0$ *and* $\limsup \beta(n-1)/\beta(n) = 1$, *then there is an infinite–dimensional Banach space of hypercyclic vectors for the backward shift* B *on* $H^p(\beta)$.

A question which now arises and we study in the rest of this paper is: Which operators in the commutant of the backward shift B on $H^p(\beta)$, denoted by $\{B\}'$, are hypercyclic?

Theorem 3.4. *If* $0 \neq A \in \{B\}'$ *such that* $A1 = 0$, *then there is a dense subset* $X \subseteq H^p(\beta)$ *and a right inverse* R *for* $A(AR = I_X$, *the identity on* $X)$ *such that* $||A^n x|| \to 0$ *for every* $x \in X$.

Proof. Let $f_k(z) = z^k$ for every $k \geq 0$. Indeed, the space X is the linear span of $f_k, k \geq 0$. To prove that $||A^n x|| \to 0$ for every $x \in X$, it is enough to show that $A^k(f_k) = 0$ for all k. Assume that this is true for all $j < k$, and since $BA^{k-1}(f_k) = A^{k-1}B(f_k) = A^{k-1}f_{k-1} = 0$ it follows that $A^{k-1}(f_k) = \lambda$ for a constant λ and therefore $A^k(f_k) = AA^{k-1}(f_k) = 0$. To prove that there is a right inverse R for A, let n be the smallest integer such that $Af_n(0) \neq 0$ (this n exists because $A \neq 0$); thus

$$Af_n = \sum_{k=0}^{n} Af_k(0)B^k f_n = Af_n(0),$$

and so $Ag_0 = 1$, where $g_0 = f_n/Af_n(0)$. Suppose that there exists an element g_i in $H^p(\beta)$ such that $Ag_i = f_i$ for $0 \leq i \leq m$. Now,

$$Af_{n+m+1} = \sum_{k=0}^{n+m+1} Af_k(0)B^k f_{n+m+1} = \sum_{i=0}^{m+1} Af_{n+i}(0)f_{m+1-i}\,.$$

Thus,

$$\begin{aligned} f_{m+1} &= \frac{Af_{n+m+1}}{Af_n(0)} - \sum_{i=1}^{m+1} \frac{Af_{n+i}(0)}{Af_n(0)} f_{m+1-i} \\ &= \frac{Af_{n+m+1}}{Af_n(0)} - \sum_{i=1}^{m+1} \frac{Af_{n+1}(0)}{Af_n(0)} Ag_{m+1-i} \\ &= A\Big(\frac{f_{n+m+1}}{Af_n(0)} - \sum_{i=1}^{m+1} \frac{Af_{n+1}(0)}{Af_n(0)} g_{m+1-i}\Big). \end{aligned}$$

Put

$$g_{m+1} = \frac{f_{n+m+1}}{Af_n(0)} - \sum_{i=1}^{m+1} \frac{Af_{n+i}(0)}{Af_n(0)} g_{m+1-i}. \tag{3.2}$$

Hence by induction we conclude that there exists $g_i \in H^p(\beta)$ such that $Ag_i = f_i$ for every $i \geq 0$. If $Rf_i = g_i, i \geq 0$, then $AR = I_X$. ∎

Corollary 3.5. *If* $\lim_{n\to+\infty} \beta(n) = 0$ *then the operator* $A = B^i$ *is hypercyclic for every* $i \geq 1$.

Proof. Let f_m and R be as in the proof of the previous theorem. By the Hypercyclicity Criterion ([10] or [11]) and Theorem 3.4 it is sufficient to show that $\lim_{n\to+\infty} ||R^n f_m|| = 0$ for every $m \geq 0$. Applying (3.2) we have $Rf_m = g_m = f_{m+i}$, for $m \geq 0$ and so $R^n f_m = f_{m+in}$. Therefore, $\lim_{n\to+\infty} ||R^n f_m|| = \lim_{n\to+\infty} \beta(m+in) = 0$. ∎

Remark. Note that another proof of the previous corollary is obtained by considering a result in [12] and a result of S. Ansari [1].

Corollary 3.6. *For a nonzero constant* α *and* $i \geq 1$, *if* $\lim_{n\to+\infty} \beta(n)/\alpha^{n/i} = 0$ *then* $A = \alpha B^i$ *is hypercyclic.*

Proof. Applying (3.2), we have $Rf_m = g_m = f_{i+m}/\alpha$, for $m \geq 0$ and so $R^n f_m = f_{in+m}/\alpha^n$. Hence

$$\lim_{n\to+\infty} ||R^n f_m|| = \lim_{n\to+\infty} \frac{\beta(m+ni)}{\alpha^n}$$

$$= \lim_{n\to\infty} \frac{\beta(m+ni)}{\alpha^{\frac{m+ni}{i}}} \frac{\alpha^{\frac{m+ni}{i}}}{\alpha^n} = \lim_{n\to+\infty} \frac{\beta(m+ni)}{\alpha^{\frac{m+ni}{i}}} \alpha^{\frac{m}{i}} = 0. \qquad ∎$$

Remark. If $|\alpha| > 1$ and $\beta(n) = 1$ for all n, then we conclude that αB^i satisfies the Hypercyclicity Criterion. For $i = 1$ this was proved by Gethner and Shapiro [5].

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