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MULTIDIMENSIONAL EXTENSIONS OF THE BERNOULLI AND APPELL POLYNOMIALS

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Abstract. Multidimensional extensions of the Bernoulli and Appell polynomials are defined generalizing the corresponding generating functions, and using the Hermite-Kampé de Fériet (or Gould-Hopper) polynomials. Furthermore the differential equations satisfied by the corresponding 2D polynomials are derived exploiting the factorization method, introduced in [15].

1. Introduction

The Hermite-Kampé de Fériet (or Gould-Hopper) polynomials [2, 11, 18], have been recently used in order to construct addition formulas for different classes of generalized Gegenbauer polynomials [6].

They are defined by the generating function:

$$e^{xt+yt^j} = \sum_{n=0}^{\infty} H_n^{(j)}(x,y)\frac{t^n}{n!} \tag{1.1}$$

or by the explicit form

$$H_n^{(j)}(x,y) = n!\sum_{s=0}^{[\frac{n}{j}]} \frac{x^{n-js}y^s}{(n-js)!s!}, \tag{1.2}$$

where $j \geq 2$ is an integer. The case when $j = 1$ is not considered, since the corresponding 2D polynomials are simply expressed by the Newton binomial formula.

It is worth recalling that the polynomials $H_n^{(j)}(x,y)$ are a natural solution of the generalized heat equation:

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$$\begin{cases} \dfrac{\partial}{\partial y} F(x,y) = \dfrac{\partial^j}{\partial x^j} F(x,y), \\ F(x,0) = x^n. \end{cases}$$

The case when $j = 2$ is then particularly important (see [20]), and it was recently used in order to define 2D extensions of the Bernoulli and Euler polynomials [8].

Further generalizations including the $H_n^{(j)}(x,y)$ polynomials as a particular case, are given by

$$e^{x_1 t + x_2 t^2 + \cdots + x_r t^r} = \sum_{n=0}^{\infty} H_n(x_1, x_2, \ldots, x_r) \frac{t^n}{n!}. \tag{1.3}$$

Note that the generating function of eq. (1.3) can be written in the form:

$$\begin{aligned} e^{x_1 t + x_2 t^2 + \cdots + x_r t^r} &= \sum_{k=0}^{\infty} \frac{\left(x_1 t + x_2 t^2 + \cdots + x_r t^r\right)^k}{k!} \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{k_1 + k_2 + \cdots + k_r = k} \frac{k!}{k_1! k_2! \cdots k_r!} x_1^{k_1} x_2^{k_2} \cdots x_r^{k_r} t^{k_1 + 2k_2 + \cdots + r k_r} \qquad (1.4) \\ &= \sum_{n=0}^{\infty} \left(\sum_{\pi_k(n|r)} n! \frac{x_1^{k_1} x_2^{k_2} \cdots x_r^{k_r}}{k_1! k_2! \cdots k_r!} \right) \frac{t^n}{n!}, \end{aligned}$$

where $k := k_1 + k_2 + \cdots + k_r$, $n := k_1 + 2k_2 + \cdots + r k_r$, and the sum runs over all the restricted partitions $\pi_k(n|r)$ (containing at most r sizes) of the integer n, k denoting the number of parts of the partition and k_i the number of parts of size i. Note that, using the ordinary notation for the partitions of n, i.e. $n = k_1 + 2k_2 + \cdots + n k_n$, we have to assume $k_{r+1} = k_{r+2} = \cdots = k_n = 0$.

From eq. (1.4) the following explicit form of the multidimensional Hermite-Kampé de Fériet polynomials follows

$$H_n(x_1, x_2, \ldots, x_r) = \sum_{\pi_k(n|r)} n! \frac{x_1^{k_1} x_2^{k_2} \cdots x_r^{k_r}}{k_1! k_2! \cdots k_r!}.$$

Furthermore, they satisfy for all n the *isobaric property* (of weight n):

$$H_n(t x_1, t^2 x_2, \ldots, t^r x_r) = t^n H_n(x_1, x_2, \ldots, x_r) \tag{1.5}$$

and consequently, they are solutions of the first order partial differential equation:

$$x_1 \frac{\partial H_n}{\partial x_1} + 2 x_2 \frac{\partial H_n}{\partial x_2} + \cdots + r x_r \frac{\partial H_n}{\partial x_r} = n H_n. \tag{1.6}$$

The multivariate Hermite-Kampé de Fériet polynomials appear as an interesting tool for introducing and studying multidimensional generalizations of the Appell polynomials too, including the Bernoulli and Euler ones, starting from the corresponding generating functions. A first approach in this direction was given in [7].

In this article, we will study in some detail properties of the generalized 2D Appell polynomials, considering first the case of the 2D Bernoulli polynomials, in order to introduce the subject in a more friendly way. The relevant extensions to the multidimensional Bernoulli and Appell case can be derived almost straightforwardly, but the relevant equations are rather involved.

We will show that for every integer $j \geq 2$ it is possible to define a class of 2D Bernoulli polynomials denoted by $B_n^{(j)}(x,y)$ generalizing the classical Bernoulli polynomials.

Furthermore, in sect. 5, the bivariate Appell polynomials $R_n^{(j)}(x,y)$ are introduced, by means of the generating function

$$A(t)\ e^{xt+yt^j} = \sum_{n=0}^{\infty} R_n^{(j)}(x,y)\frac{t^n}{n!}.$$

Exploiting the factorization method, introduced in [15] and recalled in [13], we derive differential equations satisfied by these 2D polynomials. The differential equation for the classical Appell polynomials was first obtained in [17], and recently recovered in [14], by using the factorization method.

It is worth noting that, in general, the differential operators satisfied by the multidimensional Appell polynomials are of infinite order. This is a quite general situation, since the Appell type polynomials satisfying a differential operator of finite order can be considered as an exceptional case [7].

In a forthcoming article we will consider further generalizations as the multi-index polynomials defined by means of the generating functions

$$A(t,\tau)e^{xt^l+y\tau^j} = \sum_{n,m}^{0,+\infty} R_{n,m}^{(l,j)}(x,y)\frac{t^n}{n!}\frac{\tau^m}{m!}$$

or, more generally:

$$A(t_1,\ldots,t_r)e^{x_1t_1^{j_1}+\cdots+x_rt_r^{j_r}} = \sum_{n_1,\ldots,n_r}^{0,+\infty} R_{n_1,\ldots,n_r}^{(j_1,\ldots,j_r)}(x_1,\ldots,x_r)\frac{t_1^{n_1}}{n_1!}\cdots\frac{t_r^{n_r}}{n_r!},$$

which belong to the set of multidimensional special functions recently introduced by G. Dattoli and his group.

2. Recalling Bernoulli and Appell Polynomials

The Bernoulli polynomials $B_n(x)$ are defined (see [12], p. xxix) by the generating function:

$$G(x,t) := \frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \qquad |t| < 2\pi, \tag{2.1}$$

and consequently, the Bernoulli numbers $B_n := B_n(0)$ satisfy

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}. \tag{2.2}$$

It is well known that

$$\begin{aligned} &B_n(0) = B_n(1) = B_n, \quad n \neq 1, \\ &B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k}, \\ &B_n'(x) = nB_{n-1}(x). \end{aligned}$$

The Bernoulli numbers (see [4-16]) enter in many mathematical formulas, such as

- the Taylor expansion in a neighborhood of the origin of the circular and hyperbolic tangent and cotangent functions,
- the sums of powers of natural numbers,
- the residual term of the Euler-MacLaurin quadrature rule.

The Bernoulli polynomials, first studied by Euler [10], are employed in the integral representation of differentiable periodic functions, since they are employed for approximating such functions in terms of polynomials. They are also used for representing the remainder term of the composite Euler-MacLaurin quadrature rule (see [19]).

The Appell polynomials [1] can be defined by the generating function

$$G_A(x,t) = A(t)\ e^{xt} = \sum_{n=0}^{\infty} \frac{R_n(x)}{n!} t^n \tag{2.3}$$

where

$$A(t) = \sum_{k=0}^{\infty} \frac{\mathcal{R}_k}{k!} t^k, \ (A(0) \neq 0) \tag{2.4}$$

is analytic function at $t = 0$, and $R_k := R_k(0)$.

It is easy to see that for any $A(t)$ the derivatives of $R_n(x)$ satisfy

$$R_n'(x) = nR_{n-1}(x), \tag{2.5}$$

and furthermore

- if $A(t) = \frac{t}{e^t - 1}$, then $R_n(x) = B_n(x)$,
- if $A(t) = \frac{2}{e^t + 1}$, then $R_n(x) = E_n(x)$, i.e. the Euler polynomials,
- if $A(t) = \alpha_1 \cdots \alpha_m t^m [(e^{\alpha_1 t} - 1) \cdots (e^{\alpha_m t} - 1)]^{-1}$, then the $R_n(x)$ are the Bernoulli polynomials of order m [3],
- if $A(t) = 2^m [(e^{\alpha_1 t} + 1) \cdots (e^{\alpha_m t} + 1)]^{-1}$, then the $R_n(x)$ are the Euler polynomials of order m [3],
- If $A(t) = e^{\xi_0 + \xi_1 t + \cdots \xi_{d+1} t^{d+1}}, \xi_{d+1} \neq 0$, then the $R_n(x)$ are the generalized Gould-Hopper polynomials [9] including the Hermite polynomials when $d = 1$ and classical 2-orthogonal polynomials when $d = 2$.

3. The 2D Bernoulli Polynomials $B_n^{(2)}(x, y)$

Starting from the Hermite–Kampé de Fériet (or Gould-Hopper) polynomials $H_n^{(2)}(x, y)$, we define the 2D Bernoulli polynomials $B_n^{(2)}(x, y)$ by means of the generating function:

$$G^{(2)}(x, y; t) := \frac{t}{e^t - 1} e^{xt + yt^2} = \sum_{n=0}^{\infty} B_n^{(2)}(x, y) \frac{t^n}{n!} \tag{3.1}$$

It is possible to find the explicit form of the polynomials $B_n^{(2)}(x, y)$ in terms of the Hermite–Kampé de Fériet polynomials $H_n^{(2)}(x, y)$ and vice-versa.

Theorem 3.1. *The following representation formulas hold true:*

$$\begin{aligned} B_n^{(2)}(x, y) &= \sum_{h=0}^{n} \binom{n}{h} B_{n-h} H_h^{(2)}(x, y) = \\ &= n! \sum_{h=0}^{n} \frac{B_{n-h}}{(n-h)!} \sum_{s=0}^{[\frac{h}{2}]} \frac{x^{h-2s} y^s}{(h-2s)! s!}, \end{aligned} \tag{3.2}$$

where B_k *denote the Bernoulli numbers;*

$$H_n^{(2)}(x,y) = \sum_{h=0}^{n} \binom{n}{h} \frac{1}{n-h+1} B_h^{(2)}(x,y). \tag{3.3}$$

Proof. Eq. (3.2) is obtained starting from the generating function (3.1) by using the Cauchy product of the series expansions (2.2) and (1.1), for $j = 2$, and then using the identity principle of power series.

Eq. (3.3) is obtained in the same way, starting from the equation

$$e^{xt+yt^2} = \frac{e^t - 1}{t} \sum_{n=0}^{\infty} B_n^{(2)}(x,y) \frac{t^n}{n!}. \qquad \blacksquare$$

A recurrence relation for the polynomials $B_n^{(2)}$ is given by the following theorem

Theorem 3.4. *For any integral $n \geq 1$ the following linear homogeneous recurrence relation for the generalized Bernoulli polynomials $B_n^{(2)}(x,y)$ holds true:*

$$\begin{aligned} B_0^{(2)}(x,y) &= 1, \\ B_n^{(2)}(x,y) &= -\tfrac{1}{n} \sum_{k=0}^{n-2} \binom{n}{k} B_{n-k} B_k^{(2)}(x,y) + \left(x - \frac{1}{2}\right) B_{n-1}^{(2)}(x,y) \\ &\quad + 2(n-1) y B_{n-2}^{(2)}(x,y), \end{aligned} \tag{3.4}$$

where B_h denote the Bernoulli numbers.

Proof. Differentiating both sides of eq. (3.1) with respect to t, recalling the generating functions (2.2) of the Bernoulli numbers, and using some elementary algebra and the identity principle of power series, recursion (3.4) easily follows. $\blacksquare$

We are now in condition to prove the following theorem, which gives differential equations satisfied by the $B_n^{(2)}(x,y)$:

Theorem 3.4. *The 2D Bernoulli polynomials $B_n^{(2)}(x,y)$ satisfy the differential or integro-differential equations:*

$$\begin{aligned} &\left[\frac{B_n}{n!} D_x^n + \cdots + \frac{B_4}{4!} D_x^4 \right. \\ &\left. + \left(\frac{B_2}{2!} - 2y \right) D_x^2 + \left(\frac{1}{2} - x \right) D_x + n \right] B_n^{(2)}(x,y) = 0, \end{aligned} \tag{3.5}$$

$$\begin{aligned} &\left[\left(x - \frac{1}{2} \right) D_y + 2 D_x^{-1} D_y + 2y D_x^{-1} D_y^2 \right. \\ &\left. - \sum_{k=1}^{n-1} \frac{B_{n-k+1}}{(n-k+1)!} D_x^{-(n-k)} D_y^{n-k+1} - (n+1) D_x \right] B_n^{(2)}(x,y) = 0, \end{aligned} \tag{3.6}$$

$$\begin{aligned}(3.7)\qquad &\Bigg[\left(x-\frac{1}{2}\right)D_x^{(n-1)}D_y+(n-1)D_x^{n-2}D_y+2D_x^{(n-2)}\left(D_y+yD_y^2\right)\\ &-\sum_{k=1}^{n-1}\frac{B_{n-k+1}}{(n-k+1)!}D_x^{k-1}D_y^{n-k+1}-(n+1)D_x^n\Bigg]B_n^{(2)}(x,y)=0,(n\geq 2).\end{aligned}$$

Proof. We start proving eq. (3.5), which gives a differential equation satisfied by the $B_n^{(2)}(x,y)$ with respect to the x variable, assuming y as a parameter.

It is easily seen that a lowering operator L_n^- for the polynomials $B_n^{(2)}(x,y)$ is given by

$$L_n^-=\frac{1}{n}\frac{\partial}{\partial x}.$$

In fact, for any fixed y, the $B_n^{(2)}(x,y)$ belong to the Appell class, assuming in eq. (2.3) $A(t)=\frac{t}{e^t-1}e^{yt^2}$, and therefore eq. (2.5) holds true.

Furthermore, by using the recurrence relation of Theorem 3.2, we find the corresponding increasing operator L_n^+ (see [13], which is given by

$$L_n^+=\left(x-\frac{1}{2}\right)+2yD_x-\sum_{k=0}^{n-1}\frac{B_{n-k+1}}{(n-k+1)!}D_x^{n-k}.$$

Then, by exploiting the factorization method (see [15], [13]), equation (3.5) immediately follows using

$$L_{n+1}^-L_n^+B_n^{(2)}(x,y)=B_n^{(2)}(x,y).$$

In order to find the integro-differential equation (3.6), including derivatives with respect to y, note that differentiating the generating function (1.1) (assuming $j=2$), with respect to y, yields:

$$\frac{\partial B_n^{(2)}(x,y)}{\partial y}=n(n-1)B_{n-2}^{(2)}(x,y)=n\frac{\partial B_{n-1}^{(2)}(x,y)}{\partial x}, \tag{3.8}$$

so that

$$D_x^{-1}D_yB_n^{(2)}(x,y)=nB_{n-1}^{(2)}(x,y) \tag{3.9}$$

and therefore, we can assume:

$$\mathcal{L}_n^-:=\frac{1}{n}D_x^{-1}D_y. \tag{3.10}$$

Using again the recurrence relation of Theorem 3.2, we obtain the corresponding increasing operator $\mathcal{L}_n^+$:

$$\mathcal{L}_n^+:=\left(x-\frac{1}{2}\right)+2yD_x^{-1}D_y-\sum_{k=0}^{n-1}\frac{B_{n-k+1}}{(n-k+1)!}D_x^{-(n-k)}D_y^{n-k} \tag{3.11}$$

and consequently the integro-differential equation (3.6) follows.

Therefore, the differential equation (3.7) immediately follows differentiating both sides of eq. (3.6) $(n-1)$-times with respect to x. ∎

4. The 2D Bernoulli Polynomials $B_n^{(j)}(x,y)$

In a similar way, starting from the Hermite–Kampé de Fériet (or Gould-Hopper) polynomials $H_n^{(j)}(x,y)$, we define the 2D Bernoulli polynomials $B_n^{(j)}(x,y)$ by means of the generating function:

$$G^{(j)}(x,y;t) := \frac{t}{e^t-1} e^{xt+yt^j} = \sum_{n=0}^{\infty} B_n^{(j)}(x,y)\frac{t^n}{n!} \tag{4.1}$$

It is worth noting that the polynomial $H_n^{(j)}(x,y)$, being *isobaric of weight* n, cannot contain the variable y, for every $n = 0, 1, ..., j-1$.

Using the same procedure as before, the following results for the $B_n^{(j)}(x,y)$ polynomials can be easily derived.

- **Explicit forms of the polynomials $B_n^{(j)}$ in terms of the Hermite–Kampé de Fériet polynomials $H_n^{(j)}$ and vice-versa.**

Theorem 4.1. *The following representation formulas hold true:*

$$\begin{aligned} B_n^{(j)}(x,y) &= \sum_{h=0}^{n} \binom{n}{h} B_{n-h} H_h^{(j)}(x,y) \\ &= n! \sum_{h=0}^{n} \frac{B_{n-h}}{(n-h)!} \sum_{r=0}^{[\frac{h}{j}]} \frac{x^{h-jr} y^r}{(h-jr)!r!}, \end{aligned}$$

where B_k *denote the Bernoulli numbers*:

$$H_n^{(j)}(x,y) = \sum_{h=0}^{n} \binom{n}{h} \frac{1}{n-h+1} B_h^{(j)}(x,y).$$

- **Recurrence relation.**

Theorem 4.2. *For any integral $n \geq 1$ the following linear homogeneous recurrence relation for the generalized Bernoulli polynomials $B_n^{(j)}(x,y)$ holds true:*

$$\begin{aligned} B_0^{(j)}(x,y) &= 1, \\ B_n^{(j)}(x,y) &= -\tfrac{1}{n} \sum_{k=0}^{n-2} \binom{n}{k} B_{n-k} B_k^{(j)}(x,y) \\ &+ \left(x - \frac{1}{2}\right) B_{n-1}^{(j)}(x,y) + jy\frac{(n-1)!}{(n-j)!} B_{n-j}^{(j)}(x,y). \end{aligned}$$

• **Shift operators.**

$$L_n^- := \frac{1}{n} D_x$$

$$L_n^+ := \left(x - \frac{1}{2}\right) - \sum_{k=0}^{n-1} \frac{B_{n-k+1}}{(n-k+1)!} D_x^{n-k} + jyD_x^{j-1}$$

$$\mathcal{L}_n^- := \frac{1}{n} D_x^{-(j-1)} D_y$$

$$\mathcal{L}_n^+ := \left(x - \frac{1}{2}\right) + jyD_x^{-(j-1)^2} D_y^{j-1} - \sum_{k=0}^{n-1} \frac{B_{n-k+1}}{(n-k+1)!} D_x^{-(j-1)(n-k)} D_y^{n-k}$$

• **Differential or integro-differential equations.**

Theorem 4.3. *The 2D Bernoulli polynomials* $B_n^{(j)}(x,y)$ *satisfy the differential or integro-differential equations:*

$$\left[\frac{B_n}{n!} D_x^n + \cdots + \frac{B_{j+1}}{(j+1)!} D_x^{j+1} + \left(\frac{B_j}{j!} - jy\right) D_x^j\right.$$
$$\left. + \frac{B_{j-1}}{(j-1)!} D_x^{j-1} + \cdots + \left(\frac{1}{2} - x\right) D_x + n\right] B_n^{(j)}(x,y) = 0,$$

$$\left[\left(x - \frac{1}{2}\right) D_y + jD_x^{-(j-1)^2} D_y^{j-1}\right.$$
$$+ jyD_x^{-(j-1)^2} D_y^j - \sum_{k=1}^{n-1} \frac{B_{n-k+1}}{(n-k+1)!} D_x^{-(j-1)(n-k)} D_y^{n-k+1}$$
$$\left. -(n+1) D_x^{(j-1)}\right] B_n^{(j)}(x,y) = 0,$$

$$\left[\left(x - \frac{1}{2}\right) D_x^{(j-1)(n-1)} D_y + (j-1)(n-1) D_x^{(j-1)(n-1)-1} D_y\right.$$
$$+ jD_x^{(j-1)(n-j)} \left(D_y^{j-1} + yD_y^j\right) - \sum_{k=1}^{n-1} \frac{B_{n-k+1}}{(n-k+1)!} D_x^{(j-1)(k-1)} D_y^{n-k+1}$$
$$\left. -(n+1) D_x^{(j-1)n}\right] B_n^{(j)}(x,y) = 0, \quad (n \geq j).$$

Note that the last equation is derived by differentiating $(j-1)(n-1)$-times with respect to x both sides of the preceding one, and does not contain anti-derivatives for $n \geq j$.

5. 2D Appell Polynomials

For any $j \geq 2$, the 2D Appell polynomials $R_n^{(j)}(x,y)$ are defined by means of the generating function:

$$G_A^{(j)}(x,y;t) := A(t)\ e^{xt+yt^j} = \sum_{n=0}^{\infty} R_n^{(j)}(x,y)\frac{t^n}{n!} \tag{5.1}$$

Even in this more general case, the polynomial $R_n^{(j)}(x,y)$, is *isobaric of weight* n, so that it does not contain the variable y, for every $n = 0, 1, ..., j-1$.

- **Explicit forms of the polynomials $R_n^{(j)}$ in terms of the Hermite–Kampé de Fériet polynomials $H_n^{(j)}$ and vice-versa.**

Theorem 5.1. *The following representation formulas hold true:*

$$\begin{aligned} R_n^{(j)}(x,y) &= \sum_{h=0}^{n} \binom{n}{h} \mathcal{R}_{n-h} H_h^{(j)}(x,y) \\ &= n! \sum_{h=0}^{n} \frac{\mathcal{R}_{n-h}}{(n-h)!} \sum_{r=0}^{[\frac{h}{j}]} \frac{x^{h-jr}y^r}{(h-jr)!r!}, \end{aligned}$$

where the $\mathcal{R}_k$ are the "Appell numbers" appearing in eq. (2.4),

$$H_n^{(j)}(x,y) = \sum_{k=0}^{n} \binom{n}{k} Q_{n-k} R_k^{(j)}(x,y),$$

where the Q_k are the coefficients of the Taylor expansion in a neighborhood of the origin of the reciprocal function $1/A(t)$.

- **Recurrence relation.**

It is suitable to introduce the coefficients of the Taylor expansion:

$$\frac{A'(t)}{A(t)} = \sum_{n=0}^{\infty} \alpha_n \frac{t^n}{n!}. \tag{5.2}$$

Theorem 5.2. *For any integral $n \geq 1$ the following linear homogeneous recurrence relation for the generalized Appell polynomials $R_n^{(j)}(x,y)$ holds true:*

$$\begin{aligned} R_0^{(j)}(x,y) &= 1, \\ R_n^{(j)}(x,y) &= (x+\alpha_0)R_{n-1}^{(j)}(x,y) + \binom{n-1}{j-1} jy\ R_{n-j}^{(j)}(x,y) \\ &+ \sum_{k=0}^{n-2} \binom{n-1}{k} \alpha_{n-k-1} R_k^{(j)}(x,y). \end{aligned}$$

• **Shift operators.**

$$
\begin{aligned}
L_n^- : &= \frac{1}{n} D_x \\
L_n^+ : &= (x+\alpha_0) + \frac{j}{(j-1)!}\, y\, D_x^{j-1} + \sum_{k=0}^{n-1} \frac{\alpha_{n-k}}{(n-k)!}\, D_x^{n-k} \\
\mathcal{L}_n^- : &= \frac{1}{n} D_x^{-(j-1)} D_y \\
\mathcal{L}_n^+ : &= (x+\alpha_0) + \frac{j}{(j-1)!}\, y\, D_x^{-(j-1)^2} D_y^{j-1} \\
&\quad + \sum_{k=0}^{n-1} \frac{\alpha_{n-k}}{(n-k)!}\, D_x^{-(j-1)(n-k)} D_y^{n-k}.
\end{aligned}
$$

• **Differential or integro-differential equations.**

Theorem 5.3. *The 2D Appell polynomials* $R_n^{(j)}(x,y)$ *satisfy the differential or integro-differential equations:*

$$
\left[\frac{\alpha_{n-1}}{(n-1)!} D_x^n + \cdots + \frac{\alpha_j}{j!} D_x^{j+1} + \left(\frac{\alpha_{j-1} + jy}{(j-1)!}\right) D_x^j \right.
$$

$$
\left. + \frac{\alpha_{j-2}}{(j-2)!} D_x^{j-1} + \cdots + (x+\alpha_0)\, D_x - n \right] R_n^{(j)}(x,y) = 0,
$$

$$
\left[(x+\alpha_0)\, D_y + \frac{j}{(j-1)!} D_x^{-(j-1)^2} \left(y D_y^j + D_y^{j-1}\right)\right.
$$

$$
\left. + \sum_{k=1}^{n-1} \frac{\alpha_{n-k}}{(n-k)!} D_x^{-(j-1)(n-k)} D_y^{n-k+1} - (n+1) D_x^{j-1} \right] R_n^{(j)}(x,y) = 0,
$$

$$
\left[(x+\alpha_0)\, D_x^{(j-1)(n-1)} D_y + (j-1)(n-1) D_x^{(j-1)(n-1)-1} D_y \right.
$$

$$
+ \frac{j}{(j-1)!} D_x^{(j-1)(n-j)} \left(y D_y^j + D_y^{j-1}\right) + \sum_{k=1}^{n-1} \frac{\alpha_{n-k}}{(n-k)!} D_x^{(j-1)(k-1)} D_y^{n-k+1}
$$

$$
\left. -(n+1) D_x^{n(j-1)} \right] R_n^{(j)}(x,y) = 0, \quad (n \geq j).
$$

6. Example

We give in this section a particular example of 2D Appell polynomials, corresponding to a suitable choice of the function $A(t)$, already considered in [5].

Fix the integral N, and define:

$$A(t) := \frac{1}{e}\exp\left(\exp(-\frac{t}{N})\right),$$

so that the normalizing condition $A(0) = \mathcal{R}_0 = 1$ is satisfied.

Then, recalling eq. (5.2), the numerical values

$$\alpha_k = \frac{(-1)^{k+1}}{N^{k+1}}$$

are easily found. Furthermore the Appell numbers can be computed by means of the recurrence relation:

$$\mathcal{R}_{h+1} = -\frac{1}{N}\sum_{k=0}^{h}\binom{h}{k}\frac{(-1)^{h-k}}{N^{h-k}}\mathcal{R}_k$$

The first values of the $\mathcal{R}_k$ are consequently:

$$\mathcal{R}_0 = A(0) = 1, \quad \mathcal{R}_1 = -\frac{1}{N}, \quad \mathcal{R}_2 = \frac{2}{N^2}, \quad \mathcal{R}_3 = -\frac{5}{N^3}, \quad \ldots$$

and so on.

Assuming $j = 2$, the first 2D relevant Appell polynomials are

$$\begin{aligned}
R_0^{(2)}(x,y) &= 1,\\
R_1^{(2)}(x,y) &= x - \frac{1}{N},\\
R_2^{(2)}(x,y) &= x^2 - \frac{2}{N}x + 2y + \frac{2}{N^2},\\
R_3^{(2)}(x,y) &= x^3 - \frac{3}{N}x^2 + 6xy + \frac{6}{N^2}x - \frac{6}{N}y - \frac{5}{N^3},
\end{aligned}$$

and so on.

Following methods of the above sections we have found the recurrence relation:

$$\begin{aligned}
R_0^{(2)}(x,y) &= 1,\\
R_n^{(2)}(x,y) &= \left(x - \tfrac{1}{n}\right)R_{n-1}^{(2)}(x,y) + 2(n-1)yR_{n-2}^{(2)}(x,y)\\
&\quad + \sum_{k=0}^{n-1}\frac{(-1)^{n-k}}{N^{n-k}}R_k^{(2)}(x,y),
\end{aligned}$$

and the differential equations:

$$\left[\frac{(-1)^n}{(n-1)!\ N^n}\ D_x^n + \cdots - \frac{1}{2!\ N^3} D_x^3 + \left(\frac{1}{N^2} + 2y\right) D_x^2 + \left(x - \frac{1}{N}\right) D_x - n\right] R_n^{(2)}(x,y) = 0,$$

and, for $n \geq 2$:

$$\left[\left(x - \frac{1}{N}\right) D_x^{n-1} D_y + (n-1) D_x^{n-2} D_y + 2 D_x^{n-2}\left(y D_y^2 + D_y\right) + \frac{(-1)^n}{(n-1)!\ N^n} D_y^n + \cdots + \frac{1}{N^2} D_x^{n-2} D_y^2 - (n+1) D_x^n\right] R_n^{(2)}(x,y) = 0.$$

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