

COMMUTING MAPS: A SURVEY

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Abstract. A map f on a ring \mathcal{A} is said to be commuting if $f(x)$ commutes with x for every $x \in \mathcal{A}$. The paper surveys the development of the theory of commuting maps and their applications. The following topics are discussed: commuting derivations, commuting additive maps, commuting traces of multi-additive maps, various generalizations of the notion of a commuting map, and applications of results on commuting maps to different areas, in particular to Lie theory.

1. INTRODUCTION

Let \mathcal{A} be a ring and let \mathcal{X} be a subset of \mathcal{A} . A map $f : \mathcal{X} \rightarrow \mathcal{A}$ is called *commuting (on \mathcal{X})* if

$$(1) \quad f(x)x = xf(x) \quad \text{for all } x \in \mathcal{X}.$$

In the sequel we shall usually write $[x, y]$ for the commutator $xy - yx$ of $x, y \in \mathcal{A}$. Accordingly (1) will be written as $[f(x), x] = 0$.

The usual goal when treating a commuting map is to describe its form. Therefore we first point out two basic and obvious examples of commuting maps: these are the identity map and every map having its range in the center $Z_{\mathcal{A}}$ of \mathcal{A} . Further, the sum and the pointwise product of commuting maps are again commuting maps. So, for example, the map

$$(2) \quad f(x) = \lambda_0(x)x^n + \lambda_1(x)x^{n-1} + \dots + \lambda_{n-1}(x)x + \lambda_n(x), \quad \lambda_i : \mathcal{A} \rightarrow Z_{\mathcal{A}}$$

Received June 10, 2003; Accepted June 12, 2003.

Communicated by P. H. Lee.

2000 *Mathematics Subject Classification*: 16R50, 16W10, 16W25, 16N60, 46H40, 47B47

Key words and phrases: commuting map, functional identity, prime ring, Banach algebra, derivation, Lie theory, linear preservers.

Partially supported by MŠZŠ Grant P0-0501-0101.

is commuting for any choice of central maps λ_i . Of course there are other examples; namely, elements commuting with x may not necessarily be equal to a polynomial in x (with central coefficients) and so in most rings there is a variety of possibilities of how to find commuting maps different from those described in (2). Nevertheless, a typical result does say that a commuting map must be necessarily of (some version of) the form (2). Obviously we cannot consider just arbitrary set-theoretic maps to prove something like this, so some restrictions must be imposed.

The first important result on commuting maps is Posner's theorem [153] from 1957. This theorem says that the existence of a nonzero commuting derivation on a prime ring \mathcal{A} implies that \mathcal{A} is commutative. Considering this theorem from some distance it is not entirely clear to us what was Posner's motivation for proving it and for which reasons he was able to conjecture that the theorem is true. Anyhow, it is a fact that the theorem has been extremely influential and at least indirectly it initiated many issues discussed in this paper. In Section 2 we shall consider commuting derivations, i.e. the topic arising directly from Posner's theorem. In spite of the purely algebraic nature of the present paper a part of this section will be devoted to derivations on Banach algebras, since in our opinion this may give a better insight on the meaning of the notion of a commuting map.

Much more recently it has been found out that it is possible to characterize a commuting map f (by assertions in the spirit of (2)) without assuming how f acts on the product of elements (as in the case of derivations), but assuming only the additivity of f (the theme of Section 3), or even more generally assuming that f is the trace of a multiadditive map, i.e. $f(x) = M(x, x, \dots, x)$ where M is a multiadditive map in n variables (Section 4). The initial results on such maps were obtained in the beginning of the 90's by the author. Since then there has been a lot of activity on this subject. Important contributions have been made by Ara, Banning, Beidar, Chebotar, Fong, P.-H. Lee, T.-K. Lee, Lin, Martindale, Mathieu, Miers, Mikhalev, Villena, Wang, Wong, and others.

The main reason for describing commuting traces of multiadditive maps is a wide variety of applications. One of them is the solution of a long-standing open problem by Herstein on Lie isomorphisms of associative rings. We shall consider this and other applications in Section 5. Most of them are connected with nonassociative (especially Lie) algebras. Commuting maps also naturally appear in some linear preserver problems. This is another relevant area of applications.

Throughout the paper we shall also briefly discuss various extensions of the notion of a commuting map. The most general and important one among them is the notion of a *functional identity*. An introductory account on functional identities is given in our preceding survey paper [55], which however does not cover the most recent developments of this theory, especially the powerful theory of *d-free sets* by Beidar and Chebotar [26,27] which has to some extent changed our comprehension

of functional identities. The concepts of a commuting map and a functional identity are intimately connected. The theory of functional identities originated from the results on commuting maps, and from another point of view, commuting maps give rise to the most basic and important examples of functional identities. A similar interaction holds with regard to applications: various problems can be solved at some basic level of generality by using results on commuting maps, while in order to solve some more sophisticated versions of these problems one has to apply deeper results on functional identities. We have decided, however, not to examine functional identities in greater detail in this paper. A partial reason for this is that we would like to avoid the paper being too lengthy, and another reason is that we would like to keep the exposition at an introductory level and accessible to a wider audience. Occasionally we will make some digressions primarily more interesting for specialists, and in such instances we shall omit stating definitions and results precisely. But otherwise the paper is fairly self-contained; in order to understand the core of the exposition no specific knowledge is required. Moreover, we shall deliberately avoid presenting the results in their most general forms, and only some simple proofs illustrating the general methods will be presented or outlined. The reader having a deeper interest in some particular topic considered in this paper shall not find complete answers here. However, in that case the comprehensive list of references together with our comments could be of some help.

2. COMMUTING DERIVATIONS

We start with some general remarks and basic definitions. By a ring (algebra) we shall mean an associative ring (algebra). We shall also consider some nonassociative rings and algebras, but this will be always pointed out. Here it should be remarked that by a *nonassociative ring* we mean a ring in which the multiplication is *not necessarily* associative, while by a *noncommutative ring* we mean an (associative) ring in which the multiplication is *not* commutative. We are primarily interested in noncommutative rings; usually the noncommutativity will not be assumed, but most of our results are trivial in the commutative case. Further, the existence of unity is not assumed in advance, so the assumption that a ring is unital shall be explicitly mentioned.

Recall that a ring \mathcal{A} is said to be *prime* if the product of any two nonzero ideals of \mathcal{A} is nonzero. Equivalently, $a\mathcal{A}b = 0$ with $a, b \in \mathcal{A}$ implies $a = 0$ or $b = 0$. A ring \mathcal{A} is called *semiprime* if it has no nonzero nilpotent ideals. Equivalently, $a\mathcal{A}a = 0$ with $a \in \mathcal{A}$ implies $a = 0$.

An additive map d from a ring \mathcal{A} into itself is called a *derivation* if $d(xy) = d(x)y + xd(y)$ for all $x, y \in \mathcal{A}$. When speaking about a derivation of an algebra we assume additionally that d is linear. A simple example is of course the usual

derivative on various algebras consisting of differentiable functions. Basic examples in noncommutative rings are quite different. Noting that $[a, xy] = [a, x]y + x[a, y]$ for all $a, x, y \in \mathcal{A}$ we see that for every fixed $a \in \mathcal{A}$, the map $d : x \mapsto [a, x]$ is a derivation. Such maps are called *inner derivations*. In some rings and algebras the inner derivations are in fact the only derivations.

2.1 Posner's Theorem and Its Generalizations

We restate Posner's theorem already mentioned above as follows.

Theorem 2.1. *If d is a commuting derivation on a noncommutative prime ring, then $d = 0$.*

It should be mentioned that Posner in fact proved this theorem under the more general condition that d satisfies $[d(x), x] \in Z_{\mathcal{A}}$ for every $x \in \mathcal{A}$. Maps satisfying this condition are usually called *centralizing* in the literature. It has turned out that under rather mild assumptions a centralizing map is necessarily commuting (see for example [48, Proposition 3.1]). Therefore (and also for the sake of simplicity of the exposition) we shall not treat centralizing maps in this paper.

A typical example of a ring that is not prime is the direct product $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$ of two nonzero rings \mathcal{A}_1 and \mathcal{A}_2 . If \mathcal{A}_1 is a commutative ring having a nonzero derivation d_1 and \mathcal{A}_2 is a noncommutative ring, then \mathcal{A} is a noncommutative ring and $d : (x_1, x_2) \mapsto (d_1(x_1), 0)$ is a nonzero commuting derivation on \mathcal{A} . This is a trivial example, but it explains well why the assumption of primeness is natural in Theorem 2.1. Many results in this paper will be stated for prime rings, and often similar simple-minded examples can be constructed to show why this restriction is necessary.

As we shall now see, the proof of Posner's theorem is short, simple and elementary. The original proof from [153] is longer, but the main idea to make substitutions and then manipulate the identities obtained is the same.

Proof of Theorem 2.1. Linearizing $[d(x), x] = 0$ (i.e. replacing x by $x + y$ in this identity) we get

$$(3) \quad [d(x), y] = [x, d(y)] \quad \text{for all } x, y \in \mathcal{A}.$$

In particular,

$$(4) \quad [d(x), yx] = [x, d(yx)] = [x, d(y)x + yd(x)] \quad \text{for all } x, y \in \mathcal{A}.$$

Since $d(x)$ and x commute we have $[d(x), yx] = [d(x), y]x$. By (3) it follows that $[d(x), yx] = [x, d(y)]x$, which is further equal to $[x, d(y)x]$. Therefore (4)

reduces to $[x, yd(x)] = 0$ for all $x, y \in \mathcal{A}$. This can be rewritten as $[x, y]d(x) = 0$. Substituting zy for y and using $[x, zy] = [x, z]y + z[x, y]$ we then get $[x, z]yd(x) = 0$ for all $x, y, z \in \mathcal{A}$. Since \mathcal{A} is prime it follows that for every $x \in \mathcal{A}$ we have either $x \in Z_{\mathcal{A}}$ or $d(x) = 0$. In other words, \mathcal{A} is the set-theoretic union of its additive subgroups $Z_{\mathcal{A}}$ and the kernel of d . However, since a group cannot be the union of its two proper subgroups, and since $\mathcal{A} \neq Z_{\mathcal{A}}$ by assumption, it follows that $d = 0$. ■

Posner's theorem has been generalized by a number of authors in several ways. Let us briefly describe some of them. For details the reader should consult the papers mentioned and references therein.

• **Derivations that are commuting on some additive subgroups of (semi)prime rings.**

Typical subgroups that one studies in this context are ideals, Lie ideals, one-sided ideals, and the sets of all symmetric elements $\{x \in \mathcal{A} \mid x^* = x\}$ and all skew elements $\{x \in \mathcal{A} \mid x^* = -x\}$ in the case the ring is equipped with an involution $*$ [7, 37, 106, 110, 111]. The usual conclusion is that Posner's theorem remains true in these more general situations, unless the ring is very special (say, its characteristic is 2 or it satisfies some special polynomial identity).

• **More general conditions with derivations.**

This is a very broad subject. Numerous identities satisfied by derivations, all of them more general than $[d(x), x] = 0$, have been studied; so the list [74, 84, 107, 108, 109, 112, 115, 117, 148, 170, 173] is far from complete. As a sample result we state a simplified version of a theorem of Lanski [108]: If d is a derivation of a noncommutative prime ring \mathcal{A} such that for some positive integer n we have $[d(x), x]_n = [[\dots [d(x), x], x], \dots, x] = 0$ for all $x \in \mathcal{A}$ (here x appears n times), then $d = 0$. In the papers mentioned the authors often combine elementary combinatorial arguments (as presented in the proof of Theorem 2.1) with the much more profound Kharchenko's theory of differential identities [99] (see also [36, 100]). We also mention the paper [13] which in the last part deals with some related conditions, but uses a different approach, based on a version of the density theorem for outer derivations [13, 68].

• **Commuting automorphisms.**

In 1970 Luh [124], extending an earlier result by Divinsky [86], proved an analogue of Theorem 2.1 for automorphisms: If α is a commuting automorphism on a noncommutative prime ring, then $\alpha = 1$. This result has also been extended in various directions [37, 38, 93, 121, 139, 140, 141]. The reader might think that treating a commuting automorphism α must be quite different than treating a commuting derivation. However, note that $\Delta = \alpha - 1$ is also commuting and

satisfies a condition similar to the derivation law: $\Delta(xy) = \Delta(x)y + \alpha(x)\Delta(y) = \Delta(x)\alpha(y) + x\Delta(y)$. So in fact the treatment is quite similar and in particular the result of Luh can be proved by just modifying the proof of Theorem 2.1.

• **Measuring the size of $\mathcal{T} = \{[d(x), x] \mid x \in \mathcal{A}\}$.**

Theorem 2.1 tells us that $\mathcal{T} \neq 0$ in the case when d is a nonzero derivation of a noncommutative prime ring \mathcal{A} . In the series of papers [73, 76, 78, 80, 81, 85, 175] it was shown that much more can be said about the size of \mathcal{T} . For example, combining the results of [80] and [85] the following assertion can be stated: If \mathcal{A} is prime, noncommutative, $\text{char}(\mathcal{A}) \neq 2$ and d is a nonzero derivation of \mathcal{A} , then the additive subgroup generated by \mathcal{T} contains a noncentral Lie ideal of \mathcal{A} .

2.2 Commuting Derivations in Banach Algebras

By a Banach algebra we shall mean a complex normed algebra \mathcal{A} whose underlying vector space is a Banach space. By $\text{rad}(\mathcal{A})$ we denote the *Jacobson radical* of \mathcal{A} .

It is easy to find examples of nonzero derivations on commutative rings and algebras. Say, just take the usual derivative on the polynomial algebra $\mathbb{C}[X]$. In the Banach algebra context the situation is quite different. In 1955 Singer and Wermer [161] proved that every continuous derivation on a commutative Banach algebra \mathcal{A} has its range in $\text{rad}(\mathcal{A})$. So in particular, it must be 0 when \mathcal{A} is semisimple (i.e. when $\text{rad}(\mathcal{A}) = 0$). Of course the same result does not hold in noncommutative Banach algebras (say, because of inner derivations), but there are many ways how to obtain noncommutative versions of Singer-Wermer theorem. One of them is another classical result, the so-called Kleinecke-Shirokov theorem [102, 159], which in one of its forms considers the local version of the condition that a derivation is commuting: If d is a continuous derivation of a Banach algebra \mathcal{A} and $a \in \mathcal{A}$ is such that $[d(a), a] = 0$, then $d(a)$ is quasinilpotent. We also remark that the Kleinecke-Shirokov theorem is an analytic extension of Jacobson's lemma [95] from 1935 which treats this condition in finite dimensional algebras. Another noncommutative extension of the Singer-Wermer theorem was proved by Sinclair [160] in 1969: Every continuous derivation of a Banach algebra \mathcal{A} leaves primitive ideals of \mathcal{A} invariant. As it is evident from the arguments below, this theorem can indeed be regarded as a generalization of the Singer-Wermer theorem.

Results of this kind make it possible for us to obtain more information about commuting derivations in Banach algebras than in usual algebras. To our knowledge, the first result in this direction was obtained in our joint work with Vukman [72]. It treats the condition that is somewhat more general than a derivation being commuting and is somehow more natural in this context:

Theorem 2.2. *Let d be a continuous derivation of a Banach algebra \mathcal{A} . If*

$[d(x), x] \in \text{rad}(\mathcal{A})$ for all $x \in \mathcal{A}$, then d maps \mathcal{A} into $\text{rad}(\mathcal{A})$.

Proof. Let \mathcal{P} be a primitive ideal of \mathcal{A} . Since $d(\mathcal{P}) \subseteq \mathcal{P}$ by Sinclair's theorem, d induces a derivation $d_{\mathcal{P}}$ on \mathcal{A}/\mathcal{P} defined by $d_{\mathcal{P}}(x + \mathcal{P}) = d(x) + \mathcal{P}$. Note that $d_{\mathcal{P}}$ is commuting. Since \mathcal{A}/\mathcal{P} is a primitive and hence a prime algebra, Theorem 2.1 tells us that either $d_{\mathcal{P}} = 0$ or \mathcal{A}/\mathcal{P} is commutative. However, since \mathbb{C} is the only commutative primitive Banach algebra, $d_{\mathcal{P}} = 0$ in every case (this follows immediately from the fact that every derivation maps the unity $\mathbf{1}$ into 0, which is a consequence of $\mathbf{1} = \mathbf{1}^2$ and the derivation law). Accordingly, $d(\mathcal{A}) \subseteq \mathcal{P}$ for every primitive ideal \mathcal{P} of \mathcal{A} and hence $d(\mathcal{A}) \subseteq \text{rad}(\mathcal{A})$. ■

Is the assumption of continuity superfluous in Theorem 2.2? This is a very interesting and deep question. Let us give some comments about its background.

Already in [161] Singer and Wermer conjectured that the assumption of continuity is superfluous in their theorem. This became known as the Singer-Wermer conjecture and it stood open for over thirty years till it was finally confirmed by Thomas [167]. A natural conjecture that now appears is that Sinclair's theorem also holds without assuming continuity, that is, that every (possibly discontinuous) derivation on a Banach algebra \mathcal{A} leaves primitive ideals of \mathcal{A} invariant. This is usually called the *noncommutative Singer-Wermer conjecture* in the literature. A number of mathematicians have tried to prove it, but, to our knowledge, without success so far. It is known that for every derivation d there can be only finitely many primitive ideals which are not invariant under d , and each of them has finite codimension [168]. But it is not known whether such ideals actually exist. The conjecture that Theorem 2.2 holds without assuming the continuity of d is equivalent to the noncommutative Singer-Wermer conjecture. For details we refer the reader to Mathieu's survey article [135] where in particular one can find other different versions of this conjecture (see also [71] for some new results).

Various partial answers to this conjecture have been obtained. For example, Mathieu and Runde [138] proved that every centralizing derivation of a Banach algebra \mathcal{A} has its range in $\text{rad}(\mathcal{A})$. We shall prove this only for commuting derivations; the argument in this particular case is somewhat different and more direct.

For every subset \mathcal{S} of \mathcal{A} we let $C(\mathcal{S}) = \{x \in \mathcal{A} \mid [s, x] = 0 \text{ for every } s \in \mathcal{S}\}$ denote the *centralizer* of \mathcal{S} in \mathcal{A} (in the Banach algebra theory this set is more often called the *commutant*). We shall write $C(a)$ for $C(\{a\})$.

Theorem 2.3. *Every commuting derivation of a Banach algebra \mathcal{A} has its range in $\text{rad}(\mathcal{A})$.*

Proof. Let d be a commuting derivation on \mathcal{A} and let \mathcal{S} be an arbitrary non-empty subset of \mathcal{A} . For every $x \in C(\mathcal{S})$ we have $0 = d([s, x]) = [d(s), x] + [s, d(x)]$. On the other hand, from the linearized form of $[d(x), x] = 0$ (cf. (3)) we see that

$[d(s), x] = [s, d(x)]$. Accordingly $[s, d(x)] = 0$. That is to say, $C(\mathcal{S})$ is invariant under d . In particular, for every $a \in \mathcal{A}$ we have $d(C(C(a))) \subseteq C(C(a))$. It is straightforward to check that $C(C(a))$ is a commutative Banach algebra. Therefore, we can apply Thomas' theorem [167] for the restriction of d to $C(C(a))$, and conclude that $d(C(C(a))) \subseteq \text{rad}(C(C(a)))$. In particular, $d(C(C(a)))$ consists of quasinilpotent elements. Since $a \in C(C(a))$ we see that $d(a)$ is quasinilpotent.

Let \mathcal{P} be a primitive ideal of \mathcal{A} . Since $d(\mathcal{A})$ contains only quasinilpotent elements in \mathcal{A} , $d(\mathcal{A}) + \mathcal{P}$ contains only quasinilpotent elements in \mathcal{A}/\mathcal{P} . Therefore, for every $p \in \mathcal{P}$ and $x \in \mathcal{A}$ we see that $(d(p) + \mathcal{P})(x + \mathcal{P}) = d(px) + \mathcal{P}$ is a quasinilpotent element in \mathcal{A}/\mathcal{P} . By a well-known characterization of the radical it follows that $d(p) + \mathcal{P} \in \text{rad}(\mathcal{A}/\mathcal{P})$. However, as a primitive algebra \mathcal{A}/\mathcal{P} is semisimple, and so it follows that $d(p) \in \mathcal{P}$. That is, every primitive ideal is invariant under d , and now the same argument as in the proof of Theorem 2.2 works. ■

For more related results see for example [12, 52, 63, 70, 136, 137, 158, 171].

3. COMMUTING ADDITIVE MAPS

Our aim now is to investigate arbitrary additive maps that are commuting. Since derivations are just very special additive maps, this of course appears to be a much more entangled problem than the one treated in the previous section. Fortunately, at least at some basic level of generality, the problem is not so difficult. It may come as a surprise to the reader that the notion of a derivation will also play an important role in this section. But the reason is simple: whenever we consider a condition involving commutators (as is the case with the notion of a commuting map) we can express it through (inner) derivations. Sometimes this point of view is useful, and perhaps the treatment of additive commuting maps is a typical example for this. Let us now describe the main idea of our approach.

Let \mathcal{A} be a ring and let $f : \mathcal{A} \rightarrow \mathcal{A}$ be an additive commuting map. A linearization of (1) gives

$$(5) \quad [f(x), y] = [x, f(y)] \quad \text{for all } x, y \in \mathcal{A}.$$

Hence we see that the map $(x, y) \mapsto [f(x), y] (= [x, f(y)])$ is an inner derivation in each argument. This gives rise to the following definition: a biadditive map $\Delta : \mathcal{A}^2 \rightarrow \mathcal{A}$ is called a *biderivation* on \mathcal{A} if it is a derivation in each argument, that is, for every $y \in \mathcal{A}$ the maps $x \mapsto \Delta(x, y)$ and $x \mapsto \Delta(y, x)$ are derivations. For example, for every $\lambda \in Z_{\mathcal{A}}$, $(x, y) \mapsto \lambda[x, y]$ is a biderivation. We shall call such maps *inner biderivations*. It is easy to construct non-inner biderivations on commutative rings. For instance, if d is nonzero derivation of a commutative domain \mathcal{A} , then $\Delta : (x, y) \mapsto d(x)d(y)$ is such an example. In noncommutative

rings, however, it happens quite often that all biderivations are inner. If \mathcal{A} is such a ring, then every additive commuting map f on \mathcal{A} is of the form

$$(6) \quad f(x) = \lambda x + \mu(x), \quad \lambda \in Z_{\mathcal{A}}, \quad \mu : \mathcal{A} \rightarrow Z_{\mathcal{A}}$$

with μ being an additive map. Indeed, since $(x, y) \mapsto [f(x), y]$ is a biderivation it follows that there is $\lambda \in Z_{\mathcal{A}}$ such that $[f(x), y] = \lambda[x, y]$ for all $x, y \in \mathcal{A}$, from which it clearly follows that $\mu(x) = f(x) - \lambda x$ lies in $Z_{\mathcal{A}}$.

Thus, in order to show that every commuting additive map on a ring \mathcal{A} is of the form (6), it is enough to show that every biderivation is inner. To establish this the following simple lemma will be of crucial importance.

Lemma 3.1. *Let Δ be a biderivation on a ring \mathcal{A} . Then*

$$(7) \quad \Delta(x, y)z[u, v] = [x, y]z\Delta(u, v) \quad \text{for all } x, y, z, u, v \in \mathcal{A}.$$

Proof. Consider $\Delta(xu, yv)$ for arbitrary $x, y, u, v \in \mathcal{A}$. Since Δ is a derivation in the first argument, we have

$$\Delta(xu, yv) = \Delta(x, yv)u + x\Delta(u, yv),$$

and since it is also a derivation in the second argument it follows that

$$\Delta(xu, yv) = \Delta(x, y)vu + y\Delta(x, v)u + x\Delta(u, y)v + xy\Delta(u, v).$$

On the other hand, first using the derivation law in the second and after that in the first argument we get

$$\begin{aligned} \Delta(xu, yv) &= \Delta(xu, y)v + y\Delta(xu, v) \\ &= \Delta(x, y)uv + x\Delta(u, y)v + y\Delta(x, v)u + yx\Delta(u, v). \end{aligned}$$

Comparing both relations we obtain $\Delta(x, y)[u, v] = [x, y]\Delta(u, v)$ for all $x, y, u, v \in \mathcal{A}$. Replacing v by zv and using $[u, zv] = [u, z]v + z[u, v]$, $\Delta(u, zv) = \Delta(u, z)v + z\Delta(u, v)$, the desired identity follows. ■

The next result illustrates the utility of this lemma.

Theorem 3.2. *Let \mathcal{A} be a unital ring such that the ideal of \mathcal{A} generated by all commutators in \mathcal{A} is equal to \mathcal{A} . Then every biderivation on \mathcal{A} is inner. Accordingly, every commuting additive map f on \mathcal{A} is of the form (6).*

Proof. By assumption there are $z_i, u_i, v_i, w_i \in \mathcal{A}$ such that $\sum_i z_i[u_i, v_i]w_i = \mathbf{1}$. Lemma 3.1 implies that

$$\Delta(x, y) = \sum_i \Delta(x, y)z_i[u_i, v_i]w_i = \sum_i [x, y]z_i\Delta(u_i, v_i)w_i.$$

That is, $\Delta(x, y) = [x, y]\lambda$ for all $x, y \in \mathcal{A}$ where $\lambda = \sum_i z_i \Delta(u_i, v_i) w_i \in \mathcal{A}$. We claim that $\lambda \in Z_{\mathcal{A}}$. Indeed, we have

$$\begin{aligned} [x, y]z\lambda + y[x, z]\lambda &= [x, yz]\lambda = \Delta(x, yz) \\ &= \Delta(x, y)z + y\Delta(x, z) = [x, y]\lambda z + y[x, z]\lambda, \end{aligned}$$

showing that $[x, y][z, \lambda] = 0$ for all $x, y, z \in \mathcal{A}$. Replacing z by zw and using $[zw, \lambda] = [z, \lambda]w + z[w, \lambda]$ we obtain $[\mathcal{A}, \mathcal{A}]\mathcal{A}[\lambda, \mathcal{A}] = 0$. Using $\sum_i z_i [u_i, v_i] w_i = \mathbf{1}$ again it follows that $[\lambda, \mathcal{A}] = 0$, i.e. $\lambda \in Z_{\mathcal{A}}$. ■

Corollary 3.3. *Let \mathcal{A} be a simple unital ring. Then every commuting additive map f on \mathcal{A} is of the form (6).*

Proof. If \mathcal{A} is commutative this is trivial (just take $\lambda = 0$ and $\mu = f$). If \mathcal{A} is noncommutative then this follows from Theorem 3.2. ■

The idea to describe commuting additive maps through the commutator ideal was used for the first time in [46] where the main goal was to show that the conclusion of Corollary 3.3 holds for von Neumann algebras. Unfortunately, this idea has a limited applicability, it works only in rather special rings. Before describing a more common approach we point out the delicate nature of the problem. First of all, the assumption that \mathcal{A} is unital can not be removed in Corollary 3.3. Namely, taking a simple ring \mathcal{A} with $Z_{\mathcal{A}} = 0$ we see that, for instance, the identity map is certainly commuting, but it cannot be expressed by (6). But suppose that \mathcal{A} is unital, and even that $Z_{\mathcal{A}}$ is a field. Is it possible to prove Corollary 3.3 for some more general classes of rings? The following example shows that even for rings that are close to simple ones the expected form (6) is not entirely sufficient.

Example 3.4. Let V be an infinite dimensional vector space over a field E and let $\mathcal{F}(V)$ be the algebra of all finite rank E -linear operators on V . Note that $\mathcal{F}(V)$ is a simple algebra with $Z_{\mathcal{F}(V)} = 0$. Let F be a proper subfield of E , and let \mathcal{A} be the algebra over F consisting of all operators of the form $u + \alpha$ where $u \in \mathcal{F}(V)$ and $\alpha \in F$ (here elements in F are identified by corresponding scalar operators). Pick $\lambda \in E \setminus F$ and define $f : \mathcal{A} \rightarrow \mathcal{A}$ by $f(u + \alpha) = \lambda u$ for all $u \in \mathcal{F}(V)$, $\alpha \in F$. Clearly f is an additive commuting map. However, since $Z_{\mathcal{A}} = F$ it is clear that f is not of the form (6). On the other hand, f can be written as $f(x) = \lambda x + \mu(x)$ for all $x \in \mathcal{A}$ where μ is defined by $\mu(u + \alpha) = -\lambda\alpha$. But here λ and $\mu(x)$ do not lie in $Z_{\mathcal{A}}$ but in the field extension E of $Z_{\mathcal{A}}$. Similarly, $\Delta : (x, y) \mapsto \lambda[x, y]$ is a biderivation on \mathcal{A} which is not inner in the sense defined above.

The reader may feel that the map f is “essentially” of the form (6), just formally this is not true. The example suggests that in order to describe commuting maps it will sometimes be necessary to deal with some extensions of the center of the

ring . In the context of (semi)prime rings, the so-called *extended centroid* is the most appropriate extension for our purposes. We shall now recall just a few facts about it, and refer the reader to [36] for details. Let \mathcal{A} be a prime ring. The extended centroid $C_{\mathcal{A}}$ of \mathcal{A} is usually defined as the center of the right (or left, or symmetric) Martindale ring of quotients of \mathcal{A} . We omit giving definitions of these rings of quotients, let us just say that these are prime rings containing \mathcal{A} as their subring. Instead we point out a characteristic property of elements belonging to the extended centroid. Let \mathcal{I} be a nonzero ideal of \mathcal{A} . One can regard \mathcal{I} and \mathcal{A} as $(\mathcal{A}, \mathcal{A})$ -bimodules. If $f : \mathcal{I} \rightarrow \mathcal{A}$ is an $(\mathcal{A}, \mathcal{A})$ -bimodule homomorphism then there exists $\lambda \in C_{\mathcal{A}}$ such that $f(x) = \lambda x$ for all $x \in \mathcal{I}$. Conversely, giving $\lambda \in C_{\mathcal{A}}$ there is a nonzero ideal \mathcal{I} of \mathcal{A} such that $\lambda \mathcal{I} \subseteq \mathcal{A}$ and so $x \mapsto \lambda x$ is a bimodule homomorphism from \mathcal{I} into \mathcal{A} . It turns out that $C_{\mathcal{A}}$ is a field containing $Z_{\mathcal{A}}$ as a subring. In many important instances $C_{\mathcal{A}}$ coincides with $Z_{\mathcal{A}}$. In particular this is true in simple unital rings. Incidentally we mention that it is also true in various significant Banach algebras (e.g. in unital primitive Banach algebras and unital prime C^* -algebras) which often makes this algebraic theory applicable in the analytic setting. If $Z_{\mathcal{A}}$ is not a field then of course it cannot coincide with $C_{\mathcal{A}}$. But in such case it sometimes turns out (e.g. in PI prime rings) that $C_{\mathcal{A}}$ is the field of fractions of $Z_{\mathcal{A}}$. In general, however, $C_{\mathcal{A}}$ can be larger. For instance, this is true for the algebra \mathcal{A} from Example 3.4. We leave as an exercise for the reader to show that $C_{\mathcal{A}} = E$ while, as already mentioned, $Z_{\mathcal{A}} = F$. In view of this example and Corollary 3.3 it seems natural that

$$(8) \quad f(x) = \lambda x + \mu(x), \quad \lambda \in C_{\mathcal{A}}, \mu : \mathcal{A} \rightarrow C_{\mathcal{A}}$$

is the expected form of a commuting additive map on a prime ring \mathcal{A} . This is true indeed. To show this we only need to know an extremely useful property of the extended centroid, discovered by Martindale in his classical work [130]. This property is concerned (at least in its simplest version) with the situation when $a, b \in \mathcal{A}$, $a \neq 0$, satisfy

$$(9) \quad axb = bxa \quad \text{for all } x \in \mathcal{A}.$$

The conclusion is that there exists $\lambda \in C_{\mathcal{A}}$ such that $b = \lambda a$. The idea of the proof goes back to Amitsur [3, p. 215]. We define $\varphi : \mathcal{A}a\mathcal{A} \rightarrow \mathcal{A}$ by

$$\varphi\left(\sum_i x_i a y_i\right) = \sum_i x_i b y_i,$$

and claim that φ is an $(\mathcal{A}, \mathcal{A})$ -bimodule homomorphism. Everything is clear except that φ is well-defined. To show this, assume that $\sum_i x_i a y_i = 0$ for some $x_i, y_i \in \mathcal{A}$. Multiplying this identity from the left by bx and using (9) it follows that

$\sum_i ax_i by_i = 0$ for every $x \in \mathcal{A}$. That is, $a\mathcal{A}(\sum_i x_i by_i) = 0$ and hence, since \mathcal{A} is prime, $\sum_i x_i by_i = 0$. Thus our claim is true and so there is $\lambda \in C_{\mathcal{A}}$ such that $\varphi(u) = \lambda u$ for every $u \in \mathcal{A}a\mathcal{A}$, from which $b = \lambda a$ follows.

Now assume that \mathcal{A} is a noncommutative prime ring and Δ is a biderivation on \mathcal{A} . Picking $u, v \in \mathcal{A}$ such that $[u, v] \neq 0$ and applying (7) with $x = u$, $y = v$, it follows from what we have just discussed that $\Delta(u, v) = \lambda[u, v]$ for some $\lambda \in C_{\mathcal{A}}$. Again using (7), this time in its full generality, it follows that $(\Delta(x, y) - \lambda[x, y])\mathcal{A}[u, v] = 0$. Consequently, the following is true.

Theorem 3.5. *Let \mathcal{A} be a noncommutative prime ring and let Δ be a biderivation on \mathcal{A} . Then there exists $\lambda \in C_{\mathcal{A}}$ such that $\Delta(x, y) = \lambda[x, y]$ for all $x, y \in \mathcal{A}$.*

Theorem 3.5 was published in our paper [61] with Martindale and Miers. However, Professor I. P. Shestakov pointed out to us that it was already obtained somewhat earlier by Skosyrskii [162] who treated biderivations for different reasons, namely, in connection with noncommutative Jordan algebras. Moreover, Theorem 3.5 was rediscovered by Farkas and Letzter [89] in their study of Poisson algebras. So it seems that biderivations appear naturally in different areas.

Our interest in biderivations of course proceeds from their connection with commuting additive maps. Note that Theorem 3.5 yields the following basic result.

Theorem 3.6. *Let \mathcal{A} be a prime ring. Then every commuting additive map f on \mathcal{A} is of the form (8).*

Theorem 3.6 appeared for the first time in our paper [48]. The original proof also makes use of derivations, but it is longer and somewhat different. A short cut based on biderivations was found a little bit later. A different and very short proof was obtained also in [114].

One can define the extended centroid also for semiprime rings (though it is no longer always a field), and it turns out that Theorem 3.6 is also true in the semiprime case. This was proved by Ara and Mathieu [4]. A shorter proof based on the description of biderivations of semiprime rings was given in [50]. In arbitrary rings, however, the problem to describe the structure of additive commuting maps seems to be unapproachable. The next example, a modification of the one found by Cheung [82, p. 123], shows that in general not much can be said. In [82] this example was used to show that not every additive commuting map of the so-called triangular algebra is of the form (6). As we shall see, the example shows considerably more. From the result just mentioned (or more directly, just by using Lemma 3.1 for a biderivation arising from an additive commuting map) the following is clear: If an additive map f on a semiprime ring \mathcal{A} is commuting (in other words, $f(x) \in C(x)$ for every $x \in \mathcal{A}$), then $f(x) \in C(C(x))$ for every $x \in \mathcal{A}$. Let us show that this is not true in every ring.

Example 3.7. Let \mathcal{T} be a ring containing an element a such that the ideal

\mathcal{U} of \mathcal{T} generated by a is commutative and $t_1 a t_2 \neq t_2 a t_1$ for some $t_1, t_2 \in \mathcal{T}$. A concrete example is the ring of all 2×2 upper triangular matrices over a field and $a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Further, let \mathcal{A} be the algebra of all matrices of the form $\begin{pmatrix} u & t \\ 0 & v \end{pmatrix}$ where $u, v \in \mathcal{U}$ and $t \in \mathcal{T}$. Define $f : \mathcal{A} \rightarrow \mathcal{A}$ according to the rule

$$\begin{pmatrix} u & t \\ 0 & v \end{pmatrix} \mapsto \begin{pmatrix} ta & 0 \\ 0 & at \end{pmatrix}.$$

(If one prefers the context of unital rings, then one can adjoin $\mathbf{1}$ to \mathcal{A} and define $f(\mathbf{1}) = 0$ – everything that follows is then still true). Note that f is commuting. However, it is not true that $f(x) \in C(C(x))$ for every $x \in \mathcal{A}$. Namely, the element $x = \begin{pmatrix} 0 & t_1 \\ 0 & 0 \end{pmatrix}$ commutes with $y = \begin{pmatrix} 0 & t_2 \\ 0 & 0 \end{pmatrix}$, but $f(x)$ and y do not commute.

The results presented so far in this section have been generalized in many different ways. We shall now superficially describe the main directions of these generalizations, but only those that are concerned with additive maps and are closely related to the concept of an additive commuting map.

- **Additive commuting maps on other rings and algebras.**

There are other rings and algebras in which additive commuting maps can be characterized by the forms similar to (6) or (8). One type of algebras that have been studied are triangular algebras [15, 82]. Another significant example are C^* -algebras [4, 46]. An account on commuting maps in C^* -algebras is given in the recently published book [5] by Ara and Mathieu.

- **Additive maps that are commuting on some additive subgroups of (semi)prime rings.**

Let \mathcal{X} be an additive subgroup of a prime (or semiprime) ring \mathcal{A} and let $f : \mathcal{X} \rightarrow \mathcal{A}$ be a commuting map. Then one can hope to prove that f is of the form (8) (restricted to \mathcal{X} of course) provided that \mathcal{X} is “large enough”. Specifically, this has been established in the case when \mathcal{X} is a Lie ideal [65, 120], a one-sided ideal [65, 120], the set of all symmetric [34, 113] and all skew elements [34, 61, 116] in the case the ring has an involution. Most of these conclusions, however, can be obtained only under some restrictions which we shall not discuss here. We also remark that some of these results were recently superseded by more general theorems on d -free sets [26, 27].

- **More general conditions with additive maps.**

Numerous conditions concerning an additive map f which are more general than f being commuting, but usually implying the same conclusion, have been studied. It would occupy too much space to discuss at greater length all of them, so we just list

some references: [11, 33, 47, 53, 54, 90, 119]. As an example we state the result of [54], an extension of a theorem from [108] mentioned earlier: If \mathcal{A} is a prime ring and an additive map $f : \mathcal{A} \rightarrow \mathcal{A}$ satisfies $[f(x), x]_n = [[\dots [f(x), x], x], \dots, x] = 0$ for all $x \in \mathcal{A}$ and some fixed $n \geq 1$, then f is commuting (and hence of the form (8)), provided that $\text{char}(\mathcal{A}) = 0$ or $\text{char}(\mathcal{A}) > n$. We remark that a considerably more general result was obtained somewhat later in [33]. We also point out the result from [90] which provides an appropriate extension of Theorem 3.6 to prime associative superalgebras. The problem to study more general functional identities in superalgebras now naturally appears.

• **Related conditions with derivations.**

In view of Posner's theorem and Theorem 3.6 one might wonder what can be said about an additive map $f : \mathcal{A} \rightarrow \mathcal{A}$ such that $[f(x), d(x)] = 0$ for all $x \in \mathcal{A}$, where d is some nonzero derivation of a prime ring \mathcal{A} . The expected result is of course that $f(x) = \lambda d(x) + \mu(x)$ for some $\lambda \in C_{\mathcal{A}}$ and $\mu : \mathcal{A} \rightarrow C_{\mathcal{A}}$. This has turned out to be true indeed, the only restriction being that $\text{char}(\mathcal{A}) \neq 2$. A partial solution was first obtained in [65], and recently the complete one was given in [17]. The proof, however, is no longer elementary and it relies heavily on advanced results on functional identities. Even more recently [174] a related result stating that $d([f(x), x]) = 0$ for all $x \in \mathcal{A}$ implies that f is commuting was obtained. In view of several similarities between these two results and their proofs one can conjecture that both are special cases of some more general and global phenomenon.

• **Associating additive maps in Jordan algebras.**

Let \mathcal{J} be a Jordan algebra with product \cdot and set $[x, y, z] = (x \cdot y) \cdot z - x \cdot (y \cdot z)$ for the associator of $x, y, z \in \mathcal{J}$. We say that $f : \mathcal{J} \rightarrow \mathcal{J}$ is an *associating map* if $[f(x), \mathcal{J}, x] = 0$ for all $x \in \mathcal{J}$. For example, if \mathcal{A} is an associative algebra, \mathcal{A}^+ is its associated Jordan algebra (i.e. \mathcal{A}^+ is the linear space of \mathcal{A} equipped with the symmetrized product $x \cdot y = \frac{1}{2}(xy + yx)$), and f is a commuting map on \mathcal{A} , then f is readily seen to be an associating map on \mathcal{A}^+ . Therefore, the concept of an associating map can be regarded as a Jordan analogue of the concept of a commuting map. In [58] it was shown that if \mathcal{J} is a prime nondegenerate algebra, then every associating additive map $f : \mathcal{J} \rightarrow \mathcal{J}$ is of the standard form $f(x) = \lambda x + \mu(x)$ (we remark that one can define the extended centroid also in this setting). The proof uses Zelmanov's classification theorem [176].

• **Range-inclusive maps.**

In [101] Kissin and Shulman initiated the study of additive maps f on a ring \mathcal{A} satisfying $[f(x), \mathcal{A}] \subseteq [x, \mathcal{A}]$ for every $x \in \mathcal{A}$. They call them *range-inclusive maps* (the exact definition in [101] is different, we stated only a simplified version). From (5) we see that every additive commuting map is also range-inclusive. In [101] it

was shown that linear range-inclusive maps on certain C^* -algebras are necessarily commuting and of the form (6). In [57] we investigated analogous algebraic problems. It has turned out that even in division rings a range-inclusive map may not be commuting, so it is a challenging problem to find a reasonable class of rings in which range-inclusive and commuting additive maps coincide. In primitive GPI rings and algebraic prime algebras this is true [57]. We conjecture that it is also true in prime rings containing nontrivial idempotents; perhaps the existence of one nontrivial idempotent is not enough, maybe more nontrivial orthogonal idempotents should exist. Anyway, this is an open problem.

4. COMMUTING TRACES OF MULTIADDITIVE MAPS

Passing from the study of commuting derivations to the study of arbitrary additive commuting maps has been of course an important step. We shall now take a step further. The topic of this section considerably exceeds the preceding ones by the level of generality, difficulty, and especially by the significance of applications. Anyhow, the results and the methods presented above show us the way how to approach this more complicated setting.

We start with the fundamental, yet the simplest, topic of this section. For the first time it was treated in the author's paper [49] from 1993.

A map q from a ring \mathcal{A} into itself is said to be the *trace of a biadditive map* if there exists a biadditive map $B : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$ such that

$$q(x) = B(x, x) \quad \text{for all } x \in \mathcal{A}.$$

Another (and perhaps better) name is a *quadratic map* (cf. [5]), but we shall not use it since traces of multiadditive maps in more than two variables will also appear in the sequel. Assume that q is commuting. The results in the preceding section suggest to us what is the expected form of q in this case. Our basic theorem is a "quadratic analogue" of Theorem 3.6.

Theorem 4.1. *Let \mathcal{A} be a prime ring with $\text{char}(\mathcal{A}) \neq 2$ and let $q : \mathcal{A} \rightarrow \mathcal{A}$ be the trace of a biadditive map. If q is commuting then it is of the form*

$$(10) \quad q(x) = \lambda x^2 + \mu(x)x + \nu(x), \quad \lambda \in C_{\mathcal{A}}, \mu, \nu : \mathcal{A} \rightarrow C_{\mathcal{A}},$$

where μ is an additive map and ν is the trace of a biadditive map into $C_{\mathcal{A}}$.

Let us briefly sketch the proof. Now there are different proofs available, but we shall follow the original approach from [49]. We have assumed that $\text{char}(\mathcal{A}) \neq 2$ (in problems like this one is often forced to exclude rings having some small characteristic). This in particular allows us to assume that B is a symmetric map,

i.e. $B(x, y) = B(y, x)$ for all $x, y \in \mathcal{A}$, since otherwise we can replace B by the symmetric map $(x, y) \mapsto B(x, y) + B(y, x)$. Using a standard linearization process we see that $[B(x, x), x] = 0$, $x \in \mathcal{A}$, yields

$$[B(x, y), z] + [B(z, x), y] + [B(y, z), x] = 0 \quad \text{for all } x, y, z \in \mathcal{A}.$$

So again inner derivations appear, but the situation is much more unclear than in the case of additive maps when we arrive, after a linearization, at a rather tractable situation with a biderivation. We have to take a step further and show (it is easy but we omit details) that for any fixed $z, w \in \mathcal{A}$ the map $\tilde{B} : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$ defined by

$$\tilde{B}(x, y) = [B(x, zw), y] + z[y, B(x, w)] + [y, B(x, z)]w$$

satisfies

$$\tilde{B}(x, y) = -\tilde{B}(y, x) \quad \text{for all } x, y \in \mathcal{A}.$$

By the definition we see that the map $y \mapsto \tilde{B}(x, y)$ is the sum of compositions of inner derivations and multiplications with fixed elements z and w , and from the last identity we see that the same is true for the map $y \mapsto \tilde{B}(y, x)$. This is of course still much more complicated than in the biderivation situation, but at least there is some similarity. Based on these observations one can after a rather long computation involving several substitutions derive the crucial identity

$$\begin{aligned} & ([w^2, z]y[w, z] - [w, z]y[w^2, z])uq(x) \\ (11) \quad & = f(w, y, z)ux^2 + g(x, w, y, z)ux + h(x, w, y, z)u \\ & \text{for all } u, w, x, y, z \in \mathcal{A}, \end{aligned}$$

where f, g, h are certain maps arising from B (we could express them explicitly but their role is insignificant in the sequel). Now we have to assume that $a = [w^2, z]y[w, z] - [w, z]y[w^2, z] \neq 0$ for some $w, y, z \in \mathcal{A}$. Rewriting (11) with w, z, y fixed we have

$$(12) \quad auq(x) = bux^2 + \tilde{g}(x)ux + \tilde{h}(x)u \quad \text{for all } u, x \in \mathcal{A}$$

and some $\tilde{g}, \tilde{h} : \mathcal{A} \rightarrow \mathcal{A}$. So far the assumption on primeness has not been used. The relation (12) makes it possible for us to use it in an efficient way. Again the clue is Martindale's result concerning the identity (9). Making some manipulations with (12) one can show that $(bva - avb)\mathcal{A}a = 0$ for all $v \in \mathcal{A}$, hence $bva = avb$ by the primeness of \mathcal{A} , which yields $b = \lambda a$ for some $\lambda \in C_{\mathcal{A}}$. Accordingly, (12) can now be written as $au(q(x) - \lambda x^2) = \tilde{g}(x)ux + \tilde{h}(x)u$. This is the same kind of relation as (12), just that the first summand on the right-hand side is missing. Repeating the same computational tricks one can then easily complete the proof.

Of course this was done under the additional condition that $[w^2, z]y[w, z] - [w, z]y[w^2, z] \neq 0$ for some $w, z, y \in \mathcal{A}$. It is known by standard PI theory that this condition is not fulfilled if and only if \mathcal{A} satisfies St_4 , the standard polynomial identity of degree 4, i.e. $\sum_{\sigma \in S_4} (-1)^\sigma x_{\sigma(1)} x_{\sigma(2)} x_{\sigma(3)} x_{\sigma(4)} = 0$ for all $x_1, x_2, x_3, x_4 \in \mathcal{A}$. Equivalently, \mathcal{A} can be embedded in the ring $M_2(F)$ of 2×2 matrices over some field F . Another equivalent condition is that $\deg(\mathcal{A}) \leq 2$ where $\deg(\mathcal{A}) \leq n$ means that every element in \mathcal{A} is algebraic of degree at most n over $C_{\mathcal{A}}$.

So far we followed the proof from [49]. In this first paper on this subject Theorem 4.1 was proved under the additional assumption that \mathcal{A} does not satisfy the conditions from the preceding paragraph. Such rings are indeed very special and moreover their structure can be rather precisely described by Posner's theorem on prime PI rings [154] (see also [155, 157]). Nevertheless, it has been a mystery for the author for quite some time whether the exclusion of these rings in Theorem 4.1 is really necessary or not. Just recently in our paper with Šemrl [69] we have found out that the answer is no, that is, that Theorem 4.1 holds true also when $\deg(\mathcal{A}) \leq 2$. The proof in this special case, however, is quite different. It is obtained as an application of two results. The first one is an elementary linear algebraic result obtained in the same paper [69], and the second one is a much deeper result by Lee, Lin, Wang and Wong [114] which we shall now formulate. But first we state explicitly the following definition: A map t from an additive group \mathcal{X} into an additive group \mathcal{Y} is called the trace of an n -additive map if there exists a map $M : \mathcal{X}^n \rightarrow \mathcal{Y}$ which is additive in every argument and such that $t(x) = M(x, x, \dots, x)$ for all $x \in \mathcal{X}$.

Theorem 4.2. *Let \mathcal{A} be a prime ring, let n be a positive integer, and suppose that $\text{char}(\mathcal{A}) = 0$ or $\text{char}(\mathcal{A}) > n$. Let $t : \mathcal{A} \rightarrow \mathcal{A}$ be the trace of an n -additive map. If t is commuting then the following holds:*

- (i) *For every $x \in \mathcal{A}$ there exist $\lambda_i(x) \in C_{\mathcal{A}}$, $i = 0, 1, \dots, n$, such that*

$$t(x) = \lambda_0(x)x^n + \lambda_1(x)x^{n-1} + \dots + \lambda_{n-1}(x)x + \lambda_n(x).$$
- (ii) *If $\deg(\mathcal{A}) \not\leq n$, then we can choose $\lambda_i(x)$ so that $\lambda_0 = \lambda_0(x)$ is independent of x and for each $i = 1, \dots, n$ the map $x \mapsto \lambda_i(x)$ is the trace of an i -additive map into $C_{\mathcal{A}}$.*

We remark that the assertion (i), which is of course the one that we used when proving the St_4 case of Theorem 4.1 in [69], is of special interest also because the results of this type do not follow by using the general functional identities machinery. The assertion (ii) is a generalization of our result in [49] from $n = 2$ to an arbitrary n . In view of [69] one can now of course conjecture that the exclusion of rings with $\deg(\mathcal{A}) \leq n$ is not necessary in (ii). From the arguments in [69] it does not seem obvious how to prove this for $n \geq 3$, so in our opinion this is a challenging problem.

Commuting traces were studied also on some other rings. Banning and Mathieu [9] extended the result from [49] to semiprime rings; taking into account [69] one can now ask whether their theorem can be improved. Further results concerning semiprime (as well as some other) rings can be extracted (they are not stated explicitly) from [16]. Commuting traces of multilinear maps on the algebra of all upper triangular matrices were characterized in [15], and commuting traces of biadditive maps in C^* -algebras were studied in [5, 64].

There has been a considerable interest in commuting traces of multiadditive maps in rings with involution. The first result in this context was obtained by Beidar, Martindale and Mikhalev [35] who considered commuting traces of 3-additive maps on the Lie subring \mathcal{K} of skew elements of a (non-GPI and centrally closed) prime ring with involution. Commuting additive maps on \mathcal{K} were studied previously in [61], and the reader might wonder why the simpler biadditive case was not the next one. This will be revealed in the next section on applications. The last result in this section that we explicitly state is due to Beidar and Martindale [34, Corollary 5.6] (combined with [26, Lemma 2.2]) which is obtained as an application of more general results on functional identities in prime rings with involution.

Theorem 4.3. *Let \mathcal{A} be a prime ring with involution, and let \mathcal{X} be either the set of all symmetric or the set of all skew elements in \mathcal{A} . Let n be a positive integer and suppose that $\deg(\mathcal{A}) \not\leq 2(n+1)$ and $\text{char}(\mathcal{A}) = 0$ or $\text{char}(\mathcal{A}) > n$ (and $\text{char}(\mathcal{A}) \neq 2$ if $n = 1$). Let $t : \mathcal{X} \rightarrow \mathcal{A}$ be the trace of an n -additive map. If t is commuting then there exist $\lambda_0 \in C_{\mathcal{A}}$ and traces of i -additive maps $\lambda_i : \mathcal{X} \rightarrow C_{\mathcal{A}}$, $i = 1, \dots, n$ such that*

$$t(x) = \lambda_0 x^n + \lambda_1(x)x^{n-1} + \dots + \lambda_{n-1}(x)x + \lambda_n(x) \quad \text{for all } x \in \mathcal{X}.$$

In the special case when $n = 2$ and \mathcal{X} is the set of symmetric elements Lee and Lee [113] proved a result similar to (i) in Theorem 4.2 without any restriction on $\deg(\mathcal{A})$. So it is possible to predict that Theorem 4.3 could be strengthened. We also mention that the method by Lee and Lee was used in characterizing associating traces of biadditive maps on nondegenerate Jordan algebras [59].

It is about time to say a few words about the theory of *functional identities*. The name “functional identity” was introduced by the author in [52]. This paper and [53] were predecessors of the general theory. The major breakthroughs were done by Beidar [11] and Chebotar [77]. The basic functional identity considered in the general theory is

$$\sum_{i=1}^n E_i(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)x_i + \sum_{j=1}^n x_j F_j(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n) = 0$$

where E_i, F_j are maps from \mathcal{X}^{n-1} into \mathcal{A} (here of course \mathcal{X} is a subset of the ring \mathcal{A}). For example, if t is a commuting trace of a symmetric $(n-1)$ -additive map

M , then by linearizing $[t(x), x] = 0$ we get this identity with $E_i = M = -F_j$ for each i and j . On the other hand, it is easy to see that every multilinear polynomial identity can be interpreted as the identity of this type. One can consider even far more general functional identities, for example such involving summands of the form

$$x_{i_1} \dots x_{i_r} F(x_{j_1}, \dots, x_{j_s}) x_{k_1} \dots x_{k_t},$$

or such that also involve some fixed elements from the ring (the so-called generalized functional identities) etc. In fact, all identities mentioned in this paper can be regarded as special examples of functional identities.

The usual goal when studying a functional identity is to describe the form of the maps involved or to show that a ring admitting a functional identity in which maps can not be described must have a special structure (often the conclusion is that it is a PI ring of some special degree). A more precise description of the main results and methods of the theory of functional identities would necessarily occupy a lot of space, so we feel that it is better to resist the temptation to expose this subject in greater detail and instead refer to [55] for an introductory account and to some of the most recent articles [14, 16, 17, 19, 26, 27, 56] for the advanced theory.

5. APPLICATIONS

The result on commuting traces of biadditive maps, which has been discussed in the previous section, particularly stimulated the further development of the theory because of various applications that were found already in [49]. Before encountering some specific topics we point out a different aspect from which the condition treated in this result may be viewed.

Let \mathcal{A} be a ring. A biadditive map from \mathcal{A}^2 into \mathcal{A} can be regarded as another multiplication $(x, y) \mapsto x * y$ on \mathcal{A} under which the additive group of \mathcal{A} becomes a nonassociative ring. The condition that the trace of this biadditive map is commuting, i.e.

$$(12) \quad (x * x)x = x(x * x) \quad \text{for all } x \in \mathcal{A},$$

thus means that the square (with respect to the new multiplication) of each element in \mathcal{A} commutes (with respect to the original multiplication) with this element. This point of view indicates why several applications lie in the meeting place of the associative and the nonassociative algebra.

5.1 Lie Isomorphisms

Let \mathcal{A} be a ring. If we replace the original product by the *Lie product* $[x, y] = xy - yx$, the additive group of \mathcal{A} becomes a Lie ring. If $\text{char}(\mathcal{A}) = 2$ then the

Lie product coincides with the *Jordan product* $x \circ y = xy + yx$ which makes the treatment of these notions rather muddled. We shall therefore usually assume that our rings have characteristic different from 2.

An additive subgroup of \mathcal{A} closed under the Lie product is called a *Lie subring* of \mathcal{A} . Let \mathcal{L}' be a Lie subring of the ring \mathcal{A}' and let \mathcal{L} be a Lie subring of the ring \mathcal{A} . A bijective additive map $\theta : \mathcal{L}' \rightarrow \mathcal{L}$ is called a *Lie isomorphism* if

$$\theta([u, v]) = [\theta(u), \theta(v)] \quad \text{for all } u, v \in \mathcal{L}',$$

that is, θ is an isomorphism between Lie rings \mathcal{L}' and \mathcal{L} . In his 1961 AMS Hour Talk [92] Herstein formulated several conjectures on various “Lie type” maps in associative rings. Roughly speaking, he conjectured that these maps arise from appropriate “associative” maps, so for example that Lie isomorphisms can be expressed through (anti)isomorphisms between \mathcal{A}' and \mathcal{A} . In the classical case of finite dimensional algebras the results of this kind have been known for a long time (see e.g. [96, Chapter 10]), and Herstein proposed the problem to extend them to a much more general level. There have been numerous publications by several mathematicians on Herstein’s conjectures, but we mention Martindale as a major force in this program. Until the 90’s all solutions had been obtained under the assumption that the rings contain some nontrivial idempotents (see e.g. Martindale’s survey [134] from 1976). We also mention that similar problems have also been considered in operator algebras [8, 91, 142–147] where idempotents also play an important role. Roughly speaking, there are many important rings that contain nontrivial idempotents, but there are also many that do not (say, domains and in particular division rings). The problem whether the assumptions on idempotents can be removed in the results of Martindale and others was open for a long time. Rather recently it was finally solved by making use of commuting maps and more general functional identities. The great advantage of this approach is that it is independent of some local properties of rings; say, the existence of some special elements such as idempotents is irrelevant.

We first consider the simplest case when $\mathcal{L}' = \mathcal{A}'$ and $\mathcal{L} = \mathcal{A}$. Isomorphisms between \mathcal{A}' and \mathcal{A} are of course also Lie isomorphisms. Other basic examples are maps of the form $\theta = -\psi$ where ψ is an antiisomorphism. Moreover, if a map $\tau : \mathcal{A}' \rightarrow Z_{\mathcal{A}}$ vanishes on commutators then $\theta + \tau$ also preserves the Lie product for every Lie isomorphism θ . Thus, a typical example of a Lie isomorphism $\theta : \mathcal{A}' \rightarrow \mathcal{A}$ is $\theta = \varphi + \tau$ where φ is either an isomorphism or the negative of an antiisomorphism and τ is a central additive map such that $\tau([\mathcal{A}', \mathcal{A}']) = 0$. It has been known for a long time that in the fundamental case when $\mathcal{A}' = \mathcal{A} = \mathcal{M}_n(F)$ with F a field these are also the only possible examples of Lie isomorphisms. In 1951 Hua [94] generalized this by proving that the same is true if $\mathcal{A}' = \mathcal{A} = M_n(\mathcal{D})$ where $n \geq 3$ and \mathcal{D} is a division ring. Herstein [92] conjectured that this should be true in all simple and perhaps even prime rings. This problem was studied by Martindale

in [126, 128, 129, 131]. The culminating result of this series of papers is that a Lie isomorphism θ between unital prime rings \mathcal{A}' and \mathcal{A} is of the expected form $\theta = \varphi + \tau$, provided, however, that \mathcal{A} contains an idempotent $e \neq 0, 1$. Here, τ does not necessarily map into the center $Z_{\mathcal{A}}$ but into the extended centroid $C_{\mathcal{A}}$, and the range of φ lies in the so-called *central closure* \mathcal{A}_C of \mathcal{A} , that is, the subring of the right (or left, or symmetric) Martindale ring of quotients of \mathcal{A} generated by \mathcal{A} and $C_{\mathcal{A}}$. An example in [126] shows that the range of φ need not be contained in \mathcal{A} . In fact, it was the Lie isomorphism problem which motivated Martindale to introduce the concept of the extended centroid.

Recently the following generalization of Martindale's theorem, giving the complete solution of Herstein's conjecture, was proved.

Theorem 5.1. *Let \mathcal{A}' and \mathcal{A} be noncommutative prime rings of characteristic not 2. Then every Lie isomorphism θ of \mathcal{A}' onto \mathcal{A} is of the form $\theta = \varphi + \tau$, where φ is either an isomorphism or the negative of an anti-isomorphism of \mathcal{A}' onto the subring of \mathcal{A}_C , and τ is an additive map of \mathcal{A}' into $C_{\mathcal{A}}$ sending commutators to 0.*

Theorem 5.1 was proved by the author in [49], however, under the additional technical assumption that \mathcal{A} and \mathcal{A}' do not satisfy St_4 . This assumption was removed by Blau [44] who used the classical structure theory of PI rings together with Martindale's result [129]. Another more straightforward proof based only on commuting maps was found recently in [69].

The main idea of the proof can be easily described. Every element commutes with its square and so θ satisfies $[\theta(u^2), \theta(u)] = 0$ for every $u \in \mathcal{A}'$. Setting $x = \theta(u)$ we can rewrite this as

$$[q(x), x] = 0 \text{ for all } x \in \mathcal{A}, \text{ where } q : x \mapsto \theta(\theta^{-1}(x)^2).$$

That is, q is a commuting map and clearly it is the trace of a biadditive map $B : (x, y) \mapsto \theta(\theta^{-1}(x)\theta^{-1}(y))$. So we are in a position to apply Theorem 4.1. Hence there are $\lambda \in C_{\mathcal{A}}$ and $\mu, \nu : \mathcal{A} \rightarrow C_{\mathcal{A}}$ with μ additive such that

$$q(x) = \lambda x^2 + \mu(x)x + \nu(x) \quad \text{for all } x \in \mathcal{A}.$$

Setting $\eta = \mu\theta : \mathcal{A}' \rightarrow C_{\mathcal{A}}$ and writing u for $\theta^{-1}(x)$ it follows that

$$\theta(u^2) - \lambda\theta(u)^2 - \eta(u)\theta(u) \in C_{\mathcal{A}} \quad \text{for all } u \in \mathcal{A}'.$$

So we now have some control concerning the action of θ on squares, and hence (linearization!) also on the Jordan product; by the initial assumption we know how θ acts on the Lie product and so it should not be of surprise anymore that we are able to describe the action of θ on the original product $xy = \frac{1}{2}([x, y] + x \circ y)$.

The main breakthrough has already been made, but there is more to the proof. Let us briefly sketch what remains to be done. We have to divide the proof by

considering separately two cases, the one that none of \mathcal{A}' and \mathcal{A} satisfies St_4 , and another one when one of them does satisfy St_4 . In the first case (cf. [49]) we define $\varphi : \mathcal{A}' \rightarrow \mathcal{A}_C$ by

$$\varphi(u) = \lambda\theta(u) + \frac{1}{2}\eta(u),$$

and then prove that either $\lambda = 1$ and φ is an isomorphism or $\lambda = -1$ and φ is an antiisomorphism. The argument in the second case is somewhat shorter (cf. [69]). Assume, for example, that \mathcal{A} satisfies St_4 , that is, $\deg(\mathcal{A}) \leq 2$. Then for every $u \in \mathcal{A}'$ there is $\rho(u) \in C_{\mathcal{A}}$ such that $\theta(u)^2 - \rho(u)\theta(u) \in C_{\mathcal{A}}$. Moreover, one can show that $\rho(u)$ can be chosen so that the map $u \mapsto \rho(u)$ is additive. Without loss of generality we may assume that $\lambda = 0$. We define $\varphi : \mathcal{A}' \rightarrow \mathcal{A}_C$ by

$$\varphi(u) = \theta(u) - \frac{1}{2}(\rho(u) - \eta(u))$$

and then prove that φ is an isomorphism (antiisomorphisms do not appear in the St_4 case, since in this very special situation they can be expressed by isomorphisms and central maps).

Theorem 5.1 settles only the simplest one among Herstein's conjectures on Lie isomorphisms between Lie subrings of associative rings. Let us consider another important case when \mathcal{A}' and \mathcal{A} are rings with involution and $\mathcal{L}' = \mathcal{K}'$ and $\mathcal{L} = \mathcal{K}$ are their Lie subrings of skew elements. This problem is considerably more difficult, in particular since in certain finite dimensional algebras there are counterexamples to the expected and usual conclusion [133, pp. 942-943]. We shall assume that \mathcal{A}' and \mathcal{A} are prime rings and that involutions are of the first kind, meaning that they are linear over the extended centroid (see [36, Section 9.1] for a more detailed explanation; we also mention that an involution is said to be of the second kind if it is not of the first kind). This problem was considered in the 70's by Martindale for rings containing idempotents [132, 133]. The approach avoiding idempotents is based on the observation that the cube of every skew element is skew again, and so a Lie isomorphism $\theta : \mathcal{K}' \rightarrow \mathcal{K}$ satisfies $[\theta(l^3), \theta(l)] = 0$ for all $l \in \mathcal{K}'$. Note that this can be interpreted as

$$[t(k), k] = 0 \text{ for all } k \in \mathcal{K}, \text{ where } t : k \mapsto \theta(\theta^{-1}(k)^3).$$

Thus, t is a commuting trace of a 3-additive map on \mathcal{K} , and so Theorem 4.3 can be applied. This approach was used by Beidar, Martindale and Mikhalev in [35]. Actually Theorem 4.3 did not yet exist in this form at that time, so they had to consider commuting traces of 3-additive maps on \mathcal{K} first. Their work was continued in [45] and [79]. In the result that we are now going to state we shall also take into account a technical improvement of their result obtained by Chebotar [79] who also gave a shorter proof.

Theorem 5.2. *Let \mathcal{A}' and \mathcal{A} be prime rings with involutions of the first kind and of characteristic not 2. Let \mathcal{K}' and \mathcal{K} denote respectively the skew elements of \mathcal{A}' and \mathcal{A} . Assume that the dimension of the central closure of \mathcal{A}' over $C_{\mathcal{A}'}$ is different from 1, 4, 9, 16, 25 and 64. Then any Lie isomorphism θ of \mathcal{K}' onto \mathcal{K} can be extended uniquely to an associative isomorphism of $\langle \mathcal{K}' \rangle$ onto $\langle \mathcal{K} \rangle$, the associative subrings generated by \mathcal{K}' and \mathcal{K} respectively.*

If Φ was the negative of an antiisomorphism from $\langle \mathcal{K}' \rangle$ onto $\langle \mathcal{K} \rangle$ which coincides with θ on \mathcal{K}' , then $x \mapsto -\Phi(x^*)$ would be an isomorphism from $\langle \mathcal{K}' \rangle$ onto $\langle \mathcal{K} \rangle$ which also extends θ . This explains why the presence of antiisomorphisms can be avoided in Theorem 5.2.

There are other important examples of Lie subrings that have not been considered so far, for example $[\mathcal{A}, \mathcal{A}]$ or $[\mathcal{K}, \mathcal{K}]$ (i.e., the additive subgroups generated by all commutators in \mathcal{A} and \mathcal{K} respectively), and more generally Lie ideals of \mathcal{A} and \mathcal{K} (i.e., ideals of the Lie rings \mathcal{A} and \mathcal{K}). Since these Lie subrings are not necessarily closed under some powers of elements, the same tricks as above, which basically reduce the Lie isomorphism problems to the commuting map problems, do not always work. But in this case more profound functional identities methods can be used. In the recent series of papers [20, 22, 29] Lie epimorphisms (not only isomorphisms) between various Lie subrings of associative rings have been characterized; in particular, all of Herstein's Lie map conjectures have been settled (see the last paper in the series [22]). The proofs in these papers are based on advanced results on functional identities, especially the concept of d -freeness [26, 27] plays an important role. Explaining this in greater detail exceeds the scope of this paper.

We also mention that certain modified versions of the results presented in Section 4 have been used in the study of Lie isomorphisms in Banach algebras [5, 41, 42, 60, 64]. Some of these works are concerned with automatic continuity problems. For instance, Berenguer and Villena [41] proved that the separating space of every Lie isomorphism from a semisimple Banach algebra \mathcal{A}' onto a semisimple Banach algebra \mathcal{A} lies in the center of \mathcal{A} .

The reader might wonder whether analogous Jordan map problems could also be handled by using the methods exposed in this paper. Indeed Theorem 4.1 was used in the solution of Herstein's problem [92] on n -Jordan homomorphisms [62]. To treat some similar but more difficult problems, the more general functional identities approach has to be used [20, 23, 32].

5.2 Other Applications to Lie Theory

Let \mathcal{L} be a Lie subring of a ring \mathcal{A} . An additive map $\delta : \mathcal{L} \rightarrow \mathcal{L}$ is called a *Lie derivation* if

$$\delta([x, y]) = [\delta(x), y] + [x, \delta(y)] \quad \text{for all } x, y \in \mathcal{L}.$$

A typical example is the sum of a derivation and a central map sending commutators to 0. Lie derivations can be studied similarly as were Lie isomorphisms, and almost all results on Lie isomorphisms have their Lie derivation parallels. The history of both problems is also similar: Lie derivations are also a part of Herstein's program [92] and with the presence of idempotents they have been examined by Martindale [127] and others (apparently the first result in this direction was obtained in an unpublished work of Kaplansky, cf. [92, p. 529]). We shall therefore examine this topic very briefly, pointing out only some initial ideas.

Suppose that $\mathcal{L} = \mathcal{A}$ is a prime ring. It is clear every Lie derivation δ on \mathcal{A} satisfies $[\delta(x), x^2] + [x, \delta(x^2)] = 0$. Since $[\delta(x), x^2] = [\delta(x)x + x\delta(x), x]$, we can rewrite the last identity as

$$[q(x), x] = 0 \text{ for all } x \in \mathcal{A}, \text{ where } q : x \mapsto \delta(x^2) - \delta(x)x - x\delta(x).$$

Clearly q is the trace of a biadditive map and so we are in a position to apply Theorem 4.1. This gives us an information how δ acts on the Jordan product. This is the idea upon which the proof of the following theorem is based. For rings not satisfying St_4 this theorem was proved in [49], while the St_4 case was covered recently in [69].

Theorem 5.3. *Let \mathcal{A} be a prime ring with $\text{char}(\mathcal{A}) \neq 2$. Then every Lie derivation δ of \mathcal{A} is of the form $\delta = d + \tau$, where d is a derivation of \mathcal{A} into \mathcal{A}_C and τ is an additive map of \mathcal{A} into $C_{\mathcal{A}}$ sending commutators to 0.*

An analogue of Theorem 5.2 for Lie derivations was proved by Swain in [165]. The critical observation in the proof is that every Lie derivation δ of \mathcal{K} satisfies

$$[t(k), k] = 0 \text{ for all } k \in \mathcal{K}, \text{ where } t : k \mapsto \delta(k^3) - \delta(k)k^2 - k\delta(k)k - k^2\delta(k).$$

For more recent results on Lie derivations on rings we refer to [9, 21, 22, 28, 166]. The most complete results are contained in the last paper [22] of this series.

Commuting maps have also been successfully applied to the analytic study of Lie derivations [1, 5, 40, 41, 43, 169]. For example, Johnson's result on the structure of continuous Lie derivations from C^* -algebras into their bimodules [97] was proved in [1] without assuming the continuity for some classes of C^* -algebras (in particular for von Neumann algebras). In this context we mention that commuting maps from a ring into its arbitrary bimodule can also be described in some cases.

We continue by presenting the initial ideas from the paper [25] by Beidar and Chebotar. For further development see [31].

Let $(\mathcal{U}, +, *)$ be a nonassociative algebra. We say that \mathcal{U} is *Lie-admissible* if \mathcal{U} becomes a Lie algebra when replacing the original product in \mathcal{U} by the product $[x, y] = x * y - y * x$. Of course, associative algebras are Lie-admissible. These algebras were introduced by Albert [2] in 1949 and they have been studied by a number of authors (see [25] for references).

In this context we introduce another concept. Let \mathcal{A} be an associative algebra over a field F . Suppose there exists an additional nonassociative multiplication $*$: $\mathcal{A}^2 \rightarrow \mathcal{A}$ such that for some nonzero $\gamma \in F$ we have $y * x - x * y = \gamma[y, x]$ for all $x, y \in \mathcal{A}$. In this case we say that $*$ is *Lie-compatible* (of course $(\mathcal{A}, +, *)$ is then a Lie-admissible algebra). Assume further that $*$ is *third-power associative*, that is, $(x * x) * x = x * (x * x)$ holds for all $x \in \mathcal{A}$. Replacing y by $x * x$ in the former identity we get

$$[q(x), x] = 0 \text{ for all } x \in \mathcal{A}, \text{ where } q : x \mapsto x * x.$$

So again we have arrived at the situation when Theorem 4.1 is applicable. As a corollary we obtain the result which we are now going to state; we omit giving details of the proof, but actually they are quite simple. It should be pointed out, however, that this is just one of the simplest results in this setting, and an interested reader should consult [25] and [31].

We say that a prime algebra \mathcal{A} over a field F is a *centrally closed prime algebra over F* if it is unital and $Z_{\mathcal{A}} = C_{\mathcal{A}} = F\mathbf{1}$. In this case \mathcal{A} of course coincides with its central closure.

Theorem 5.4. *Let F be a field with $\text{char}(F) \neq 2$ and let \mathcal{A} be a centrally closed prime algebra over F . Let $*$ be a Lie-compatible multiplication on \mathcal{A} . Then $*$ is third-power associative if and only if there exist $\lambda_1, \lambda_2 \in F$, $\lambda_1 \neq \lambda_2$, an F -linear map $\mu : \mathcal{A} \rightarrow F$, and a symmetric F -bilinear map $\tau : \mathcal{A}^2 \rightarrow F$ such that*

$$x * y = \lambda_1 xy + \lambda_2 yx + \mu(x)y + \mu(y)x + \tau(x, y)\mathbf{1} \quad \text{for all } x, y \in \mathcal{A}.$$

For the matrix algebra $\mathcal{A} = M_n(F)$ this theorem was proved by Benkart and Osborn [39] who used completely different methods. We also remark that in [25, Theorem 1.3] this result is stated under the additional assumption that \mathcal{A} does not satisfy St_4 . The reason for this is that when [25] was published it was not yet known that in Theorem 4.1 this assumption is superfluous.

Finally we give just a hint how to tackle another Lie theoretic topic. We say that $(\mathcal{P}, +, \cdot, \{., .\})$ is a *Poisson algebra* if $(\mathcal{P}, +, \cdot)$ is an associative algebra, $(\mathcal{P}, +, \{., .\})$ is a Lie algebra, and $\{x \cdot y, z\} = x \cdot \{y, z\} + \{x, z\} \cdot y$ for all $x, y, z \in \mathcal{P}$. These algebras originally appeared in differential geometry and have also been studied as algebraic structures [87-89, 103, 104]. First we remark that if $(\mathcal{P}, +, \{., .\})$ is a Lie subalgebra of some associative algebra \mathcal{A} so that $\{x, y\} = [x, y]$ ($= xy - yx$ where xy denotes the product of x and y in \mathcal{A}), then setting $x = y = z$ we arrive at a now familiar situation

$$[q(x), x] = 0 \text{ for all } x \in \mathcal{A}, \text{ where } q : x \mapsto x \cdot x.$$

Given a (commutative) Poisson algebra $(\mathcal{P}, +, \cdot, \{ \cdot, \cdot \})$, Dirac's problem is to find all Lie homomorphisms from \mathcal{P} to noncommutative algebras (specifically, for algebras of linear operators acting on a Hilbert space) [98, 163, 164]. Since a noncommutative algebra is a Poisson algebra under the standard Lie operation, in the framework of noncommutative prime algebras Dirac's problem includes Herstein's Lie isomorphism problem. The study of Lie homomorphisms of (not necessarily commutative) Poisson algebras onto noncommutative associative algebras is the main theme of another work of Beidar and Chebotar [30]. The identity $[q(x), x] = 0$ plays an important role in their study.

5.3 Linear Preservers

The theory of linear preservers deals with maps on algebras which, roughly speaking, preserve some properties of some elements in an algebra. The usual goal is to describe such maps. Since automorphisms, as well as antiautomorphisms, preserve algebraic properties of elements, they appear very often in these descriptions. The list of publications on linear preservers is voluminous and so we refer only to some survey papers [6, 67, 122, 125]. Most of results on linear preserver treat algebras of matrices or operators. We shall see that the approach based on commuting maps allows us to obtain ring-theoretic generalizations of some of these results.

One of the most thoroughly studied problems is that on commutativity preservers. Consider a bijective linear map θ between unital algebras \mathcal{A}' and \mathcal{A} such that $\theta(x)$ and $\theta(y)$ commute whenever x and y commute. Lie isomorphisms clearly have this property. The standard conclusion is that $\theta(x) = \lambda\varphi(x) + \mu(x)\mathbf{1}$ where λ is a nonzero scalar, μ is a linear functional, and φ is either an isomorphism or an antiisomorphism. In 1976 Watkins [172] proved that this is true in the case when $\mathcal{A}' = \mathcal{A} = M_n(F)$, $n \geq 4$. Moreover, he constructed a counterexample for $n = 2$. On the other hand, it has turned out that the $n = 3$ case is not exceptional [10, 151]. Somewhat later, in the 80's, this result was extended to infinite dimensional algebras: [83] considers the algebra of all bounded linear operators on a Hilbert space, [150] considers the algebra of all bounded linear operators on a Banach space, and [149] considers von Neumann factors. All these algebras are centrally closed prime algebras over \mathbb{C} (or \mathbb{R}). In [49] we proved the following theorem which generalizes and unifies the results of these papers.

Theorem 5.5. *Let \mathcal{A}' and \mathcal{A} be centrally closed prime algebras over a field F with $\text{char}(F) \neq 2, 3$, and suppose that none of them satisfies $St_{\mathcal{A}}$. Let $\theta : \mathcal{A}' \rightarrow \mathcal{A}$ be a bijective linear map satisfying $[\theta(x^2), \theta(x)] = 0$ for all $x \in \mathcal{A}'$. Then $\theta(x) = \lambda\varphi(x) + \mu(x)\mathbf{1}$ where $\lambda \in F$, $\lambda \neq 0$, μ is a linear functional, and φ is either an isomorphism or an antiisomorphism from \mathcal{A}' onto \mathcal{A} .*

So, in particular the assumption that commutativity is preserved can be weakened by assuming only that $\theta(x^2)$ always commutes with $\theta(x)$. The idea of the proof is the

same as in the Lie isomorphism case: we can interpret the condition $[\theta(x^2), \theta(x)] = 0$ as

$$[q(x), x] = 0 \text{ for all } x \in \mathcal{A}, \text{ where } q : x \mapsto \theta(\theta^{-1}(x)^2).$$

For various extensions of Theorem 5.5, which are based on the commuting map approach, we refer to [5, 9, 15, 16, 18, 24, 64, 123].

Linear maps preserving the commutativity of symmetric matrices (and operators) have also been studied [75, 83, 156]. A ring-theoretic generalization [18] of these results is based on the identity

$$[q(s), s] = 0 \text{ for all } s \in \mathcal{S}, \text{ where } q : s \mapsto \theta(\theta^{-1}(s)^2);$$

here, \mathcal{S} is the set of symmetric elements in a ring with involution, and so Theorem 4.3 can be applied.

The last topic we are going to consider concerns normal-preservers. Linear maps preserving the set of normal matrices and operators were treated in [83, 105, 152] where methods completely different from those presented in this paper were used, and also in [66] where it was noted for the first time that commuting maps can be used in this problem. The recent paper [18] contains ring-theoretic generalizations of these results. We recall that an element x in a ring with involution $*$ is said to be *normal* if x commutes with x^* . We now state the simpler one among two results from [18] on normal-preservers.

Theorem 5.6. *Let \mathcal{A}' and \mathcal{A} be centrally closed prime algebras over a field F with involutions of the second kind. Suppose that $\text{char}(F) \neq 2, 3$, and suppose that none of \mathcal{A}' and \mathcal{A} satisfies St_4 . Let $\theta : \mathcal{A}' \rightarrow \mathcal{A}$ be a bijective linear map with the property that $\theta(x)$ is normal whenever $x \in \mathcal{A}'$ is normal. Then $\theta(x) = \lambda\varphi(x) + \mu(x)\mathbf{1}$ where $\lambda \in F$, $\lambda \neq 0$, μ is a linear functional, and φ is either a $*$ -isomorphism or a $*$ -antiisomorphism from \mathcal{A}' onto \mathcal{A} .*

Theorem 5.6 is actually a corollary to Theorem 5.5 – the main goal of its proof is to show that θ satisfies the condition $[\theta(x^2), \theta(x)] = 0$ for all $x \in \mathcal{A}'$. Another result in [18] treats the case when the involution is of the first kind. This case is much more involved. One of the identities that one has to face here is

$$[q(k), k] = 0 \text{ for all } k \in \mathcal{K}, \text{ where } q : k \mapsto \theta(\theta^{-1}(k)^2);$$

here \mathcal{K} is the set of skew elements (see [18] for details).

We have thereby examined the main areas of applications. Hopefully, more possible areas exist, but they still have to be discovered.

ACKNOWLEDGEMENTS

Most parts of this paper were presented in the series of the author's lectures at the National Cheng Kung University, the National Taiwan University and the

National Sun Yat-Sen University during his several visits of Taiwan over the last years. The author would like to express his sincere thanks to K. I. Beidar, Y. Fong and W.-F. Ke from Tainan, C.-L. Chuang, P.-H. Lee and T.-K. Lee from Taipei, and N.-C. Wong and T.-L. Wong from Kaohsiung for arranging his visits and lectures and for the warm hospitality which they extended to him.

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