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ORLICZ SPACES THAT ARE UNIFORMLY ROTUND IN THE DIRECTION OF WEAKLY COMPACT SETS

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Abstract. Sufficient and necessary condition of Orlicz spaces equipped with Orlicz norm that are uniformly rotund in the direction of weakly compact sets using only conditions on generated function of the space are given.

1. Introduction

Let X be a Banach space and let $S(X)$ and $B(X)$ be the unit sphere and the unit ball of X. X is said to be uniformly rotund in the direction of weakly compact sets (URWC) if $\|x_n\| \to 1$, $\|y_n\| \to 1$, $\|x_n + y_n\| \to 2$, and $x_n - y_n \overset{w}{\to} z$ (in weak topology) imply that $z = 0$ [7]. X is said to be uniformly rotund in every direction (URED) if $\|x_n\| \to 1$, $\|y_n\| \to 1$, $\|x_n + y_n\| \to 2$, and $x_n - y_n \to z$ (in norm topology) imply that $z = 0$. X is said to be uniformly weak* rotund (W*UR) if $\|x_n\| \to 1$, $\|y_n\| \to 1$, and $\|x_n + y_n\| \to 2$ imply that $x_n - y_n \overset{w}{\to} 0$. X is said to be rotund (R) if $\|x\| = 1$, $\|y\| = 1$, and $\|x + y\| = 2$ imply that $x = y$. Clearly,

$$\text{W*UR} \Longrightarrow \text{URWC} \Longrightarrow \text{URED} \Longrightarrow \text{R}.$$

Banach spaces with these types of rotundity were studied in [7], [8] and have been applied to fixed point theory. For Orlicz spaces with Luxemburg norm, W*UR is equivalent to R. But for Orlicz spaces with Orlicz norm, W*UR and URED have much different criteria [11], [12]. All known characterizations of URWC for Orlicz spaces with Orlicz norm have been described by reference both to elements in the Orlicz space and to the generated function M [5], [10], [14]. Up to now, no

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characterization of URWC by using only conditions on the generated function M has been given. As stated in [9], "some new methods and techniques are needed to solve this kind of difficult problems." In this paper, we give a characterization of URWC by using only conditions on the generated function M. As a consequence, we show that no criterion of URWC for Orlicz spaces with Orlicz norm can be obtained by using only the classical conditions of M, such as $M \in$ UC, $M \in \Delta_2$, and $M \in \nabla_2$. The proof of our result is relatively complicated.

In the sequel, let $\Re$ be the set of all real numbers. A function $M : \Re \to \Re_+$ is called an N-function if M is convex and even, $\lim_{u\to 0} \frac{M(u)}{u} = 0$, and $\lim_{u\to\infty} \frac{M(u)}{u} = \infty$. The complemented function N of M is defined in the sense of Young by

$$N(v) = \sup_{u\in\Re}\{uv - M(u)\}.$$

It is known that if M is an N-function, then its complemented function N is also an N-function. Let p and q be the derivatives on the right-hand of M and N, respectively. M is said to be strictly convex (SC) if $M(\frac{u+v}{2}) < \frac{M(u)+M(v)}{2}$ for $u \neq v$. M is said to be uniformly convex if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for given u and v, if $|u - v| \geq \varepsilon \max(|u|, |v|)$, then $M(\frac{u+v}{2}) < (1-\delta)\frac{M(u)+M(v)}{2}$. M is said to satisfy the Δ_2 condition for large u ($M \in \Delta_2$) if for some $u_0 > 0$ there exists $K > 0$ such that for all $u \geq u_0$, $M(2u) \leq KM(u)$. M is said to satisfy the ∇_2 condition ($M \in \nabla_2$) if $N \in \Delta_2$. Let G be a bounded set in $\Re^n$ and let (μ, Σ, G) be a finite atomless measure space. For a real-valued measurable function $x(t)$ on G, let $\rho_M(x) = \int_G M(x(t))\,d\mu$, called the modular of x. The Orlicz function space L_M generated by M is the Banach space

$$L_M = \{x(t) : \rho_M(\lambda x) < \infty \text{ for some } \lambda\},$$

equipped with the Orlicz norm

$$\|x\|_M = \sup_{\rho_N(y)\leq 1} \int_G x(t)y(t)\,d\mu = \inf_{k>0} \frac{1}{k}\left(1 + \rho_M(kx)\right).$$

See [3], [6] for references to Orlicz function spaces.

We firstly state several lemmas.

Lemma 1. ([3, 13]) *For* $x \in L_M$ *and for* $k \in K(x) = [K^*, K^{**}]$, *where* $K^* = \inf\{k : \rho_N(p(kx)) \geq 1\}$ *and* $K^{**} = \sup\{k : \rho_N(p(kx)) \leq 1\}$, *the Orlicz norm* $\|x\|_M$ *is given by*

$$\|x\|_M = \frac{1}{k}\left(1 + \rho_M(kx)\right).$$

Lemma 2. ([11]) *Suppose* $2 \ge \|x_n\|_M = \frac{1}{k_n}\left(1+\rho_M(k_nx_n)\right)$ $(n = 1, 2, \dots)$ *and* $k_n \to \infty$. *Then* $x_n(t)$ *is convergent to zero in measure.*

Lemma 3. ([11]) *Suppose* $M \in \Delta_2$, *and* $\|x_n\|_M = \frac{1}{k_n}\left(1+\rho_M(k_nx_n)\right) \to 1$, $\|y_n\|_M = \frac{1}{h_n}\left(1+\rho_M(h_ny_n)\right) \to 1$ $(n \to \infty)$, *with* $\{k_n\}$ *and* $\{h_n\}$ *bounded. If* $\|x_n + y_n\|_M \to 2$ *then* $k_nx_n(t) - h_ny_n(t) \xrightarrow{\mu} 0$ *in measure.*

Lemma 4. ([11]) *Let* $\|x_n\|_M = \frac{1}{k_n}\left(1+\rho_M(k_nx_n)\right) \to 1$, $\|y_n\|_M = \frac{1}{h_n}\left(1+\rho_M(h_ny_n)\right) \to 1$ $(n \to \infty)$, *with* $\{k_n\}$ *and* $\{h_n\}$ *bounded. For* $v_n \in L_N$, $\rho_N(v_n) \le 1$ *and* $\int_G (x_n(t) + y_n(t))v_n(t)\,d\mu \to 2$, *then there hold uniformly for all sets* $G_n \in \Sigma$,

$$\lim_{n\to\infty}\int_{G_n}\left(k_nx_n(t) - h_ny_n(t)\right)v_n(t)\,d\mu = \lim_{n\to\infty}\int_{G_n}\left(M(k_nx_n(t)) - M(h_ny_n(t))\right)d\mu.$$

Next, we prove some lemmas with elementary arguments.

Lemma 5. *If* $M \in SC$, *then* $0 < \lambda < 1$, $\varphi(t) = \frac{M(\lambda u+(1-\lambda)t)}{\lambda M(u)+(1-\lambda)M(t)}$ *is increasing* on $[0, u]$.

Proof. Because $M \in SC$, $p(u)$ is increasing on $[0, \infty)$, we get

$$\varphi'(t) = (1-\lambda)\frac{p\left(\lambda u+(1-\lambda)t\right)\left(\lambda M(u)+(1-\lambda)M(t)\right) - M\left(\lambda u+(1-\lambda)t\right)p(t)}{[\lambda M(u)+(1-\lambda)M(t)]^2}$$
$$> 0$$ ■

Lemma 6. *For* $\varepsilon \in (0, 1)$, $\lambda \in [\alpha, \beta] \subset (0, 1)$, *and* u *in* $\Re$, *let*

$$x = \frac{M\left(\frac{u + (1-\varepsilon)u}{2}\right)}{\frac{M(u) + M((1-\varepsilon)u)}{2}}$$

and

$$y = \frac{M\left(\lambda u + (1-\lambda)(1-\varepsilon)u\right)}{\lambda M(u) + (1-\lambda)M((1-\varepsilon)u)}.$$

Then

$$\lim_{x\to1} y = 1 \quad \text{uniformly for } \lambda \in [\alpha, \beta] \qquad \text{if and only if} \qquad \lim_{y\to1} x = 1.$$

Proof. Let $f(\theta) = M\big(\theta u + (1-\theta)(1-\varepsilon)u\big) - \theta M(u) - (1-\theta) M\big((1-\varepsilon)u\big)$. By [6], $f(\theta)$ is convex and for $\theta', \theta'' \in [\alpha, \beta]$, $\theta' < \theta''$,

$$\frac{\theta''}{\theta'} f(\theta') \le f(\theta'') \le \frac{1-\theta''}{1-\theta'} f(\theta').$$

Thus

$$\begin{aligned}
&\frac{\theta''}{\theta'}\Big[M\big(\theta' u + (1-\theta')(1-\varepsilon)u\big) - \theta' M(u) - (1-\theta') M\big((1-\varepsilon)u\big)\Big]\\
&\le M\big(\theta'' u + (1-\theta'')(1-\varepsilon)u\big) - \theta'' M(u) - (1-\theta'') M\big((1-\varepsilon)u\big)\\
&\le \frac{1-\theta''}{1-\theta'}\Big[M\big(\theta' u + (1-\theta')(1-\varepsilon)u\big) - \theta' M(u) + (1-\theta') M\big((1-\varepsilon)u\big)\Big].
\end{aligned}$$

Hence

$$\begin{aligned}
&\frac{\theta''}{\theta'}\Big[\theta' M(u) + (1-\theta') M\big((1-\varepsilon)u\big)\Big]\left[\frac{M\big(\theta' u + (1-\theta')(1-\varepsilon)u\big)}{\theta' M(u) + (1-\theta') M\big((1-\varepsilon)u\big)} - 1\right]\\
&\le \Big[\theta'' M(u) + (1-\theta'') M\big((1-\varepsilon)u\big)\Big]\left[\frac{M\big(\theta'' u + (1-\theta'')(1-\varepsilon)u\big)}{\theta'' M(u) + (1-\theta'') M\big((1-\varepsilon)u\big)} - 1\right]\\
&\le \frac{1-\theta''}{1-\theta'}\Big[\theta' M(u) + (1-\theta') M\big((1-\varepsilon)u\big)\Big]\left[\frac{M\big(\theta' u + (1-\theta')(1-\varepsilon)u\big)}{\theta' M(u) + (1-\theta') M\big((1-\varepsilon)u\big)} - 1\right]
\end{aligned}$$

and so

$$\begin{aligned}
&\frac{\theta''}{\theta'}\left[\theta' + (1-\theta')\frac{M\big((1-\varepsilon)u\big)}{M(u)}\right]\left[\frac{M\big(\theta' u + (1-\theta')(1-\varepsilon)u\big)}{\theta' M(u) + (1-\theta') M\big((1-\varepsilon)u\big)} - 1\right]\\
&\le \left[\theta'' + (1-\theta'')\frac{M\big((1-\varepsilon)u\big)}{M(u)}\right]\left[\frac{M\big(\theta'' u + (1-\theta'')(1-\varepsilon)u\big)}{\theta'' M(u) + (1-\theta'') M\big((1-\varepsilon)u\big)} - 1\right]\\
&\le \frac{1-\theta''}{1-\theta'}\left[\theta' + (1-\theta')\frac{M\big((1-\varepsilon)u\big)}{M(u)}\right]\left[\frac{M\big(\theta' u + (1-\theta')(1-\varepsilon)u\big)}{\theta' M(u) + (1-\theta') M\big((1-\varepsilon)u\big)} - 1\right].
\end{aligned}$$

Note that $\alpha < \theta' + (1-\theta')\frac{M((1-\varepsilon)u)}{M(u)} < 1$, $\alpha < \theta'' + (1-\theta'')\frac{M((1-\varepsilon)u)}{M(u)} < 1$,

$\frac{\alpha}{\beta} \le \frac{\lambda''}{\lambda'} \le \frac{\beta}{\alpha}$, $\frac{1-\beta}{1-\alpha} \le \frac{1-\lambda''}{1-\lambda'} \le \frac{1-\alpha}{1-\beta}$. Hence,

$$\frac{M\left(\lambda' u+(1-\lambda')(1-\varepsilon)u\right)}{\lambda' M(u)+(1-\lambda')M\left((1-\varepsilon)u\right)} \to 1 \iff \frac{M\left(\lambda'' u+(1-\lambda'')(1-\varepsilon)u\right)}{\lambda'' M(u)+(1-\lambda'')M\left((1-\varepsilon)u\right)} \to 1.$$

Replacing λ'' by $\frac{1}{2}$ and λ' by λ, respectively, leads to the conclusion. ■

Lemma 7. *Let* $u > 0$. *If* $\frac{\frac{M(u)+M((1-\varepsilon)u)}{2}}{M\left(\frac{u+(1-\varepsilon)u}{2}\right)} \le 1+\delta$, *then there exists,* $(1-\frac{\varepsilon}{2})u \le t \le u$ *such that*

$$p(t-\frac{\varepsilon}{2}u) \ge \left(1-2\delta\frac{2-\varepsilon}{\varepsilon}\right)p(t).$$

Proof. From $\frac{\frac{M(u)+M((1-\varepsilon)u)}{2}}{M\left(\frac{u+(1-\varepsilon)u}{2}\right)} \le 1+\delta$, we have

$$\begin{aligned} 2\delta M\left((1-\frac{\varepsilon}{2})u\right) &\ge M(u)+M\left((1-\varepsilon)u\right)-M\left((1-\frac{\varepsilon}{2})u\right)-M\left((1-\frac{\varepsilon}{2})u\right)\\ &= \int_{(1-\frac{\varepsilon}{2})u}^{u} p(s)\,ds-\int_{(1-\varepsilon)u}^{(1-\frac{\varepsilon}{2})u} p(s)\,ds\\ &= \int_{(1-\frac{\varepsilon}{2})u}^{u} [p(s)-p(s-\frac{\varepsilon}{2}u)]\,ds\\ &\ge \frac{\varepsilon}{2}u[p(t)-p(t-\frac{\varepsilon}{2}u)] \end{aligned}$$

where $(1-\frac{\varepsilon}{2})u \le t \le u$. From $(1-\frac{\varepsilon}{2})up\left((1-\frac{\varepsilon}{2})u\right) \ge M\left((1-\frac{\varepsilon}{2})u\right)$, we have

$$\frac{4\delta}{\varepsilon}(1-\frac{\varepsilon}{2})p(t) \ge p(t)-p(t-\frac{\varepsilon}{2}u).$$

Hence

$$p(t-\frac{\varepsilon}{2}u) \ge \left(1-2\delta\frac{2-\varepsilon}{\varepsilon}\right)p(t).$$ ■

In [11], necessary and sufficient conditions of URED are given in terms of derivatives of M. In Lemma 8, necessary and sufficient conditions of URED in terms of M directly are given.

Lemma 8. ([11]) L_M *is URED if and only if*

(i) $M \in SC$;

(ii) *Let* $[\alpha,\beta] \subset (0,1)$, *and* $\varepsilon, \varepsilon' \in (0,1)$, *there exists* $u_0 > 0$, $D = D(\varepsilon,\varepsilon') > 0$ *and* $\gamma = \gamma(\varepsilon,\varepsilon') > 0$ *such that for all* $\lambda \in [\alpha,\beta]$, *and for all* $|u| \geq u_0$, *if* $\lambda M(u) + (1-\lambda)M\big((1-\varepsilon)u\big) \leq (1+\gamma)M\big(\lambda u + (1-\lambda)(1-\varepsilon)u\big)$, *then*

$$M(u) \leq D(\varepsilon,\varepsilon')\frac{M(\varepsilon' u)}{\varepsilon'}.$$

Proof. By [11], it is enough to show that (ii) is necessary. Otherwise, for some $\varepsilon > 0$ there exist sequences $u_n \nearrow \infty$ and $\lambda_n \in [\alpha,\beta]$ such that $M(u_n) \geq 2^n n \frac{M(\varepsilon u_n)}{\varepsilon}$ and

$$\lambda_n M(u_n) + (1-\lambda_n)M\Big((1-\varepsilon)u_n\Big) \leq \Big(1+\frac{1}{n}\Big)M\Big(\lambda_n u_n + (1-\lambda_n)(1-\varepsilon)u_n\Big).$$

By Lemma 6, there exists $(1-\frac{\varepsilon}{2})u_n \leq t_n \leq u_n$ so that $p(t_n - \frac{\varepsilon}{2}u_n) \geq \Big(1 - 2\delta\frac{2-\varepsilon}{\varepsilon}\Big)p(t_n)$. Since the function $f(\lambda)$ in the proof of Lemma 5 is convex, we can get that $\frac{M(\varepsilon t_n)}{\varepsilon M(t_n)} \to 0$. If necessary passing to a subsequence, we have that $t_n p(t_n) \geq M(t_n) > 2^n n \frac{M(\varepsilon t_n)}{\varepsilon} \geq 2^n n p(\frac{\varepsilon t_n}{2})\frac{\varepsilon t_n}{2\varepsilon}$. It leads a contradiction from the proof of Theorem in [11]. ∎

Remark 1. ([11]) By the proof of Lemma 6, we have L_M is URED if and only if

(i) $M \in$ SC;

(ii) for $0 < \varepsilon, \varepsilon' < 1$ there exist positive number $D(\varepsilon,\varepsilon') < \infty$ and $u_0 > 0$ and $\gamma = \gamma(\varepsilon') > 0$ so that for all $|u| \geq u_0$ with $M(u) + M\big((1-\varepsilon)u\big) \leq (1+\gamma)2M\big((1-\frac{\varepsilon}{2})u\big)$, we have

$$M(u) \leq D(\varepsilon,\varepsilon')\frac{M(\varepsilon' u)}{\varepsilon'}.$$

For conveniece, we let $D(\varepsilon,\varepsilon')$ be the infimum over the above inequality, i.e.,$D(\varepsilon,\varepsilon') = \inf\{K > 0 : M(u) \leq K\frac{M(\varepsilon' u)}{\varepsilon'}\}$. Then we have

Lemma 9. *If* L_M *is URWC then for* $0 < \varepsilon < 1$ *there exists* $D(\varepsilon)$, $0 < D(\varepsilon) < \infty$, *such that for all* $\varepsilon' \in (0,1)$,

$$D(\varepsilon,\varepsilon') \leq D(\varepsilon)$$

where $D(\varepsilon,\varepsilon')$ *is defined as in* (ii) *of Remark* 1.

Proof. Define $D(\varepsilon) = \sup D(\varepsilon;\varepsilon')$ where ε' taken over all $(0;1)$. Because of $\frac{M(v)}{v} < \frac{M(u)}{u}$ as $0 < v < u$, it is follows that $D(\varepsilon;\varepsilon')$ is decreasing with respect with ε'. Suppose $D(\varepsilon) = \infty$. Then there exist $\varepsilon_n \searrow 0$ with $D(\varepsilon;\varepsilon_1) < D(\varepsilon;\varepsilon_2) < \cdots < D(\varepsilon;\varepsilon_n) \nearrow \infty$. Define

$$\Re_n = \left\{ u \geq n;\ \frac{\frac{M(u)+M((1-\varepsilon)u)}{2}}{M\left(\frac{u+(1-\varepsilon)u}{2}\right)} < 1 + \frac{1}{n} : M(u) > D(\varepsilon;\varepsilon_{n-1})\frac{M\left(\varepsilon_n u\right)}{\varepsilon_n} \right\},$$

where $D(\varepsilon;\varepsilon_0) = 0$. Then $\Re_1 \cap \Re_2 \neq \emptyset$. In fact, suppose that $\Re_1 \cap \Re_2 = \emptyset$, i.e., $\Re_1^c \cup \Re_2^c = \Re$, we have that for all $u \geq 2$, $\frac{\frac{M(u)+M((1-\varepsilon)u)}{2}}{M\left(\frac{u+(1-\varepsilon)u}{2}\right)} < 1 + \frac{1}{2}$, and

$$M(u) \leq D(\varepsilon;\varepsilon_0)\frac{M\left(\varepsilon_1 u\right)}{\varepsilon_1} = 0;$$

or

$$M(u) \leq D(\varepsilon;\varepsilon_1)\frac{M\left(\varepsilon_2 u\right)}{\varepsilon_2}:$$

Thus for all $u \geq 2$, $\frac{\frac{M(u)+M((1-\varepsilon)u)}{2}}{M\left(\frac{u+(1-\varepsilon)u}{2}\right)} < 1 + \frac{1}{2}$ and

$$M(u) \leq D(\varepsilon;\varepsilon_1)\frac{M\left(\varepsilon_2 u\right)}{\varepsilon_2}:$$

Hence $D(\varepsilon;\varepsilon_2) \leq D(\varepsilon;\varepsilon_1)$, a contradiction with the fact that $D(\varepsilon;\varepsilon_1) < D(\varepsilon;\varepsilon_2)$. In general, assume that

$$\Re_1 \cap \Re_2 \cap \cdots \cap \Re_n \neq \emptyset;$$

then $\Re_1 \cap \Re_2 \cap \cdots \cap \Re_{(n+1)} \neq \emptyset$. Indeed, if $\Re_1 \cap \Re_2 \cap \cdots \cap \Re_{(n+1)} = \emptyset$, i.e., $\Re_1^c \cup \Re_2^c \cup \cdots \cup \Re_{(n+1)}^c = \Re$, we get that for all $u \geq n$, $\frac{\frac{M(u)+M((1-\varepsilon)u)}{2}}{M\left(\frac{u+(1-\varepsilon)u}{2}\right)} < 1 + \frac{1}{n}$,

$$M(u) \leq D(\varepsilon;\varepsilon_1)\frac{M\left(\varepsilon_2 u\right)}{\varepsilon_2};$$

or

$$M(u) \leq D(\varepsilon;\varepsilon_2)\frac{M\left(\varepsilon_3 u\right)}{\varepsilon_3};$$

or

$$\ldots\ldots,$$

or

$$M(u) \le D(\varepsilon, \varepsilon_n) \frac{M\left(\varepsilon_{(n+1)} u\right)}{\varepsilon_{(n+1)}}.$$

Then we have one of the following contradictions:

$$D(\varepsilon, \varepsilon_2) < D(\varepsilon, \varepsilon_1) \le D(\varepsilon, \varepsilon_2);$$

$$D(\varepsilon, \varepsilon_3) < D(\varepsilon, \varepsilon_2) \le D(\varepsilon, \varepsilon_3);$$

$$\ldots\ldots;$$

$$D(\varepsilon, \varepsilon_{n+1}) < D(\varepsilon, \varepsilon_n) \le D(\varepsilon, \varepsilon_{(n+1)}).$$

Hence it holds that

$$\Re_1 \cap \Re_2 \cap \cdots \cap \Re_n \ne \emptyset.$$

Take $u_n \in \Re_1 \cap \Re_2 \cap \cdots \cap \Re_n$ with $u_n \ge n$, and $\frac{\frac{M(u_n)+M((1-\varepsilon)u_n)}{2}}{M\left(\frac{u_n+(1-\varepsilon)u_n}{2}\right)} < 1 + \frac{1}{n}$, and

(1) $$M(u_n) > D(\varepsilon, \varepsilon_{k-1}) \frac{M\left(\varepsilon_k u_n\right)}{\varepsilon_k}, \qquad 0 \le k \le n.$$

By Lemma 7, there exists $(1 - \frac{\varepsilon}{2})u_n \le t_n \le u_n$ with

(2) $$p(t_n - \frac{\varepsilon}{2} u_n) \ge \left(1 - \frac{2}{n} \frac{2-\varepsilon}{\varepsilon}\right) p(t_n).$$

Choose two disjoint measurable subsets G and F and $c > 0$ satisfying $\mu G = \mu F$ and

$$N\left(p(c)\right) \mu G = 1 = N\left(p(c)\right) \mu F.$$

Let $E \subset F$ such that

$$N\left(p(c)\right) \mu E = \frac{1}{2}.$$

Let $G_n \subset G$ such that

$$(1 - \frac{\varepsilon}{2}) t_n p\left((1 - \frac{\varepsilon}{2}) t_n\right) \mu G_n = \frac{1}{2};$$

and $E \subset E_n \subset F$

$$N\left(p\left((1-\frac{\varepsilon}{2})t_n\right)\right)\chi_{G_n} + N\left(p(c)\right)\chi_{E_n} = 1.$$

Define

$$\begin{aligned}
k_n &= cp(c)\chi_{E_n} + t_n p(t_n)\chi_{G_n};\\
h_n &= cp(c)\chi_{E_n} + (1-\frac{\varepsilon}{2})t_n p\left((1-\frac{\varepsilon}{2})t_n\right)\chi_{G_n};\\
x_n &= \frac{1}{k_n}(c|_{E_n} + t_n|_{G_n});\\
y_n &= \frac{1}{h_n}(c|_{E_n} + (1-\frac{\varepsilon}{2})t_n|_{G_n});\\
v_n &= p(c)|_{E_n} + p\left((1-\frac{\varepsilon}{2})t_n\right)\Big|_{G_n}.
\end{aligned}$$

Then

$$\begin{aligned}
\rho_N(v_n) &= 1;\\
h_n &\le k_n \le cp(c)\chi_F + \frac{2}{2-\varepsilon}\frac{1}{(1-\frac{2}{n}\frac{2-\varepsilon}{\varepsilon})}\frac{1}{2} \le cp(c)\chi_F + \frac{2}{2-\varepsilon};\\
k_n - h_n &\ge \frac{\varepsilon}{2}t_n p\left((1-\frac{\varepsilon}{2})t_n\right)\chi_{G_n}\\
&\ge \frac{\varepsilon}{4}.
\end{aligned}$$

On the other hand, by the Theorem 1.29 of [6], we have

$$\|y_n\|_M = \langle v_n, y_n\rangle = 1;$$

and

$$\begin{aligned}
\|x_n\|_M &\le \frac{1}{k_n}\left(1 + \rho_M(k_n x_n)\right)\\
&= \frac{1}{k_n}\left(\rho_N(v_n) + M(c)\chi_{E_n} + M(t_n)\chi_{G_n}\right)\\
&\le \frac{1}{k_n}\left(cp(c)\chi_{E_n} + t_n p(t_n)\chi_{G_n}\right)\\
&\le 1;
\end{aligned}$$

but

$$\begin{aligned}
\langle v_n, k_n x_n\rangle &= cp(c)\chi_{E_n} + t_n p\left((1-\frac{\varepsilon}{2})t_n\right)\chi_{G_n}\\
&\ge cp(c)\chi_{E_n} + \frac{1}{(1-\frac{2}{n}\frac{2-\varepsilon}{\varepsilon})}t_n p(t_n)\chi_{G_n}
\end{aligned}$$

$$\geq \frac{1}{(1-\frac{2}{n}\frac{2-\varepsilon}{\varepsilon})}k_n;$$

hence $\langle v_n; x_n + y_n\rangle \to 2$ $(n \to \infty)$, $\|x_n + y_n\|_M \to 2$ and

$$M(u_n)\chi G_n \leq D(\varepsilon;\varepsilon)\frac{M(\varepsilon u_n)}{\varepsilon}\chi G_n \leq D(\varepsilon;\varepsilon)\frac{M((1-\frac{\varepsilon}{2})t_n)}{\varepsilon}\chi G_n \leq \frac{D(\varepsilon;\varepsilon)}{2\varepsilon};$$

Without loss of generality, assume $\varepsilon > \varepsilon_1$, then for arbitrary $\delta > 0$, let $\frac{1}{D(\varepsilon;\varepsilon_l)} < \delta\frac{2\varepsilon}{D(\varepsilon;\varepsilon)}$ and take k_0 such that for all $k \geq k_0$, $\sup_{1\leq i\leq l}\frac{M(\varepsilon_k u_i)\chi G_i}{\varepsilon_k} < \delta$, so we have

$$\begin{aligned}
&\sup_{1\leq i}\frac{M(\varepsilon_k u_i)\chi G_i}{\varepsilon_k}\\
&\leq \max\left\{\sup_{1\leq i\leq l}\frac{M(\varepsilon_k u_i)\chi G_i}{\varepsilon_k};\ \sup_{l<i<k}\frac{M(\varepsilon_k u_i)\chi G_i}{\varepsilon_k};\ \sup_{k\leq i}\frac{M(\varepsilon_k u_i)\chi G_i}{\varepsilon_k}\right\}\\
&\leq \max\left\{\sup_{1\leq i\leq l}\frac{M(\varepsilon_k u_i)\chi G_i}{\varepsilon_k};\ \sup_{l<i<k}\frac{M(\varepsilon_i u_i)\chi G_i}{\varepsilon_i};\ \sup_{k\leq i}\frac{M(u_i)\chi G_i}{D(\varepsilon;\varepsilon_{k-1})}\right\}\\
&\leq \max\left\{\sup_{1\leq i\leq l}\frac{M(\varepsilon_k u_i)\chi G_i}{\varepsilon_k};\ \sup_{l<i<k}\frac{M(u_i)\chi G_i}{D(\varepsilon;\varepsilon_{k-1})};\ \sup_{k\leq i}\frac{M(u_i)\chi G_i}{D(\varepsilon;\varepsilon_{k-1})}\right\}\\
&< \delta;
\end{aligned}$$

By [1], $\{u_n|_{G_n}\}_{n=1}^{\infty}$ is relatively weakly compact, but

$$\langle \varphi_E; x_n - y_n\rangle = \left(\frac{1}{k_n} - \frac{1}{h_n}\right)c\chi E \not\to 0 \qquad (n \to \infty);$$

a contradiction to that L_M is URWC. ■

Theorem 1. *An Orlicz space* L_M *equipped with Orlicz norm is URWC if and only if*

(i) $M \in$ *SC;*

(ii) *for* $[\alpha;\beta] \subset (0;1)$ *and for* $0 < \varepsilon < 1$ *there exist* $\infty > D = D(\varepsilon)$, *and* $u_0 > 0$, *such that for all* ε', $0 < \varepsilon' < 1$, *we can find* $\gamma = \gamma(\varepsilon') > 0$ *so that for all* $\lambda, \lambda_3 \in [\alpha;\beta]$, *and all* $u \geq u_0$, *with* $\lambda M(u) + (1-\lambda)M\left((1-\varepsilon)u\right) \leq (1+\gamma)M\left(\lambda u + (1-\lambda)(1-\varepsilon)u\right)$, *we have*

$$M(u) \leq D\frac{M(\varepsilon' u)}{\varepsilon'};$$

Proof. Necessity. Since URWC implies Rotundity, we get (i) $M \in SC$. By Lemma 6, we have

$$\frac{\frac{M(u)+M((1-\varepsilon)u)}{2}}{M\left(\frac{u+(1-\varepsilon)u}{2}\right)} \to 1 \iff \frac{\lambda M(u) + (1-\lambda)M((1-\varepsilon)u)}{M\left(\lambda u + (1-\lambda)(1-\varepsilon)u\right)} \to 1$$

and by Lemma 9, (ii) follows.

Sufficiency. If we suppose that L_M is not URWC, there exist sequences $\{x_n\}$ and $\{y_n\}$ satisfying $\|x_n\|_M = \frac{1}{k_n}\left(1 + \rho_M(k_n x_n)\right) \to 1$, $\|y_n\|_M = \frac{1}{h_n}\left(1 + \rho_M(h_n y_n)\right) \to 1$ $(n \to \infty)$, $\|x_n + y_n\|_M \to 2$ but $x_n - y_n \equiv z_n \xrightarrow{W} z \neq 0$.

If $x_n \xrightarrow{\mu} 0$ ($y_n \xrightarrow{\mu} 0$) in measure, set $x_n' = x_n + \frac{z_n}{4}$, $y_n' = x_n + \frac{3z_n}{4}$. It is easy to see that $\|x_n'\|_M \to 1$, $\|y_n'\|_M \to 1$, $\|x_n' + y_n'\|_M \to 2$ and $x_n' - y_n' \equiv z_n' = \frac{z_n}{2} \xrightarrow{L_N} \frac{z}{2} \neq 0$. Hence $z_n' = \frac{z_n}{2} \xrightarrow{W} \frac{z}{2} \neq 0$ $(n \to \infty)$. Clearly $x_n' \not\xrightarrow{\mu} 0$. So we assume that $x_n \not\xrightarrow{\mu} 0$ and $y_n \not\xrightarrow{\mu} 0$ if necessary replacing $\{x_n\}$ and $\{y_n\}$ by $\{x_n'\}$ and $\{y_n'\}$. By Lemma 2, we get that $\{k_n\}$ and $\{h_n\}$ are bounded, assume $k_n \to k$, $h_n \to h$ by passing to a subsequence if necessary.

Lemma 3 yields that $k_n x_n - h_n y_n \xrightarrow{\mu} 0$, i.e, $(k_n - h_n)x_n - h_n z_n \xrightarrow{\mu} 0$. If $k = h$ it follows that $z_n \xrightarrow{\mu} 0$, so $z_n \xrightarrow{w^*} 0$, a contradiction with $z \neq 0$. Hence $k \neq h$, assume $k > h$ and $k_n > h_n$, passing to a subsequence if necessary. We can do the same in the case of $k < h$. Define $\lambda_n = \frac{h_n}{k_n + h_n} \leq \frac{1}{2}$. Since $\{k_n\}$ and $\{h_n\}$ are bounded we deduce that $\lambda_n \in [\alpha, \beta]$ for some $[\alpha, \beta] \subset (0, 1)$.

Since $k_n x_n(t) - h_n y_n(t) \xrightarrow{\mu} 0$ and $z \neq 0$, by N. Riesz Theorem, there exists a subset $G_0 \supset G$ such that on G_0 there uniformly hold

$$k_n x_n(t) - h_n y_n(t) \to 0; \tag{3}$$

$$z|_{G_0} \neq 0. \tag{4}$$

For arbitrary $\varepsilon > 0$.

Since $\{z_n\}$ is weakly compact, then $\{z_n\}$ is L_N weakly compact. From [1], we take $0 < \varepsilon' < 1$ such that

$$\frac{\rho_M(\varepsilon' 2k z_n)}{\varepsilon'} < \frac{\varepsilon^2}{4D} \tag{5}$$

By (ii), there is $\gamma > 0$ such that for all $\lambda, \lambda \in [\alpha, \beta]$, and all u, v, $\max(|u|, |v|) \geq u_0$, $|u - v| \geq \varepsilon \max(|u|, |v|)$, with $\lambda M(u) + (1-\lambda)M(v) \leq (1+\gamma)M(\lambda u + (1-\lambda)v)$, by Lemma 5, we have

$$M(u) \leq D \frac{M(\varepsilon' \varepsilon u)}{\varepsilon' \varepsilon}. \tag{6}$$

By (3) and (4)

$$\rho_M\left(\frac{hz}{k-h}\Big|_{G_0}\right) > 0 \tag{7}$$

For each n, split G into the following parts:

$$\begin{aligned}
A_n &= \{t \in G \setminus G_0 : \max(|k_n x_n(t)|, |h_n y_n(t)|) < \varepsilon\};\\
B_n &= \{t \in G \setminus G_0 \setminus A_n : |k_n x_n(t) - h_n y(t)| < \varepsilon \max(|k_n x_n(t)|, |h_n y_n(t)|)\};\\
H_n &= \Big\{t \in G \setminus G_0 \setminus A_n \setminus B_n : (1+\delta)M\left(\frac{k_n h_n}{k_n+h_n}(x_n(t)+y_n(t))\right)\\
&\qquad < \frac{h_n}{k_n+h_n} M\left(k_n x_n(t)\right) + \frac{k_n}{k_n+h_n} M\left(h_n x_n(t)\right)\Big\};\\
I_n &= \{t \in G \setminus G_0 \setminus A_n \setminus B_n \setminus H_n : |x_n(t)| < |y_n(t)|\};\\
Q_n &= \{t \in G \setminus G_0 \setminus A_n \setminus B_n \setminus H_n \setminus I_n : |z_n(t)| < \varepsilon |x_n(t)|\};\\
T_n &= G \setminus G_0 \setminus A_n \setminus B_n \setminus H_n \setminus I_n \setminus Q_n\\
&= \Big\{t \in G \setminus G_0 : \max(|k_n x_n(t)|, |h_n y_n(t)|) \geq \varepsilon,\\
&\qquad |k_n x_n(t) - h_n y(t)| \geq \varepsilon \max(|k_n x_n(t)|, |h_n y_n(t)|),\\
&\qquad (1+\delta)M\left(\frac{k_n h_n}{k_n+h_n}(x_n(t)+y_n(t))\right)\\
&\qquad \geq \frac{h_n}{k_n+h_n} M\left(k_n x_n(t)\right) + \frac{k_n}{k_n+h_n} M\left(h_n x_n(t)\right),\\
&\qquad |z_n(t)| \geq \varepsilon |x_n(t)| \quad \text{and} \quad |x_n(t)| \geq |y_n(t)|\Big\}.
\end{aligned}$$

Pick $v_n \in B(L_N)$ such that $[x_n(t) + y_n(t)]v_n(t) \geq 0$ and

$$\langle v_n, x_n + y_n \rangle \to 2.$$

Then

$$\langle v_n, x_n \rangle \to 1, \qquad \langle v_n, y_n \rangle \to 1,$$

thus

$$k - h = \lim_n (k_n - h_n) = \lim_n \int_G [k_n x_n(t) - h_n y_n(t)] v_n(t)\, d\mu.$$

In the following, we estimate the integrals over the above subsets.

(a) On G_0. Since $k_n x_n(t) - h_n y_n(t) \to 0$ uniformly on G_0, for n large enough,

$$\int_{G_0} \left| \left(k_n x_n(t) - h_n y_n(t) \right) v_n(t) \right| d\mu < \varepsilon \|\chi_G\|_M.$$

(b) On A_n. Clearly, by Hölder Inequality,

$$\int_{A_n} \left| \left(k_n x_n(t) - h_n y_n(t) \right) v_n(t) \right| d\mu < 2\varepsilon \|\chi_G\|_M.$$

(c) On B_n.

$$\begin{aligned} &\int_{B_n} \left| \left(k_n x_n(t) - h_n y_n(t) \right) v_n(t) \right| d\mu \\ &\le \varepsilon \int_{B_n} \left(|k_n x_n(t)| + |h_n y_n(t)| \right) |v_n(t)| \, d\mu \\ &\le \varepsilon (k_n + h_n). \end{aligned}$$

(d) On H_n. Notice

$$\frac{\gamma}{1+\gamma} \left[\frac{h_n}{k_n + h_n} \rho_M \left(k_n x_n|_{H_n} \right) + \frac{k_n}{k_n + h_n} \rho_M \left(h_n y_n|_{H_n} \right) \right] \le 2 - \|x_n + y_n\|_M \to 0;$$

we get that for n large enough, by Lemma 4

$$\int_{H_n} \left| \left(k_n x_n(t) - h_n y_n(t) \right) v_n(t) \right| d\mu < \varepsilon.$$

(e) On I_n. For $|x_n(t)| < |y_n(t)|$.

When $x_n(t) y_n(t) \ge 0$, by $[x_n(t) + y_n(t)] v_n(t) \ge 0$, we have $x_n(t) v_n(t) \ge 0$ and $y_n(t) v_n(t) \ge 0$, so $x_n(t) z_n(t) = x_n(t)[x_n(t) - y_n(t)] < 0$, then $z_n(t) v_n(t) \le 0$. Hence

$$\begin{aligned} [k_n x_n(t) - h_n y_n(t)] v_n(t) &= (k_n - h_n) x_n(t) v_n(t) + h_n [x_n(t) - y_n(t)] v_n(t) \\ &= (k_n - h_n) x_n(t) v_n(t) + h_n z_n(t) v_n(t) \\ &\le (k_n - h_n) x_n(t) v_n(t). \end{aligned}$$

When $x_n(t) y_n(t) < 0$, by $|x_n(t)| < |y_n(t)|$, we have $y_n(t) v_n(t) \ge 0$ and $x_n(t) v_n(t) \le 0$, by $z_n(t) = x_n(t) - y_n(t)$, then $z_n(t) v_n(t) \le 0$. Hence

$$\begin{aligned} [k_n x_n(t) - h_n y_n(t)] v_n(t) &= (k_n - h_n) x_n(t) v_n(t) + h_n [x_n(t) - y_n(t)] v_n(t) \\ &= (k_n - h_n) x_n(t) v_n(t) + h_n z_n(t) v_n(t) \\ &\le (k_n - h_n) x_n(t) v_n(t). \end{aligned}$$

We have

$$\int_{I_n} [k_n x_n(t) - h_n y_n(t)] v_n(t)\, d\mu \le \int_{I_n} (k_n - h_n) x_n(t) v_n(t)\, d\mu .$$

Notice

$$\frac{1}{k_n}\rho_M\left(k_n x_n|_{G_0}\right) \ge \rho_M\left(x_n|_{G_0}\right) \to \rho_M\left(\frac{h}{k-h} z\Big|_{G_0}\right)$$

and

$$\begin{aligned} 1 \leftarrow \|x_n\|_M &= \frac{1}{k_n}\left[1 + \rho_M\left(k_n x_n|_{G_0}\right) + \rho_M\left(k_n x_n|_{G\setminus G_0}\right)\right] \\ &\ge \|x_n|_{G\setminus G_0}\|_M + \frac{1}{k_n}\rho_M\left(k_n x_n|_{G_0}\right) \\ &\ge \int_{G\setminus G_0} |x_n(t) v_n(t)|\, d\mu + \rho_M\left(\frac{h}{k-h} z\Big|_{G_0}\right); \end{aligned}$$

we have that for n large enough

$$\int_{G\setminus G_0} |x_n(t) v_n(t)|\, d\mu \le 1 - \rho_M\left(\frac{h}{k-h} z\Big|_{G_0}\right).$$

Combining $I_n \subset G \setminus G_0$

$$\int_{I_n} |x_n(t) v_n(t)|\, d\mu \le 1 - \rho_M\left(\frac{h}{k-h} z\Big|_{G_0}\right).$$

(f) On Q_n. For $|z_n(t)| \le \varepsilon |x_n(t)|$. From $|y_n(t)| \le |x_n(t)|$ and $[x_n(t) + y_n(t)] v_n(t) \ge 0$, we get $x_n(t) v_n(t) \ge 0$ and $z_n(t) v_n(t) \ge 0$,

$$\begin{aligned} [k_n x_n(t) - h_n y_n(t)] v_n(t) &= (k_n - h_n) x_n(t) v_n(t) + h_n z_n(t) v_n(t) \\ &\le (k_n - h_n) x_n(t) v_n(t) + \varepsilon h_n x_n(t) v_n(t). \end{aligned}$$

Thus

$$\begin{aligned} &\int_{Q_n} [k_n x_n(t) - h_n y_n(t)] v_n(t)\, d\mu \\ &\le (k_n - h_n + \varepsilon h_n) \int_{Q_n} x_n(t) v_n(t)\, d\mu \\ &\le (k_n - h_n + \varepsilon h_n)\left[1 - \rho_M\left(\frac{h}{k-h} z\Big|_{G_0}\right)\right]. \end{aligned}$$

(g) On T_n. For $t \in T_n$,

$$\max\{|k_n x_n(t)|, |h_n y_n(t)|\} \geq \varepsilon;$$

$$|k_n x_n(t) - h_n y_n(t)| \geq \varepsilon \max\{k_n|x_n(t)|, |h_n y_n(t)|\};$$

$$\frac{\lambda_n M\left(k_n x_n(t)\right) + (1-\lambda_n) M\left(h_n y_n(t)\right)}{M\left(\lambda_n k_n x_n(t) + (1-\lambda_n) h_n y_n(t)\right)} \leq 1 + \gamma.$$

By $\varepsilon|x_n(t)| \leq |z_n(t)|$, from (6) and Lemma 5, we get that for $t \in T_n$

$$M\left(k_n x_n(t)\right) \leq D \frac{M\left(\varepsilon'\varepsilon k_n x_n(t)\right)}{\varepsilon'\varepsilon} \leq D \frac{M\left(\varepsilon' 2k z_n(t)\right)}{\varepsilon'\varepsilon}.$$

Hence, by (5)

$$\rho_M(k_n x_n|_{T_n}) \leq D \frac{\rho_M(\varepsilon' 2k z_n|_{T_n})}{\varepsilon'\varepsilon} \leq \frac{D\varepsilon^2}{D\varepsilon} = \varepsilon.$$

Since $|x_n(t)| > |y_n(t)|$ and $k_n > h_n$, we have $|k_n x_n(t)| > |h_n y_n(t)|$, so $\rho_M(h_n y_n|_{T_n}) \leq \varepsilon$.

Since $\varepsilon > 0$ is arbitrary, from (a) to (g), this leads to a contradiction:

$$k - h \leq (k-h)\left[1 - \rho_M\left(\frac{h}{k-h} z\Big|_{G_0}\right)\right] < k - h.$$

■

By Lemma 6 and Theorem 1, we have the following:

Remark 2. L_M is URWC if and only if

(i) $M \in$ SC;

(ii) for $0 < \varepsilon < 1$ there exist $D = D(\varepsilon)$, and $u_0 > 0$ such that for all ε', $0 < \varepsilon' < 1$, we can find $\gamma = \gamma(\varepsilon') > 0$ so that for all $|u| \geq u_0$ with $M(u) + M\left((1-\varepsilon)u\right) \leq (1+\gamma) 2M\left((1-\frac{\varepsilon}{2})u\right)$, we have

$$M(u) \leq D \frac{M(\varepsilon' u)}{\varepsilon'}.$$

Example The Young function defined by

$$M(u) = \begin{cases} Au^2 & \text{as } |u| \leq 2; \\ B \exp|u| & \text{as } |u| > 2; \end{cases}$$

where A and B are constants, satisfies the condition (ii) in Theorem 1 and Remark 2. The condition (ii) for URWC in Theorem 1 and Remark 2 cannot be expressed by using classical conditions of M, such as convexity, $M \in \Delta_2$, and $M \in \nabla_2$. The condition (ii) for URED in Lemma 8 can be described as saying that, in this context, a non-uniform 'point' ('sequence') is a Δ_2 'point' ('sequence'). By an example in [11], the condition (ii) for URED in Lemma 8 is not equivalent to the Δ_2 condition. The condition (ii) for URWC in Theorem 1 is strictly stronger than the condition (ii) for URED in Lemma 8.

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