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TWIN POSITIVE SYMMETRIC SOLUTIONS FOR LIDSTONE BOUNDARY VALUE PROBLEMS

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Abstract. In this paper, we consider the Lidstone boundary value problem

$$(\Phi(y^{(2n-1)}))'(t) = f(t, y(t), y''(t), \cdots, y^{(2(n-1))}(t)), \ 0 \le t \le 1,$$
$$y^{(2i)}(0) = y^{(2i)}(1) = 0, \ 0 \le i \le n-1,$$

where $f : [0,1] \times R^n \to R$ is continuous, $\Phi(v) = |v|^{p-2}v, \ p > 1$. Growth conditions are imposed on f which yield the existence of at least two symmetric positive solutions by using a fixed point theorem in cones.

1. Introduction

In this paper, we are concerned with the existence of two positive solutions for the 2nth order Lidstone boundary value problem with a $p-$Laplacian operator

$$(1) \qquad \begin{cases} (\Phi(y^{(2n-1)}))'(t) = f(t, y(t), y''(t), \cdots, y^{(2(n-1))}(t)), \ 0 \le t \le 1, \\ y^{(2i)}(0) = y^{(2i)}(1) = 0, \ 0 \le i \le n-1, \end{cases}$$

where the nonlinear term f is allowed to change sign, and $\Phi(v) = |v|^{p-2}v, \ p > 1$. We will impose growth conditions on f which ensure the existence of at least two positive solutions for (1) by using a fixed point theorem in cones.

Fixed point theorems and their applications to nonlinear problems have a long history. Recently, there seems to be increasing interest in multiple fixed point theorems and their applications to boundary value problems for ordinary differential

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equations or finite difference equations. Such applications can be found in the papers [1-3, 6, 9-10, 12, 14-16], and the recent book by Agarwal et. al. [1] which gives a good overview of the current work. Davis et. al. [7-8] imposed conditions on f which yield at least three symmetric positive solutions to the 2mth order Lidstone boundary value problem

$$(2)\qquad \begin{cases} y^{(2m)} = f(y(t), y''(t), \cdots, y^{(2(m-1))}(t)), & t \in [0,1], \\ y^{(2i)}(0) = y^{(2i)}(1) = 0, & 0 \le i \le m-1, \end{cases}$$

where $(-1)^m f : R^m \to [0,\infty)$ is continuous, using the Leggett-Williams fixed point theorem [13] and the five functionals fixed point theorem [4]. Avery et. al. [5] applied a twin fixed point theorem to obtain at least two positive solutions for the right focal boundary value problem

$$(3)\qquad \begin{cases} y'' + f(y) = 0, \ 0 \le t \le 1, \\ \quad y(0) = y'(1) = 0, \end{cases}$$

where $f : R \to [0,\infty)$ is continuous.

In order to apply the concavity of solutions in the proofs, all the above results were obtained under the assumption that function f or $(-1)^m f$ is nonnegative. For the sign changing nonlinearity f, few results were obtained. In some sense this paper should be viewed as companion for [7-8], and the result in this paper fills a gap under the assumption that function $(-1)^m f$ is nonnegative in [7-8].

The paper is divided into three sections. In section 2, we prove a fixed point theorem in cones. In section 3, we impose growth conditions on f which allow us to apply the fixed point theorem in obtaining two symmetric positive solutions for (1).

2. The Fixed Point Theorem in Cones

For a cone K in a Banach space X with norm $\|\cdot\|$ and a constant $r > 0$, let $K_r = \{x \in K : \|x\| < r\}$, $\partial K_r = \{x \in K : \|x\| = r\}$. Suppose $\alpha : K \to R^+$ is a continuous functional, let

$$K(b) = \{x \in K : \alpha(x) < b\}, \quad \partial K(b) = \{x \in K : \alpha(x) = b\}$$

and $K_a(b) = \{x \in K : a < \|x\|, \alpha(x) < b\}$. The origin in X is denoted by θ.

Definition 1. Given a cone K in a real Banach space X, a functional $\alpha : K \to R$ is said to be concave functional on K provided

$$\alpha(tx + (1-t)y) \ge t\alpha(x) + (1-t)\alpha(y)$$

for all $x; y \in K$ and $0 \le t \le 1$.

Theorem 1. *Let* X *be a real Banach space with norm* $\| \cdot \|$ *and* $K \subset X$ *a cone. Suppose* $T : K \to K$ *is a completely continuous operator , and* $® : K \to R^+$ *a continuous concave functional satisfying* $®(x) \le ||x||$ *for all* $x \in K$ *. If there are constants* $c > b > a > 0$ *such that*

(C_1) $\|Tx\| < a$ for $x \in @K_a$;

(C_2) $®(Tx) > b$ for $x \in @K(b)$;

(C_3) $®(x) \le b$ implies $||Tx|| < c$,

then T *has at least a fixed points* y *in* K *such that*

$$a < \|y\| < c; \quad ®(y) < b:$$

Proof. Let the symbol $\deg_K$ denote the degree on the cone K. Then condition (C_1) implies

$$\deg_K\{I - T; K_a; \mu\} = 1:$$

Let $\Omega = K(b) \cap K_{2c}$, we now prove that

$$\deg_K\{I - T; \Omega; \mu\} = 0:$$

Conditions (C_2) and (C_3) imply $\inf\limits_{x \in @K(b)} ®(Tx) \ge b > 0$ and $\inf\limits_{x \in @K(b)} ||Tx|| \le c$. Let $\widetilde{T} : \overline{K}(b) \to K$ be an extension of $T|_{@K(b)} : @K(b) \to K$. Dugundji extension theorem ([17, p. 5]) ensures that $\widetilde{T}$ is completely continuous and $\widetilde{T}(\overline{K}(b)) \subset \overline{\text{convex}}T(@K(b))$. Since $\{x \in K : ®(x) \ge b\} \cap \{x \in K : ||x|| \le c\}$ is a convex set, we have

$$\inf_{x \in \overline{K}(b)} ®(\widetilde{T}x) \ge b > 0; \text{ and } \inf_{x \in \overline{K}(b)} ||\widetilde{T}x|| \le c:$$

We claim

$$\deg_K\{I - \widetilde{T}; \Omega; \mu\} = 0:$$

Clearly $@\Omega = (@K(b) \cap \overline{K}_{2c}) \cup (@K_{2c} \cap \overline{K}(b))$. For $x \in @\Omega$, $(I - \widetilde{T})(x) \ne \mu$. If it is not true, then there exists $x_0 \in @\Omega$ such that

$$x_0 = \widetilde{T}x_0:$$

If $x \in @K(b) \cap \overline{K}_{2c}$, then

$$b = ®(x_0) = ®(\widetilde{T}x_0) = ®(Tx_0) > b;$$

a contradiction. On the other hand, if $x \in @K_{2c} \cap \overline{K}(b)$; then

$$2c = ||x_0|| = ||\widetilde{T}x_0|| \le c;$$

a contradiction. For $x \in \Omega$, we have

$$®(x) < b; \text{ and } ®(\widetilde{T}x_0) \geq b:$$

Thus, $(I - \widetilde{T})(x) \neq µ$ for $x \in \Omega$. It follows that

$$\deg_K\{I - \widetilde{T}; \Omega; µ\} = 0:$$

Take a homotopy $H(x; ¸) = ¸Tx + (1 - ¸)\widetilde{T}x$. It is easy to see that

$$H(x; ¸) \neq x; \text{ for all } x \in @\Omega; ¸ \in [0; 1]:$$

Thus,

$$\deg_K\{I - T; \Omega; µ\} = \deg_K\{I - \widetilde{T}; \Omega; µ\} = 0:$$

From $®(x) \leq ||x||$, we have $K_a \subset K(b) \cap K_{2c} = \Omega$. Then

$$\begin{aligned} &\deg_K\{I - T; \Omega \setminus K_a; µ\} \\ &= \deg_K\{I - T; \Omega; µ\} - \deg_K\{I - T; K_a; µ\} \\ &= -1: \end{aligned}$$

So T has in K a fixed point y such that

$$a < \|y\| < c; \quad ®(y) < b:$$

Theorem 1 is now proved.

3. Main Result

In this section, we will impose growth conditions on f which allow us to apply Theorem 1 to obtain two symmetric positive solutions for (1). Let $G(t; s)$ be the Green's function for

$$\text{(4)} \qquad \begin{cases} u^{00} = 0; & t \in [0; 1]; \\ u(0) = u(1) = 0: & \end{cases}$$

Thus,

$$G(t; s) = -\begin{cases} (1 - t)s; & 0 \leq s \leq t \leq 1; \\ (1 - s)t; & 0 \leq t \leq s \leq 1: \end{cases}$$

Let $G_1(t; s) = G(t; s)$, then for $2 \leq j \leq n - 1$ we recursively define

$$G_j(t; s) = \int_0^1 G(t; r) G_{j-1}(r; s) dr:$$

It is easy to see that $G_j(t,s)(1 \le j \le n-1)$ is the Green's function for the boundary value problem

$$\begin{cases} y^{(2j)}(t) = 0, \ 0 \le t \le 1, \\ y^{(2i)}(0) = y^{(2i)}(1) = 0, \ 0 \le i \le j-1. \end{cases}$$

For each $1 \le j \le n-1$, we define $A_j : C[0,1] \to C[0,1]$ by

$$A_j v(s) = \int_0^1 G_j(s,\tau)v(\tau)d\tau.$$

For each $1 \le j \le n-1$, from the construction of A_j we see that

$$(A_j v)^{(2j)}(t) = v(t), 0 \le t \le 1,$$

$$(A_j v)^{(2i)}(0) = (A_j v)^{(2i)}(1) = 0, \ 0 \le i \le j-1.$$

Therefore (1) has a solution if and only if the boundary value problem

$$(5) \quad \begin{cases} (\Phi(v'))'(t) = f(t, A_{n-1}v(t), A_{n-2}v(t), \cdots, A_1 v(t), v(t)), \ 0 \le t \le 1, \\ v(0) = v(1) = 0 \end{cases}$$

has a solution. If y is a solution of (1), then $v = y^{(2(n-1))}$ is a solution of (5). Conversely, if v is a solution of (5), then $y = A_{n-1}v$ is a solution of (1). In particular, if $(-1)^{n-1}v(t) \ge 0 (\not\equiv 0)$ on $[0,1]$, then $y = A_{n-1}v$ is a positive solution of (1).

Lemma 1. $G(t,s)$ *has the following properties*

$$(6) \qquad \int_0^1 |G(t,s)|ds = \frac{t(1-t)}{2}, \ 0 \le t \le 1;$$

$$(7) \qquad \int_\delta^{1-\delta} |G(t,s)|ds = \frac{1}{2}t(1-t) - \frac{1}{2}\delta^2, \ \delta \le t \le 1-\delta.$$

Proof. From the expression of $G(t,s)$, it is easy to see that (6) and (7) hold.

Let $X = C[0,1]$, $K = \{x \in X : (-1)^{n-1}x(t) \ge 0, x(t) = x(1-t), t \in [0,1]\}$, $K' = \{x \in K : (-1)^{n-1}x$ is concave on $[\frac{\delta}{2}, 1-\frac{\delta}{2}]\}$, where $\delta \in (0, \frac{1}{2})$. Obviously, $K, K' \subset X$ are two cones with $K' \subset K$.

Let $(-1)^j[a,b] = [a,b]$ if j is even and $(-1)^j[a,b] = [-b,-a]$ if j is odd. The following conditions are satisfied throughout the rest of this paper

(H_1) $f : [0;1] \times \prod_{j=0}^{n-1} (-1)^j [0;\infty) \to R$ is continuous and for each $(u_0; \cdots ; u_{n-1})$
$\in \prod_{j=0}^{n-1} (-1)^j [0;\infty)$, $f(t; u_0; \cdots ; u_{n-1})$ is symmetric about $t = \frac{1}{2}$;

(H_2) $(-1)^n f(t; 0; 0; \cdots ; 0) \geq 0 (\not\equiv 0)$ for $t \in [0;1]$;

and there exist $a; b; d > 0$ satisfying

$$0 < \frac{1-\pm}{2} \Phi^{-1} \left[\Phi \left(\frac{2d}{\pm} \right) + \frac{M\pm}{2} \right] + d \leq a < \pm b < b$$

such that

(H_3) $(-1)^n f(t; u_0; \cdots ; u_{n-1}) \geq -M$ for $(t; u_0; \cdots ; u_{n-1}) \in [0;1] \times \prod_{j=0}^{n-1} (-1)^j [0;\infty)$;

(H_4) $(-1)^n f(t; u_0; \cdots ; u_{n-1}) \geq \frac{\pm}{1-\pm} M$ for $(t; u_0; \cdots ; u_{n-1}) \in \left[\frac{\pm}{2}; 1 - \frac{\pm}{2} \right]$
$\times \prod_{j=0}^{n-1} (-1)^j \left[\frac{1}{4^{n-1-j}} \pm^{n-1-j} (1-\pm)^{n-1-j} d; \frac{1}{8^{n-1-j}} b \right]$;

(H_5) $(-1)^n f(t; u_0; \cdots ; u_{n-1}) < 2\Phi(2a)$ for $(t; u_0; \cdots ; u_{n-1}) \in [0;1]$
$\times \prod_{j=0}^{n-1} (-1)^j \left[0; \frac{1}{8^{n-1-j}} a \right]$;

(H_6) $(-1)^n f(t; u_0; \cdots ; u_{n-1}) \geq \frac{2}{1-2\pm} (M\pm + \Phi(b))$ for $(t; u_0; \cdots ; u_{n-1})$
$\in [\pm; 1-\pm] \times \prod_{j=0}^{n-1} (-1)^j \left[\frac{1}{2^{n-1-j}} \pm^{n-j} (1-2\pm)^{n-1-j} b; \frac{1}{8^{n-1-j}} b \right]$;

(H_7) $(-1)^n f(t; u_0; \cdots ; u_{n-1}) \leq \frac{2}{\pm} \Phi \left(\frac{2a}{\pm} \right)$ for $(t; u_0; \cdots ; u_{n-1}) \in \left[0; \frac{\pm}{2} \right]$
$\times \prod_{j=0}^{n-1} (-1)^j \left[0; \frac{1}{8^{n-1-j}} b \right]$.

For $x \in K$, we define

$$®(x) = \min_{\pm \leq t \leq 1-\pm} |x(t)|;$$

$$(Tx)(t) = \begin{cases} \left(-\int_0^t \Phi^{-1} \left(\int_s^{\frac{1}{2}} f(¿; A_{n-1}x(¿); A_{n-2}x(¿); \cdots ; \right.\right. \\ \qquad \left.\left. A_1 x(¿); x(¿)) d¿ \right) ds \right)^+ ; 0 \leq t \leq \frac{1}{2}; \\ \left(-\int_t^1 \Phi^{-1} \left(\int_{\frac{1}{2}}^s f(¿; A_{n-1}x(¿); A_{n-2}x(¿); \cdots ; \right.\right. \\ \qquad \left.\left. A_1 x(¿); x(¿)) d¿ \right) ds \right)^+ ; \frac{1}{2} \leq t \leq 1; \end{cases}$$

where $(B)^+ = (-1)^{n-1}\max\{(-1)^{n-1}B; 0\}$.

$$(Ax)(t) = \begin{cases} -\int_0^t \Phi^{-1}\left(\int_s^{\frac{1}{2}} f(¿; A_{n-1}x(¿); A_{n-2}x(¿); \cdots ; \right.\\ \qquad \left. A_1x(¿); x(¿))d¿\right)ds; 0 \le t \le \frac{1}{2};\\ -\int_t^1 \Phi^{-1}\left(\int_{\frac{1}{2}}^{s} f(¿; A_{n-1}x(¿); A_{n-2}x(¿); \cdots ; \right.\\ \qquad \left. A_1x(¿); x(¿))d¿\right)ds; \frac{1}{2} \le t \le 1: \end{cases}$$

For $x \in X$, define $\mu : X \to K$ by $(\mu x)(t) = (-1)^{n-1}\max\{(-1)^{n-1}x(t); 0\}$, then $T = \mu \circ A$. For $x \in K^0$, let

$$(T^0x)(t) = \begin{cases} -\int_0^t \Phi^{-1}\left(\int_s^{\frac{1}{2}} f^*(¿; A_{n-1}x(¿); A_{n-2}x(¿); \cdots ; \right.\\ \qquad \left. A_1x(¿); x(¿))d¿\right)ds; 0 \le t \le \frac{1}{2};\\ -\int_t^1 \Phi^{-1}\left(\int_{\frac{1}{2}}^{s} f^*(¿; A_{n-1}x(¿); A_{n-2}x(¿); \cdots ; \right.\\ \qquad \left. A_1x(¿); x(¿))d¿\right)ds; \frac{1}{2} \le t \le 1; \end{cases}$$

where

$$f^*(t; u_0; \cdots ; u_{n-1}) = \begin{cases} f(t; 0; 0; \cdots ; 0); (t; u_0; \cdots ; u_{n-1}) \\ \qquad \in [0; 1] \times \prod_{j=0}^{n-1}(-1)^{j+1}[0; \infty);\\ f(t; u_0'; \cdots ; u_{n-1}'); (t; u_0; \cdots ; u_{n-1})\\ \qquad \in \left[\frac{\pm}{2}; 1 - \frac{\pm}{2}\right] \times \prod_{j=0}^{n-1}(-1)^j[0; \infty);\\ f(t; u_0^*; \cdots ; u_{n-1}^*); (t; u_0; \cdots ; u_{n-1})\\ \qquad \in \left(\left[0; \frac{\pm}{2}\right] \cup \left[1 - \frac{\pm}{2}; 1\right]\right) \times \prod_{j=0}^{n-1}(-1)^j[0; \infty): \end{cases}$$

and $u_j' = u_j$ for $u_j \in (-1)^j[\frac{1}{4^{n¡1¡j}}\pm^{n-1-j}(1-\pm)^{n-1-j}d; \frac{1}{8^{n¡1¡j}}b]$, $u_j' = (-1)^j\frac{1}{4^{n¡1¡j}}\pm^{n-1-j}(1-\pm)^{n-1-j}d$ for $u_j \in (-1)^j[0; \frac{1}{4^{n¡1¡j}}\pm^{n-1-j}(1-\pm)^{n-1-j}d)$, $u_j' = (-1)^j\frac{1}{8^{n¡1¡j}}b$ for $u_j \in (-1)^j(\frac{1}{8^{n¡1¡j}}b; \infty)$; $u_j^* = u_j$ for $u_j \in (-1)^j[0; \frac{1}{8^{n¡1¡j}}b]$, $u_j^* = (-1)^j\frac{1}{8^{n¡1¡j}}b$ for $u_j \in (-1)^j(\frac{1}{8^{n¡1¡j}}b; \infty)$.

Lemma 2. ([11, Lemma 3.5]) *If* $A : K \to X$ *is completely continuous, then* $\mu \circ A : K \to K$ *is also completely continuous.*

(H_1) implies that A and T' are well defined. From the continuity of f, it is easy to see that $A : K \to X$ is completely continuous. So $T : K \to K$ is completely continuous by using Lemma 2. For $x \in K'$, we have $|x(t)| \geq \frac{\pm}{1-\pm} \max_{\frac{\pm}{2} \leq t \leq 1-\frac{\pm}{2}} |x(t)| \geq \pm \max_{\frac{\pm}{2} \leq t \leq 1-\frac{\pm}{2}} |x(t)|$ for $t \in [\pm; 1-\pm]$ by the concavity of $(-1)^{n-1}x$ on $[\frac{\pm}{2}; 1-\frac{\pm}{2}]$. Thus,

$$\text{®}(x) \leq ||x||; \text{ and } \text{®}(x) \geq \pm \max_{\frac{\pm}{2} \leq t \leq 1-\frac{\pm}{2}} |x(t)|: \tag{8}$$

Lemma 3. *Let* $(H_1)-(H_4)$ *hold. Then* $T' : K' \to K'$ *is completely continuous.*

Proof. For all $x \in K'$, from (H_3) and (H_4), we have

$$\begin{aligned}
&(-1)^n \int_s^{\frac{1}{2}} f^*(¿; A_{n-1}x(¿); A_{n-2}x(¿); \cdots; A_1x(¿); x(¿))d¿ \\
&= \int_s^{\frac{\pm}{2}} (-1)^n f^*(¿; A_{n-1}x(¿); A_{n-2}x(¿); \cdots; A_1x(¿); x(¿))d¿ \\
&+ \int_{\frac{\pm}{2}}^{\frac{1}{2}} (-1)^n f^*(¿; A_{n-1}x(¿); A_{n-2}x(¿); \cdots; A_1x(¿); x(¿))d¿ \\
&\geq -\frac{\pm}{2}M + \frac{1-\pm}{2} \cdot \frac{\pm}{1-\pm}M \\
&= 0 \text{ for } 0 \leq t \leq \frac{\pm}{2};
\end{aligned}$$

$$(-1)^n \int_s^{\frac{1}{2}} f^*(¿; A_{n-1}x(¿); A_{n-2}x(¿); \cdots; A_1x(¿); x(¿))d¿ \geq 0 \text{ for } \frac{\pm}{2} \leq t \leq \frac{1}{2};$$

thus,

$$\begin{aligned}
(-1)^{n-1}(T'x)(t) &= \int_0^t \Phi^{-1}((-1)^n \int_s^{\frac{1}{2}} f^*(¿; A_{n-1}x(¿); \cdots; A_1x(¿); x(¿))d¿)ds \\
&\geq 0 \text{ for } 0 \leq t \leq \frac{1}{2}; \\
(-1)^{n-1}(\Phi((T'x)'))'(t) &= (-1)^{n-1}f^*(t; A_{n-1}x(t); ; \cdots; A_1x(t); x(t)) \\
&\leq 0 \text{ for } t \in \left[\frac{\pm}{2}; 1-\frac{\pm}{2}\right]:
\end{aligned}$$

So $T' : K' \to K'$. Using the continuity of f and the definition of f^*, it is easy to see that $T' : K \to K'$ is completely continuous.

Theorem 2. *Let* $(H_1)-(H_7)$ *hold. Then the boundary value problem (1) has at least two positive solutions* x_1 *and* x_2 *such that :*

$$0 < ||x_1^{(2(n-1))}|| < a < ||x_2^{(2(n-1))}||; \min_{t \in [\pm;1-\pm]} |x_2^{(2(n-1))}(t)| < \pm b:$$

Proof. At first we show that T has a fixed point $y_1 \in K$ with $0 < ||y_1|| < a$. In fact, for all $x \in \partial K_a$, we have $||x|| = a$. For $1 \le j \le n-1$ and $0 \le t \le 1$,

$$0 \le (-1)^{n-1-j}(A_j x)(t) = (-1)^{n-1-j}\int_0^1 G_j(t,\tau)x(\tau)d\tau \le a\int_0^1 |G_j(t,\tau)|d\tau \le \frac{1}{8^j}a.$$

From (H_5) we obtain

$$\begin{aligned}||Tx|| &= \max_{0\le t\le \frac{1}{2}} \left| \left(-\int_0^t \Phi^{-1}\left(\int_s^{\frac{1}{2}} f(\tau, A_{n-1}x(\tau), A_{n-2}x(\tau), \cdots,\right.\right.\right.\\ &\quad A_1x(\tau), x(\tau))d\tau)ds)^+ |\\ &= \max_{0\le t\le \frac{1}{2}} \max\{(-1)^{n-1}\left(-\int_0^t \Phi^{-1}\left(\int_s^{\frac{1}{2}} f(\tau, A_{n-1}x(\tau), A_{n-2}x(\tau), \cdots,\right.\right.\\ &\quad A_1x(\tau), x(\tau))d\tau)ds), 0\}\\ &< \frac{1}{2}\Phi^{-1}\left(\frac{1}{2}\cdot 2\Phi(2a)\right)\\ &= a.\end{aligned}$$

The existence of y_1 is proved by using the Schauder fixed point theorem.

Obviously, y_1 is a solution of (5) if and only if y_1 is a fixed point of A. Next we need to prove that y_1 is a solution of (5). Suppose the contrary, i.e., there is $t_0 \in (0,1)$ such that $y_1(t_0) \ne (Ay_1)(t_0)$. It must be $(-1)^{n-1}(Ay_1)(t_0) < 0 = y_1(t_0)$. Let (t_1, t_2) be the maximal interval such that $t_0 \in (t_1, t_2)$, and $(-1)^{n-1}(Ay_1)(t) < 0$ for $\forall t \in (t_1, t_2)$. Obviously $[t_1, t_2] \ne [0, 1]$ by (H_2). Without loss of generality, suppose $t_2 < 1$. Then $y_1(t) \equiv 0$ for $t \in [t_1, t_2]$ and $(-1)^{n-1}(Ay_1)(t) < 0$ for $t \in (t_1, t_2)$, $(Ay_1)(t_2) = 0$. Thus, $(-1)^{n-1}(Ay_1)'(t_2) \ge 0$. (H_2) implies $(-1)^{n-1}(\Phi((Ay_1)'))'(t) = (-1)^{n-1}f(t, 0, 0\cdots, 0) \le 0$ for $t \in [t_1, t_2]$. So $(-1)^{n-1}(Ay_1)'(t) \ge 0$ for $t \in [t_1, t_2]$. Therefore, $t_1 = 0$ and $(-1)^{n-1}(Ay_1)(0) \le (-1)^{n-1}(Ay_1)(t_0) < 0$. On the other hand, $(Ay_1)(0) = 0$, a contradiction.

We now show that (C_1) of Theorem 1 is satisfied. For $x \in \partial K'_a$, we have $||x|| = a$. For $1 \le j \le n-1$ and $0 \le t \le 1$, from (6) we have

$$0 \le (-1)^{n-1-j}(A_j x)(t) = (-1)^{n-1-j}\int_0^1 G_j(t,\tau)x(\tau)d\tau \le a\int_0^1 |G_j(t,\tau)|d\tau \le \frac{1}{8^j}a.$$

From (H_5) we obtain

$$\begin{aligned}||T'x|| &= \max_{0\le t\le \frac{1}{2}} \left| -\int_0^t \Phi^{-1}\left(\int_s^{\frac{1}{2}} f^*(\tau, A_{n-1}x(\tau), A_{n-2}x(\tau), \cdots,\right.\right.\\ &\quad A_1x(\tau), x(\tau))d\tau)ds|\\ &< \frac{1}{2}\Phi^{-1}\left(\frac{1}{2}\cdot 2\Phi(2a)\right)\\ &= a.\end{aligned}$$

Next we show that (C_2) of Theorem 1 is satisfied. For $x \in @K'(\pm b)$, i.e., $®(x) = \pm b$. For $\pm \le t \le 1 - \pm;$, $1 \le j \le n-1$, from (6), (7) and (8) we have

$$\pm b \le (-1)^{n-1} x(t) \le b;$$

$$(-1)^{n-1-j}(A_j x)(t) = (-1)^{n-1-j} \int_0^1 G_j(t; ¿) x(¿) d¿ \le b \int_0^1 |G_j(t; ¿)| d¿ \le \frac{1}{8^j} b;$$

$$(-1)^{n-1-j}(A_j x)(t) = (-1)^{n-1-j} \int_0^1 G_j(t; ¿) x(¿) d¿ \ge \pm b \int_{\pm}^{1-\pm} |G_j(t; ¿)| d¿ \ge \frac{1}{2^j} \pm^{j+1} (1 - 2\pm)^j b:$$

we may use conditions (H_3) and (H_6) to obtain

$$\begin{aligned} ®(T'x) &= \min_{\pm \le t \le \frac{1}{2}} | - \int_0^t \Phi^{-1} \left(\int_s^{\frac{1}{2}} f^*(¿; A_{n-1}x(¿); A_{n-2}x(¿); \cdots ; \right. \\ &\quad \left. A_1 x(¿); x(¿)) d¿ \right) ds | \\ &= \int_0^{\pm} \Phi^{-1} \left(\int_s^{\frac{1}{2}} (-1)^n f^*(¿; A_{n-1}x(¿); A_{n-2}x(¿); \cdots ; \right. \\ &\quad \left. A_1 x(¿); x(¿)) d¿ \right) ds \\ &> \pm \Phi^{-1} \left(-M\pm + \left(\frac{1}{2} - \pm \right) \frac{2}{1 - 2\pm} (M\pm + \Phi(b)) \right) = \pm b: \end{aligned}$$

Using the continuity of f and the definition of f^*, there is $c > b$ such that $||T'x|| < c$ for $®(x) \le b$. Applying Theorem 1, T' has a fixed point y_2 such that $y_2 \in K'_a(\pm b)$.

Finally, we show that $Ax = T'x$ for $x \in K'_a(\pm b) \cap \{u : T'u = u\}$. Let $x \in K'_a(\pm b) \cap \{u : T'u = u\}$, then

$$||x|| > a \ge \frac{1 - \pm}{2} \Phi^{-1} \left[\Phi\left(\frac{2d}{\pm}\right) + \frac{M\pm}{2} \right] + d:$$

We claim $||x|| = \max_{\frac{\pm}{2} \le t \le 1 - \frac{\pm}{2}} |x(t)|$. If there is $t_0 \in (0; \frac{\pm}{2})$ such that $|x(t_0)| = ||x|| > a$, then $x'(t_0) = (A'x)'(t_0) = -\Phi^{-1}(\int_{t_0}^{\frac{1}{2}} f^*(¿; A_{n-1}x(¿); \cdots ; A_1 x(¿); x(¿)) d¿) = 0$, i.e., $\int_{t_0}^{\frac{1}{2}} f^*(¿; A_{n-1}x(¿); \cdots ; A_1 x(¿); x(¿)) d¿ = 0$. From (H_7), we have

$$\begin{aligned} |x(t_0)| = ||x|| &= \Big| - \int_0^{t_0} \Phi^{-1} \left(\int_s^{\frac{1}{2}} f^*(¿; A_{n-1}x(¿); A_{n-2}x(¿); \cdots ; \right. \\ &\quad \left. A_1 x(¿); x(¿)) d¿ \right) ds \Big| \\ &= \Big| \int_0^{t_0} \Phi^{-1} \left(\int_s^{t_0} f^*(¿; A_{n-1}x(¿); A_{n-2}x(¿); \cdots ; \right. \\ &\quad \left. A_1 x(¿); x(¿)) d¿ \right) ds \Big| \end{aligned}$$

$$\le \frac{\pm}{2}\Phi^{-1}\left(\int_0^{\frac{\pm}{2}} \frac{2}{\pm}\Phi\left(\frac{2a}{\pm}\right) d¿\right)$$
$$= a;$$

a contradiction. Therefore, $||x|| = \max\limits_{\frac{\pm}{2}\le t\le 1-\frac{\pm}{2}} |x(t)|$.

Next we will prove $(-1)^{n-1}x(\frac{\pm}{2}) \ge d$. Suppose this is not true, then there exists $t_0 \in (\frac{\pm}{2}; \frac{1}{2})$ such that

$$(-1)^{n-1}x'(t_0) > \Phi^{-1}\left[\Phi\left(\frac{2d}{\pm}\right) + \frac{M\pm}{2}\right]:$$

It follows from the concavity of $(-1)^{n-1}x$ on $[\frac{\pm}{2}; 1-\frac{\pm}{2}]$ that

$$(-1)^{n-1}x'\left(\frac{\pm}{2}\right) \ge (-1)^{n-1}x'(t_0) > \Phi^{-1}\left[\Phi\left(\frac{2d}{\pm}\right) + \frac{M\pm}{2}\right]:$$

For $0 \le t \le \frac{\pm}{2}$, we have

$$\begin{aligned}(-1)^{n-1}\Phi(x'(t)) &= (-1)^{n-1}\Phi\left(x'\left(\frac{\pm}{2}\right)\right) - \int_t^{\frac{\pm}{2}}(-1)^{n-1}(\Phi(x'(s)))'ds \\ &= (-1)^{n-1}\Phi\left(x'\left(\frac{\pm}{2}\right)\right) + \int_t^{\frac{\pm}{2}}(-1)^{n} f^*(s; A_{n-1}x(s); A_{n-2}x(s); \cdots; \\ &\quad A_1x(s); x(s))ds \\ &\ge \left(\Phi\left(\frac{2d}{\pm}\right) + \frac{M\pm}{2}\right) - \frac{M\pm}{2} \\ &= \Phi(\frac{2d}{\pm});\end{aligned}$$

i.e., $(-1)^{n-1}x'(t) \ge \frac{2d}{\pm}$. Therefore,

$$0 = (-1)^{n-1}x(0) = (-1)^{n-1}x\left(\frac{\pm}{2}\right) - \int_0^{\frac{\pm}{2}}(-1)^{n-1}x'(s)ds < d - \frac{\pm}{2}\cdot\frac{2d}{\pm} = 0;$$

a contradiction. Thus, $d \le (-1)^{n-1}x(t) \le b$ for $\frac{\pm}{2} \le t \le 1-\frac{\pm}{2}$. For $1 \le j \le n-1$ and $\frac{\pm}{2} \le t \le 1-\frac{\pm}{2}$, from (6) and (7) we have

$$(-1)^{n-1-j}(A_jx)(t) = (-1)^{n-1-j}\int_0^1 G_j(t;¿)x(¿)d¿ \le b\int_0^1 |G_j(t;¿)|d¿ \le \frac{1}{8^j}b;$$

$$\begin{aligned}(-1)^{n-1-j}(A_jx)(t) = (-1)^{n-1-j}\int_0^1 G_j(t;¿)x(¿)d¿ &\ge d\int_{\frac{\pm}{2}}^{1-\frac{\pm}{2}} |G_j(t;¿)|d¿ \\ &\ge \frac{1}{4^j}\pm^j(1-\pm)^j d:\end{aligned}$$

From the definition of f^*, we have $f^*(t; A_{n-1}x(t); \cdots ; A_1x(t); x(t)) = f(t; A_{n-1}x(t); \cdots ; A_1x(t); x(t))$ for $0 \leq t \leq 1$. Then $Ax = T'x$ for $x \in K'_a(\pm b) \cap \{u : T'u = u\}$. Thus, y_2 is a solution of (5). Let $x_i(t) = (A_{n-1}y_i)(t) = \int_0^1 G_{n-1}(t; s)y_i(s)ds;\ i = 1; 2$, then x_1 and x_2 are two symmetric positive solutions of (1), and

$$0 < \left|\left|x_1^{(2(n-1))}\right|\right| < a < \left|\left|x_2^{(2(n-1))}\right|\right|; \min_{t\in[\pm;1-\pm]} \left|x_2^{(2(n-1))}(t)\right| < \pm b:$$

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