TAIWANESE JOURNAL OF MATHEMATICS
Vol. 8, No. 2, pp. 245-258, June 2004
This paper is available online at http://www.math.nthu.edu.tw/tjm/

OPTIMAL IMPULSIVE HARVEST POLICY FOR AN AUTONOMOUS SYSTEM

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Abstract. In this paper, we consider an autonomous model with impulsive harvest. We investigate the impulsive harvest policies for constant effort harvest and proportional harvest. For different harvest effort function, it is shown that there exists an unique impulsive periodic solution which is semi-stable or globally asymptotically stable. For the latter, the optimal harvest effort that maximizes the sustainable yield per unit time is determined.

1. Derivation of the Impulsive Harvest Model

After the first results about the theory of impulsive differential equations (IDE) are reported since 1960's[1-2], many theoretical research on this subject had been studied.

But the articles of which the IDE are applied practically in population dynamics are relatively rare[3-6] though IDE are suitable many mathematical models to simulate evolution of real processes. In real evolutionary processes of the population, the perturbation or the influence from out-side occur "immediately" as impulses not continuously, the duration of these perturbations is negligible compared to the duration of the whole process. For instance, as we know, fisherman can not fish day and night for 24 hours. Instead, they only fish in some time in a day. Besides, seasons will also affect the fishing period. So the problem of impulsive harvest is more practical and real compared to the continuous harvest.

However, to our knowledge, some results on impulsive harvest model[3-4] are discussed using the explicit general solution of the corresponding ODE without

Received May 21, 2002. accepted December 19, 2002.
Communicated by S. B. Hsu.
2000 *Mathematics Subject Classification*: 92D25, 34C25.
Key words and phrases: Impulsive moment; Impulsive harvest equation; Globally attractive; Optimal impulsive harvest policy.
Project Supported by the National Science Foundation of China (No. 10171010 and No. 10201005) and by Major Project of Education Ministry of China (No. 01061).

perturbation. While for most of the models of single species dynamics which are derived from differential equation in the following form

$$\dot{x} = f(x, t), \tag{1.1}$$

(1.1) mostly does not have explicit solution. That is why we study a general optimal impulsive harvest model in this paper.

Various choices of the functions f lead us to various models. Consider the following classical logistic equations,

$$\begin{cases} \dot{x} &= rx(1 - \frac{x}{K}) \\ x(0) &= x_0, \end{cases} \tag{1.2}$$

or

$$\begin{cases} \dot{x} &= r(t)x(1 - \frac{x}{K(t)}) \\ x(t_0) &= x_0, \end{cases} \tag{1.3}$$

where (1.2) is an autonomous evolutionary model, and (1.3) is a non-autonomous evolutionary model because the coefficients of (1.3) are dependent on the time t. In this paper, we consider the following equation:

$$\begin{cases} \dot{x} &= F(x) \\ x(0) &= x_0 \end{cases} \tag{1.4}$$

Suppose the following hypotheses are valid:

1. $K > 0$ is the capacity of environment (saturation level), $F : R^+ \to R$ is continuous function, where $F(0) = F(K) = 0$; $F(x) < 0$ for $x > K$;
2. there exists a unique point $\alpha \in (0, K)$ such that $F(\alpha)$ reach the maximum of $F(x)$ $x \in (0, K)$;
3. For each $x \in [0, \alpha]$ the function $F(x)$ is increasing while $F(x)$ is decreasing function in $[\alpha, K]$.

Considering the feasible operation, we suppose that we harvest once every fixed time T for the population X, which obeys equation (1.4). We shall derive a mathematical model of impulsive time harvest for equation (1.4):

$$\begin{cases} \frac{dN}{dt} = F(N) - \delta(s(t))Eh(N(t)), \\ N(t_0) = N_0. \end{cases} \tag{1.5}$$

Here, $h(N(t))$ is the function of general harvest; δ is the Dirac impulsive function, which satisfies $\delta(0) = \infty$ and $\delta(s) = 0$ for $s \neq 0$ and $\int_{-\infty}^{\infty} \delta(s)ds = 1$; $s(t)$ is defined as follows:

$$s(t) = \begin{cases} 0, & t = nT, \ n \in N; \\ -1, & t \neq nT, \ n \in N. \end{cases}$$

By our interpretation, the population X which grows according to logistic equation without exploitation and the management of the resource will be harvested $Eh(N(t))$ at time nT. For explaining the latter, we discuss the impulsive function δ deliberately. The Heaviside function $\theta(t)$.

$$\theta(t) = \begin{cases} 1, & if\ t \geq 0; \\ 0, & if\ t < 0. \end{cases}$$

Satisfies $\theta' = \delta$. Thus, if $t \neq nT$, $s(t) = -1$ and $\theta(s(t)) = 0$, namely, the management does not harvest; if $t = nT$, $s(t) = 0$ and $\theta(s(t)) = 1$, the management harvests $Q(nT)$, which satisfies

$$\begin{aligned} Q(nT) &= \int_{-\infty}^{nT} \delta(s(t))Eh(x(t))dt - \int_{-\infty}^{(n-1)T} \delta(s(t))Eh(x(t))dt \\ &= Eh(x(nT)). \end{aligned}$$

For biological considerations, we are only interested in positive solutions. In this paper, we always assume $N_0 > 0$.

2. Optimal Impulsive Harvest Policy for Constant Effort Harvest

Now, we consider single population X of size $N(t)$, which growth obeys the equation (1.4) is impulsively harvested by means of a constant effort, $h(N) \equiv 1$, namely, every time T, the management harvest constant is E. Equation of the impulsively harvested population reads

$$\begin{cases} \dfrac{dN}{dt} = F(N) - \delta(s(t))E, \\ N(t_0) = N_0. \end{cases} \tag{2.1}$$

We denote the solution of (2.1) by $N(t, t_0, N_0)$, while $x(t, t_0, x_0)$ is the solution of (1.4) without harvest.

Lemma 2.1. *Let the condition* $(1) - (3)$ *hold. Then the equation*

$$G(x) = F(x + E) - F(x) = 0 \tag{2.2}$$

where E *is a constant such that* $0 < E < \alpha < K - E < K$, *satisfied the follows:*

(1) *The equation* (2.2) *possesses a solution x=a;*

(2) $a \in [\alpha - E, \alpha]$;

(3) *The solution* $x = a$ *is unique.*

Proof. For a given constant $E,\ 0 < E < \alpha < K - E < K$, let

$$G(x) =: F(x + E) - F(x),$$

thus from the conditions (1),(3) and $0 < E < K,\ 0 < K - E < K$, we have

$$G(0) = F(E) - F(0) = F(E) > 0,$$

while

$$G(K - E) = F(K) - F(K - E) = -F(K - E) < 0$$

It is easy to see that the equation $G(x) = 0$ has at least one root $x = a$ from the property that $G(x)$ is continuous for x in the interval $[0, K - E]$.

Claim $a \in [\alpha - E, \alpha]$. From the condition (3), if $a \in [0, \alpha - E], then\ a + E \in [E, \alpha]$, then from the fact that the function $F(x)$ is increasing for x in $[0, \alpha]$, we conclude that

$$G(a) = F(a + E) - F(a) > 0.$$

Similarly, suppose $a \in [\alpha, K - E],\ a + E \in [\alpha + E, K]$, thus from the condition (3), the function $F(x)$ is decreasing in $[\alpha, K]$, and

$$G(a) = F(a + E) - F(a) < 0.$$

The above two inequalities imply that the roots of equation (2.2) locate the interval $[\alpha - E, \alpha]$.

Now we prove that the uniqueness of roots of equation (2.2). Assume that there exist two different roots of equation (2.2), namely, $G(a) = G(a^*) = 0$, Assume $\alpha - E < a < a^* < \alpha$. It is obvious that $\alpha < a + E < a^* + E < \alpha + E$. From the monotonicity of F in $[\alpha - E, \alpha]$ and $[\alpha, \alpha + E]$, we obtain that $F(a) < F(a^*)$ and $F(a + E) > F(a^* + E)$. These two inequalities yield the contradiction

$$0 = G(a) = F(a + E) - F(a) > F(a^* + E) - F(a^*) = G(a^*) = 0.$$

Therefore, we conclude that the equation $G(x) = 0$ has only unique root $x = a$. The proof is completed.

Lemma 2.2. *For a given time* $T > 0$, *the initial value* x_0 *which maximizes the increment of population of* (1.4) *in the time interval* $[0, T]$ *satisfies the equation* $F(x(T)) - F(x(0)) = 0$, *which follows that* $x(0) = x_0 = a$ *and* $x(T) = a + E$.

Proof. Let $x(t,0,x_0)$ be the solution of equation (1.4), denoted simply as $x(t)$. From the condition (1)-(3), it is obvious that $x(t)$ is a monotone increasing function of t in $[0,T]$. Define

$$g(x_0) = x(T,0,x_0) - x_0.$$

Suppose on the contrary the conclusion of Lemma 2.2 does not hold.

Let $F(x(0)) = \eta_1 < F(x(T)) = \eta_2$. If Δt is small enough, then we can expand the of solution of equation (1.4) as

$$x(t,0,x_0) = x_0 + \eta_1 t + o(\Delta t) \text{ for } t \in [0,\Delta t] \tag{2.3}$$

and

$$x(t,0,x_0) = x(T,0,x_0) + \eta_2(t-T) + o(\Delta t) \text{ for } t \in [T,T+\Delta t]. \tag{2.4}$$

From Equations (2.3) and (2.4) and $\eta_1 < \eta_2$, we have

$$\begin{aligned} x(\Delta t,0,x_0) - x_0 &= \eta_1 \Delta t + o(\Delta t) \\ &< \eta_2 \Delta t + o(\Delta t) = x(T+\Delta t,0,x_0) - x(T,0,x_0). \end{aligned}$$

Denote $\bar{x}_0 = x(\Delta t,0,x_0)$, then

$$\begin{aligned} g(\bar{x}_0) &= x(T,0,\bar{x}_0) - \bar{x}_0 = x(T+\Delta t,\Delta t,\bar{x}_0) - \bar{x}_0 \\ &= x(T+\Delta t,0,x_0) - x(\Delta t,0,x_0) \\ &> x(T,0,x_0) - x_0 = g(x_0), \end{aligned}$$

which shows that if $F(x(0)) = F(x_0) < F(x(T))$, the increment $g(x_0)$ is not maximal.

Similarly, we can prove $F(x(0)) > F(x(T))$ the increment $g(x_0)$ is also not maximal. Therefore, the increment reaches its maxima if and only if the condition $F(x(0)) = F(x(T))$ is satisfied, namely, $F(x(T)) - F(x(0)) = 0$. From Lemma 2.1 we have that $G(x) = 0$ has unique root $x = a$, therefore we obtain $x_0 = a, x(T) = a + E$.

Theorem 2.1. *For the impulsive harvest equation* (2.1), *if the initial value is the unique root* $x = a$ *of equation (2.2), then there exists a unique positive impulsive periodic solution* $\xi(t)$ *of (2.1), which satisfies* $\xi(nT) = a$, *where* T *can be estimated by* $T = \int_a^{a+E} \frac{1}{F(x)} dx$. *In addition,* $\xi(t)$ *is semi-stable in the sense that* $N(t,0,N_0) \to \xi(t)$ *if* $N_0 > a$, *but* $N(t,0,N_0) \to 0$ *if* $N_0 < a$.

Proof. For the equation without impulsive perturbations

$$\begin{cases} \dot{x} &= F(x) \\ x(0) &= a \end{cases} \tag{2.5}$$

After finite time T, the solution of equation (2.5) evolves from initial $x(0) = a$ into $x(T) = a + E$. It is easy to see that the directions of vector field of equation (2.5) are the same as at the time $t = 0$ and $t = T$ from the fact that $x = a$ is the unique root of the equation $G(x) = F(x + E) - F(x) = 0$. Integrating the equality $\frac{d(x(t))}{F(x(t))} = dt$, from 0 to T, yields

$$T = \int_a^{a+E} \frac{du}{F(u)}. \tag{2.6}$$

where $F(x) \neq 0, a < x < a + E$.

In the following, we shall show that $N(t, 0, a)$ is an impulsive periodic solution of (2.1). Since

$$N(T, 0, a) = x(T, 0, a) - E = a + E - E = a = N(0, 0, a)$$

and

$$\begin{aligned} N(2T, 0, a) &= N(2T, T, N(T, 0, a)) = x(2T, T, a) - E \\ &= x(T, 0, a) - E = a + E - E = a, \end{aligned}$$

we obtain inductively,

$$N(nT, 0, a) = a \text{ for } \forall n \in N.$$

Let $N(t, 0, a) = \xi(t)$, then $\xi(t)$ is an impulsive periodic solution of (2.1) with $\xi(nT) = a$ for $\forall n \in N$.

Next, we study the stability of the solution $\xi(t)$.

Suppose that $N_0 > a$. For convenience, we denote $N_n = N(nT, 0, N_0)$. From Lemma 2.2, it follows that

$$N_1 = N(T, 0, N_0) = x(T, 0, N_0) - E = g(N_0) - E + N_0 < N_0$$

On the other hand, $N_0 > a$ implies that

$$N_1 = x(T, 0, N_0) - E > x(T, 0, a) - E = N(T, 0, a) = \xi(T) = a.$$

Similarly, we have

$$\begin{aligned} N_2 &= N(2T, 0, N_0) = N(2T, T, N_1) = x(2T, T, N_1) - E = x(T, 0, N_1) - E \\ &= g(N_1) - E + N_1 < N_1, \end{aligned}$$

and

$$N_2 = x(T, 0, N_1) - E > x(T, 0, a) - E = a.$$

By the same arguments we can obtain a monotone decreasing sequence $\{N_n\}$ with a lower bound a. It is obvious that the sequence $\{N_n\}$ has a limit, suppose the limit is β, then $\beta \geq a$.

If $\beta > a$, then

$$\begin{aligned} N_{n+1} - N_n &= N((n+1)T, 0, N_0) - N_n = N((n+1)T, nT, N_n) - N_n \\ &= x((n+1)T, nT, N_n) - E - N_n = x(T, 0, N_n) - E - N_n \\ &= g(N_n) - E, \end{aligned}$$

thus letting $n \to \infty$ yields $g(\beta) = E$. Because $g(x_0) = x(T, 0, x_0) - x_0 = E$ has only one root $x_0 = a$, we get a contradiction. Thus $\beta = a$, that is $\lim\limits_{n\to\infty} N_n = \beta = a$.

From the continuous dependence of solution on initial value, for any given $\varepsilon > 0$ there is a $\delta \in (0, \varepsilon)$, such that $0 < |x_0 - a| < \delta$ implies $|x(t, 0, x_0) - x(t, 0, a)| < \varepsilon$ for $t \in [0, T)$. Since $\lim\limits_{n\to\infty} N_n = \beta$, there exists a nature number $\bar{N}$ such that $n \geq \bar{N}$ implies that $0 < N_n - a < \delta$, and then for any $n \geq \bar{N}$ and $t \in [nT, (n+1)T)$, we have

$$\begin{aligned} |N(t, 0, N_0) - \xi(t)| &= |N(t, 0, N_0) - N(t, 0, a)| = |N(t, nT, N_n) - N(t, nT, a)| \\ &= |x(t, nT, N_n) - x(t, nT, a)| \\ &= |x(t - nT, 0, N_n) - x(t - nT, 0, a)| < \varepsilon \end{aligned}$$

for $t \in [nT, (n+1)T)$, which implies that

$$|N(t, 0, N_0) - \xi(t)| < \varepsilon \quad \text{for } t \geq \bar{N}T.$$

Hence $|N(t, 0, N_0) - \xi(t)| \to 0$ as $t \to \infty$ for $N_0 > a$.

If $0 < N_0 < a$, we can also get a monotone decreasing sequence $\{N_n\}$, suppose the harvest effort is constant E, $N(t, 0, N_0) \to 0$ for $0 < N_0 < a$. We complete the proof.

From Theorem 2.1, we know there exists unique positive impulsive periodic solutions $\xi(t)$, it follows that $\xi(t)$ is "semi-stable". The assumption in section 2 that the harvesting effort is a constant leads to that we can not control exploitation for dangerous region and also we can not have feasible sustainable policy. Therefore, in this situation there exists no optimal harvest effort and maximal sustainable output.

3. Optimal Impulsive Harvest Policy for Proportional Harvest

In this section, we will use the phrase "*catch-per-unit-effort hypothesis*" to describe an assumption that *catc−per −unit− effort*, is proportional to the stock level, or that $h(N) = N$, E denotes effort and satisfies $0 \leq E < 1$. In other word, the management harvests $Q(nT) = EN(nT)$ in nT. Equation of the impulsively harvested population takes the form

$$\begin{cases} \dfrac{dN}{dt} &= F(N) - \delta(s(t))EN, \\ N(t_0) &= N_0. \end{cases} \tag{3.1}$$

In this section, the solution of (4.1) is still denoted by $N(t, t_0, N_0)$.

Now we investigate the optimal impulsive harvest policy, namely, the optimal harvesting effort, the maximum sustainable yield and the corresponding optimal population level.

Definition 3.1. ([7]) A solution $\tilde{\xi}(t)$ of (3.1) is globally attractive for positive initial value if any solution $N(t, 0, N_0)$ of (3.1) with $N_0 > 0$ satisfies

$$\lim_{t \to +\infty} |N(t, 0, N_0) - \tilde{\xi}(t)| = 0.$$

Lemma 3.1. *Let conditions* $(1) - (3)$ *be hold. Then the equation*

$$\tilde{G}(x) = F(\frac{x}{1-E}) - F(x) = 0, \tag{3.2}$$

where $0 < E < 1$ *and* $\dfrac{\alpha}{1-E} < K$ *satisfies the followings:*

(1) *The equation (3.2) possesses a solution* $x = \tilde{a}$;

(2) $\tilde{a} \in [(1-E)\alpha, \alpha]$;

(3) *The solution* $x = \tilde{a}$ *is unique.*

Proof. For a given constant E such that $0 < E < 1$ and $\dfrac{\alpha}{1-E} < K$, let

$$\tilde{G}(x) =: F(\frac{x}{1-E}) - F(x),$$

and $F(\alpha)$ be the maximum of $F(x)$ in $[0, K]$. Then we have

$$\tilde{G}(\alpha) = F(\frac{\alpha}{1-E}) - F(\alpha) < 0,$$

and

$$\tilde{G}((1-E)\alpha) = F(\alpha) - F((1-E)\alpha) > 0.$$

Since $\tilde{G}(x)$ is continuous in the close interval $[(1-E)\alpha, \alpha]$, $\tilde{G}(x) = 0$ has at least one root $x = \tilde{a} \in [(1-E)\alpha, \alpha]$.

Next, we prove that the uniqueness of roots of equation (3.2).

Assume that there exist two different roots of equation (3.2), namely,

$$\tilde{G}(\tilde{a}) = \tilde{G}(\tilde{a}^*) = 0.$$

Let $(1-E)\alpha < \tilde{a} < \tilde{a}^* < \alpha$. It is obvious that $\alpha < \frac{\tilde{a}}{1-E} < \frac{\tilde{a}^*}{1-E} < \frac{\alpha}{1-E} < K$. From the monotonicity of $F(x)$ in $[(1-E)\alpha, \alpha]$ and $[\alpha, \frac{\alpha}{1-E}]$, we obtain that $F(\tilde{a}) < F(\tilde{a}^*)$ and $F(\frac{\tilde{a}}{1-E}) > F(\frac{\tilde{a}^*}{1-E})$. These two inequalities yield the following contradiction

$$0 = G(\tilde{a}) = F(\frac{\tilde{a}}{1-E}) - F(\tilde{a}) > F(\frac{\tilde{a}^*}{1-E}) - F(\tilde{a}^*) = G(\tilde{a}^*) = 0.$$

Therefore, we conclude that the equation $\tilde{G}(x) = 0$ has unique root $x = \tilde{a}$. The proof is completed.

Lemma 3.2. *For a given time $T > 0$, the initial value x_0 which maximizes the increment of population of (1.4) in the time interval $[0, T]$ satisfies the equation $F(x(T)) - F(x(0)) = 0$, $x(0) = x_0 = \tilde{a}$, $x(T) = \frac{\tilde{a}}{1-E}$, and the maximal increment is $\frac{E\tilde{a}}{1-E}$.*

The proof is similar to Lemma 2.2, so we omit it.

Theorem 3.1. *There exists a unique positive impulsive periodic solution $N(t, 0, \tilde{a}) =: \tilde{\xi}(t)$ of (3.1), which satisfies $\tilde{\xi}(nT) = \tilde{a}$. In addition, $\tilde{\xi}(t)$ is globally attractive for positive initial value.*

Proof. At first, we want to show that $N(t, 0, \tilde{a})$ is impulsive periodic solution of (3.1).

For the equation without impulsive perturbations

$$\begin{cases} \dot{x} &= F(x) \\ x(0) &= \tilde{a} \end{cases} \tag{3.3}$$

After finite time T, the solution of equation (3.3) evolves from initial $x(0) = \tilde{a}$ into $x(T) = \frac{\tilde{a}}{1-E}$. It is obvious that the direction of vector field of equation (3.3) are the same as $t = 0$ and $t = T$. From the fact that $x = \tilde{a}$ is the unique root of the equation $\tilde{G}(x) = F(\frac{x}{1-E}) - F(x) = 0$. It is obvious that

$$N(T, 0, \tilde{a}) = (1-E)x(T, 0, \tilde{a}) = (1-E)\frac{\tilde{a}}{1-E} = \tilde{a}$$

and

$$\begin{aligned} N(2T,0,\tilde{a}) &= N(2T,T,N(T,0,\tilde{a})) = N(2T,T,\tilde{a}) \\ &= (1-E)x(2T,T,\tilde{a}) = (1-E)x(T,0,\tilde{a}) = \tilde{a}. \end{aligned}$$

Inductively, we prove that

$$N(nT,0,\tilde{a}) = \tilde{a} \ for \ \forall n \in N.$$

Therefore, (3.1) has a unique impulsive periodic solution $N(t,0,\tilde{a}) :=\tilde{\xi}(t)$ with $\tilde{\xi}(nT) = \tilde{a}$ for $\forall n \in N$.

In the following, we shall prove the global attractiveness of $\tilde{\xi}(t)$.

Suppose $N_0 > \tilde{a}$ and $N_n := N(nT,0,N_0),\ n \in N$. From Lemma 3.2, we conclude that the increment which is expressed $g(x_0) = x(T,0,x_0) - x_0 = \frac{E\tilde{a}}{1-E}$ reach its maxima at $x_0 = \tilde{a}$. Hence $g(x_0) < \frac{E\tilde{a}}{1-E}$ when $x_0 \neq \tilde{a}$, and we have

$$\begin{aligned} N_1 &= N(T,0,N_0) = (1-E)x(T,0,N_0) \\ &= (1-E)(g(N_0)+N_0) \\ &= (1-E)g(N_0) - EN_0 + N_0 \\ &< (1-E)\frac{E\tilde{a}}{1-E} - EN_0 + N_0 \\ &< E(\tilde{a}-N_0) + N_0 < N_0 \end{aligned}$$

and

$$\begin{aligned} N_1 &= N(T,0,N_0) = (1-E)x(T,0,N_0) > (1-E)x(T,0,\tilde{a}) \\ &= N(T,0,\tilde{a}) = \tilde{a}. \end{aligned}$$

Similarly, we can prove that $\tilde{a} < N_2 < N_1$. Thus we obtain a monotone decreasing sequence $\{N_n\}$ with a lower bound $\tilde{a}$. Assume that the sequence $\{N_n\}$ has a limit $\tilde{\beta}$, it is obvious $\tilde{\beta} \geq \tilde{a}$. Using the similar argument in Section 2, suppose $\tilde{\beta} > \tilde{a}$, then

$$\begin{aligned} N_{n+1} - N_n &= N((n+1)T,0,N_0) - N_n = N((n+1)T,nT,N_n) - N_n \\ &= (1-E)x((n+1)T,nT,N_n) - N_n = (1-E)x(T,0,N_n) - N_n \\ &= (1-E)g(N_n) - EN_n, \end{aligned}$$

Let $n \to \infty$, we obtain that $(1-E)g(\tilde{\beta}) - E\tilde{\beta} = 0$, and

$$g(\tilde{\beta}) = x(T,0,\tilde{\beta}) - \tilde{\beta} = \frac{E\tilde{\beta}}{1-E},$$

this contradicts to the fact that the above equation has a unique root $\tilde{a}$. Thus $\tilde{\beta} = \tilde{a}$ and we have proved that

$$\lim_{n\to+\infty} N_n = \tilde{\beta} = \tilde{a}.$$

Therefore, for any given $\varepsilon > 0$, there is a $\delta \in (0, \varepsilon)$ such that $n > \tilde{N}$ implies $0 < N_n - \tilde{a} < \delta$, then from the continuous dependence of solution on initial value, we have $|x(t, 0, N_n) - x(t, 0, \tilde{a})| < \varepsilon$ for $t \in [0, T)$. Thus $n \geq \tilde{N}$ and $t \in [nT, (n+1)T)$. Note that $1 - E < 1$ and

$$\begin{aligned} |N(t, 0, N_0) - \tilde{\xi}(t)| &= |N(t, nT, N_n) - N(t, nT, \tilde{a})| \\ &= |1 - E||x(t, nT, N_n) - x(t, nT, \tilde{a})| \\ &< |x(t - nT, 0, N_n) - x(t - nT, 0, \tilde{a})| < \varepsilon. \end{aligned}$$

That is,

$$\lim_{t\to\infty} |N(t, 0, N_0) - \tilde{\xi}(t)| = 0 \text{ for } N_0 > \tilde{a}.$$

By the similar argument, if $0 < N_0 < \tilde{a}$, we can write

$$\begin{aligned} (1 - E)x(T, 0, N_0) > N_0 &\iff (1 - E)(g(N_0) + N_0) > N_0 \\ &\iff (1 - E)g(N_0) > EN_0 \\ &\iff \frac{E\tilde{a}}{1 - E} > g(N_0) > \frac{EN_0}{1 - E} \iff \tilde{a} > N_0. \end{aligned}$$

thus the following holds

$$N_1 = x(T, 0, N_0) = (1 - E)x(T, 0, N_0) > N_0.$$

Therefore, we have a monotone increasing sequence $\{N_n\}$ with a upper bound $\tilde{a}$. we can prove

$$\lim_{t\to\infty} |N(t, 0, N_0) - \tilde{\xi}(t)| = 0 \text{ for } 0 < N_0 < \tilde{a}.$$

Thus we have shown that the impulsive periodic solution $\xi(t)$ is globally attractive for positive initial value. The proof is complete.

In real world, fishers would like to make a decision how to obtain maximum harvest. From Theorem 3.1, the sustainable yield per unit time is

$$Y(E) = \frac{E\tilde{a}}{T(1 - E)}. \tag{3.4}$$

The relation of T and E can be estimated as

$$T = \int_{\tilde{a}}^{\tilde{a}/(1-E)} \frac{du}{F(u)}. \tag{3.5}$$

Our object is to find an E^* such that $Y(E)$ reaches its maximum at $E = E^*$. Further we apply Lemma 3.1 and Lemma 3.2, we know $x^*(T) = \tilde{a}$. If we suppose impulsive harvest moment T is fixed, equation (3.5) means to

E^* can be solved, thus we obtain optimal sustainable yield $Y(E^*)$.

Example 1 Logistic model (general solution is explicit)

Let us consider logistic equation (1.2), it is obvious that condition (1)-(3) are satisfied. The function $F(x) = rx(1 - x/K)$ has a unique maximum at $x = \frac{1}{2}K$. From Lemma 2.1 we obtain $a = \frac{1}{2}(K - E)$ which satisfies the equation

$$G(a) = F(a + E) - F(a) = 0.$$

From Lemma 3.1 we solve

$$\tilde{a} = \frac{1 - E}{2 - E}K$$

which is the root of the equation

$$\tilde{G}(x) = F(\frac{x}{1 - E}) - F(x) = 0.$$

So in the case of proportional impulsive harvest, we conclude that

$$\tilde{a} = x^*(T) = \frac{1 - E}{2 - E}K, \tag{3.6}$$

meanwhile,

$$T = \int_{\tilde{a}}^{\tilde{a}/(1-E)} \frac{du}{ru(1 - \frac{u}{K})} = \frac{\ln(\frac{u}{u - K})}{r}\Big|_{(1-E)K/(2-E)}^{K/(2-E)}. \tag{3.7}$$

suppose T is a constant (every T time From (3.7), we have

$$E^* = 1 - e^{\frac{-rT}{2}} \tag{3.8}$$

Substituting (3.8) into (3.6) yields that

$$x^*(T) = \frac{K}{1 + e^{\frac{rT}{2}}},$$

Furthermore we have

$$Y(E^*) = \frac{e^{\frac{rT}{2}} - 1}{e^{\frac{rT}{2}} + 1}K.$$

These results coincide with the conclusions in [3].

Example 2 A generalized Logistic equation (general solution can not expressed explicitly)

In this example, we study a model which take the following form:

$$\frac{dx}{dt} = \frac{rx(1 - \frac{x}{K})}{1 + \beta x}, \tag{3.9}$$

where $\beta > 0$ is a parameter of level of that which the population utilize the environment, it is obvious that equation (3.9) possesses first integral but its solution can not be expressed explicitly. Let $F(x) = \frac{rx(1-\frac{x}{K})}{1+\beta x}$, it is not difficult to see that $\alpha = \frac{\sqrt{1+\beta K}-1}{\beta}$ is the maximum of F(x) in $[0, K]$. $F(0) = F(K) = 0$, the condition (1)-(3) are satisfied. According to Lemma 2.1 and 3.1, we can compute

$$a = \frac{\sqrt{4 + \beta^2 E^2 + 4\beta K} - (2 + \beta E)}{2\beta}$$

and

$$\tilde{a} = x(T^*) = \frac{\sqrt{(2 - E)^2 + 4\beta K(1 - E)} - (2 - E)}{2\beta}.$$

If T is fixed, we use the same method as example 1 to solving E^* and $Y(E^*)$ although it is more complicated than classical logistic equation. Therefore our results are valid.

References

1. V. Milman, A. Myshkis, On the stability of motion in the presence of impulses, *Sib. Math. J.* **1** (1960), 233-237(in Russian).
2. V. Milman, A. Myshkis, Random impulses in linear dynamical systems, approximate methods of solution of differential equations, *Ed. Acad. Sci. Ukr. SSR. Kiev*, **?** (1963), 64-81(in Russian).
3. X. Y. Zhang, Shuai Zhsh., K. Wang, Optimal impulsive harvesting policy for single population, nonlinear analysis: real world applications, **4** (2003), 639-651.
4. L. Bai, Ke Wang, Optimal impulsive harvesting policy for gompertz model, *Journal of Biomathematics*, **17** (2002), 56-67 (in Chinese).
5. I. Svetoslav, Nenov, Impulsive controllability and optimization problems in population dynamics, *Nonlinear Analysis TMA*, **36** (1999), 881-890.
6. J. Angelova, A. Dishliev, Optimization problems for one-impulsive models from population dynamics, *Nonlinear Analysis TMA*, **39** (2000), 483-497.

7. Yoshizawa T., Stability Theory and the Existence of Periodic Solutions and Almost Periodic Solutions, Springer-Verlag, New York Inc., 1975, 210-223.

8. V. Lakchmikantham, D. Bainov, P. Simeonov, *Theory of Impulsive Differential Equations, World Scientific*, Singapore, 1989.

9. D. Bainov, P. Simeonov, *Systems with Impulsive Effect: Stability Theory and Applications*, Ellis Horwood, Chiclusler, UK, 1989.

10. C. W. Clark, *Mathematical Bio-economics: The Optimal Management of Renewable Resources*, 2nd ed, Wiley, New York, 1990.

11. Meng Fan, Ke Wang, Optimal harvesting policy for single population with periodic coefficients, *Math. Biosci.*, **152** (1998), 165-177.

12. R. K. Gorge, A. K. Nandakumaran, A. Arapostathis, A note on controllability of impulsive system, *J. Math. Anal. Appl.*, **241(2)** (2000), 276-283.

13. G. V. Tsretkva, Construction of an optimal policy taking into account ecological constraints (Russian), Modelling of natural system and optimal control problems (in Russian). (Chita), Vo"Naukce", *Novosibirsk*, 1995, pp. 65-74.

14. Fan Meng, Liu Huiming, etc. The harvesting optimal problem of dispensation systems, *Bulletin of Biomathematics*, **4** (1997), 38-42 (in Chinese).

15. Liu Huiming and Wang Ke, The Harvesting Optimal Problem of Critical Dispensation Systems, *Journal of Biomathematics*, **13(4)** (1998), 479-483 (in Chinese).

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