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ON CONJECTURE OF R. BRÜCK CONCERNING THE ENTIRE FUNCTION SHARING ONE VALUE CM WITH ITS DERIVATIVE

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Abstract. In this paper, we investigate the conjecture of R. Brück, and prove that the conjecture of R. Brück holds for entire functions of infinite order and hyper order less than $\frac{1}{2}$.

1. Introduction and Results

In this paper, we shall assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna's value distribution theory of meromorphic functions (e.g. see [9, 10]). In addition, we will use the notations $\lambda(f)$ to denote the exponents of convergence of the zero-sequence of meromorphic function $f(z)$; $\sigma(f)$ to denote the order growth of $f(z)$. We recall the definition of hyper-order (see [21]), $\sigma_2(f)$ of $f(z)$ is defined by

$$\sigma_2(f) = \overline{\lim_{r\to\infty}} \frac{\log\log T(r, f)}{\log r}.$$

Let f and g be two non-constant meromorphic functions, and let a be a finite value in the complex plane. We say that f and g share the value a CM (IM) provided that $f-a$ and $g-a$ have the same zeros counting multiplicities (ignoring multiplicities). Nevanlinna four values theorem (see [16]) says that if two non-constant meromorphic functions f and g share four values CM, then $f \equiv g$ or f is a Möbius transformation of g. The condition "f and g share four values CM" has been weakened to "f and g share two values CM and two values IM" by Gundersen

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[7,8], as well as by Mues [14] and Wang [19]. But whether the condition can be weakened to "f and g share three values IM and another value CM" or not, is still an open question. In a special case, it was shown [17] that if an entire function f share two finite values CM with its derivative, then $f \equiv f'$. This result has been generalized to sharing values IM by Gundersen [6] and by Mues-Steinmetz [15] independently.

How is the relation between f with f' if an entire function f share one finite value CM with its derivative f'? In [1], R. Brück raised the following.

Conjecture. Let f be a non-constant entire function such that hyper order $\sigma_2(f) < \infty$ and $\sigma_2(f)$ isn't positive integer. If f and f' share the finite value a CM, then

$$\frac{f' - a}{f - a} = c$$

where c is a nonzero constant.

For the case that $a = 0$ had been proved by Brück in [1]. From differential equations

$$\frac{f' - 1}{f - 1} = e^{z^n}, \quad \frac{f' - 1}{f - 1} = e^{e^z},$$

we see that when the hyper order $\sigma_2(f)$ of f is a positive integer or infinite, the conjecture of Brück don't holds. For the case that the zero-points of f' are fewness, Brück obtain the following in [1].

Theorem A. *Let f be a nonconstant entire function. If f and f' share a value 1 CM, and satisfy $N(r, 0, f') = S(r, f)$, then*

$$\frac{f' - 1}{f - 1} = c$$

where c is a nonzero constant.

For entire functions with finite order, Lianzhong Yang proved following two theorems in [20].

Theorem B. *Let f be a nonconstant entire function with finite order. If f and f' share a finite value a CM, then*

$$\frac{f' - a}{f - a} = c$$

where c is a nonzero constant.

Theorem C. *Let* f *be a nonconstant entire function with finite order. If* f *and* $f^{(k)} (k \geq 1)$ *share a finite value* $a \neq 0$ *CM, then*

$$\frac{f^{(k)} - a}{f - a} = c$$

where c *is a nonzero constant,* k *is a positive integer.*

In this paper, we investigate the case that an entire function is of infinite order, and get the following theorem.

Theorem 1. *Let* $f(z)$ *be a nonconstant entire function with hyper order* $\sigma_2(f) = \alpha < \frac{1}{2}$. *If* f *and* f' *share the finite value* a *CM, then*

$$\frac{f' - a}{f - a} \equiv c$$

where c *is a nonzero constant.*

By Theorem 1, we can obtain the following corollary.

Corollary 1. *Let* f *be a nonconstant entire function with hyper order* $\sigma_2(f) < \frac{1}{2}$. *If* f *and* f' *share a finite value* a *CM, and there exists a point* z_0 *satisfying* $f'(z_0) = f(z_0) \neq a$, *then* $f \equiv f'$.

Corollary 2. *Let* f *be a nonconstant entire function with hyper order* $\sigma_2(f) < \frac{1}{2}$. *If* f *and* f' *share a finite value* a *CM and a finite value* $b (\neq a)$ *IM, then* $f \equiv f'$.

Corollary 3. *Let* f *be a nonconstant entire function with hyper order* $\sigma_2(f) < \frac{1}{2}$. *If* f *and* f' *share a finite value* a *CM, and there exists a point* z_0 *satisfying* $f^{(k)}(z_0) = f^{(k+1)}(z_0) \neq 0, k$ *is a positive integer, then* $f \equiv f'$.

2. Lemmas for the Proof of Theorem 1

The Hadamard Theorem of entire functions of infinite order can be found in [11].

Lemma 1. *Let* f *be a transcendental entire function of infinite order and hyper order* $\sigma_2(f) = \alpha < \infty$; *then* f *can be represented in*

$$f(z) = U(z)e^{V(z)},$$

where U *and* V *are entire functions such that*

$$\lambda(f) = \lambda(U) = \sigma(U); \quad \lambda_2(f) = \lambda_2(U) = \sigma_2(U);$$

$$\sigma_2(f) = \max\{\sigma_2(U); \sigma_2(e^V)\};$$

where notation $\lambda_2(f)$ *denotes the hyper exponent of convergence of zeros of entire function* f *by*

$$\lambda_2(f) = \overline{\lim_{r\to\infty}} \frac{\log\log N(r; \frac{1}{f})}{\log r}.$$

Lemma 2. [4] *Let* $g(z)$ *be an entire function of infinite order with the hyper order* $\sigma_2(g) = \sigma$, *and let* $\nu(r)$ *be the central index of* g. *Then*

$$\overline{\lim_{r\to\infty}} \frac{\log\log \nu(r)}{\log r} = \sigma_2(g) = \sigma.$$

Using the similar proof as in proof of Remark 1 of [5], we can obtain the following Lemma 3.

Lemma 3. *Let* $f(z)$ *be an entire function with* $\sigma(f) = \infty$ *and* $\sigma_2(f) = \alpha < +\infty$, *let a set* $E \subset [1, \infty)$ *have finite logarithmic measure. Then there exists* $\{z_k = r_k e^{i\theta_k}\}$ *such that* $|f(z_k)| = M(r_k, f)$, $\theta_k \in [0, 2\pi)$, $\lim_{k\to\infty} \theta_k = \theta_0 \in [0, 2\pi)$, $r_k \notin E$, $r_k \to \infty$ *and for any given* $\varepsilon > 0$, *for sufficiently large* r_k, *we have if* $\alpha > 0$ *then*

$$\exp\{r_k^{\alpha-\varepsilon}\} < \nu(r_k) < \exp\{r_k^{\alpha+\varepsilon}\};$$

if $\alpha = 0$ *then for any large* $M (> 0)$, *we have as* r_k *sufficiently large*

$$\nu(r_k) > r_k^M.$$

Lemma 4. (see [13]) *Let*

$$Q(z) = b_n z^n + b_{n-1} z^{n-1} + \cdots + b_0$$

where n *is a positive integer and* $b_n = \alpha_n e^{i\theta_n}$, $\alpha_n > 0$, $\theta_n \in [0, 2\pi)$. *For any given* ε, $0 < \varepsilon < \pi/(4n)$, *we introduce* $2n$ *opened angles*

$$S_j : -\frac{\theta_n}{n} + (2j-1)\frac{\pi}{2n} + \varepsilon < \theta < -\frac{\theta_n}{n} + (2j+1)\frac{\pi}{2n} - \varepsilon \quad (j = 0, 1, \ldots, 2n-1).$$

Then there exists a positive number $R = R(\varepsilon)$ *such that for* $|z| = r > R$,

$$\mathrm{Re}\{Q(z)\} > \alpha_n(1-\varepsilon)\sin(n\varepsilon)r^n$$

if $z \in S_j$ *where* j *is even; while*

$$\mathrm{Re}\{Q(z)\} < -\alpha_n(1-\varepsilon)\sin(n\varepsilon)r^n$$

if $z \in S_j$ *where* j *is odd.*

Now for any given $\theta \in [0, 2\pi)$. If $\theta \neq -\frac{\theta_n}{n} + (2j-1)\frac{\pi}{2n}$, $(j = 0, 1, \ldots, 2n-1)$, then we take ε sufficiently small, there is some S_j, $j \in \{0, 1, \ldots, 2n-1\}$ such that $\theta \in S_j$.

Lemma 5. [2] *Let* $h(z)$ *is an entire function with order* $\sigma(h) = \sigma < \frac{1}{2}$, *set*

$$A(r) = \inf_{|z|=r} \log|h(z)|, \; B(r) = \sup_{|z|=r} \log|h(z)|.$$

If $\sigma < \alpha < 1$, *then*

$$\underline{\log\text{dens}}\{r : A(r) > (\cos \pi\alpha) B(r)\} \geq 1 - \frac{\sigma}{\alpha},$$

where the lower logarithmic density $\underline{\log\text{dens}} H$ *of subset* $H \subset (1, +\infty)$ *is defined by*

$$\underline{\log\text{dens}} H = \underline{\lim}_{r \to \infty} \left(\int_1^r (\chi_H(t)/t) dt \right) / \log r,$$

and the upper logarithmic density $\overline{\log\text{dens}} H$ *of subset* $H \subset (1, +\infty)$ *is defined by*

$$\overline{\log\text{dens}} H = \overline{\lim}_{r \to \infty} \left(\int_1^r (\chi_H(t)/t) dt \right) / \log r,$$

where $\chi_H(t)$ *is the characteristic function of set* H.

Lemma 6. [3] *Let* $h(z)$ *is an entire function with lower order* $\mu = \mu(h) < \frac{1}{2}$, *and* $\mu < \sigma = \sigma(h)$. *If* $\mu \leq \delta < \min(\sigma, \frac{1}{2})$ *and* $\delta < \alpha < \frac{1}{2}$, *then*

$$\underline{\log\text{dens}}\{r : A(r) > (\cos \pi\alpha) B(r) > r^{\delta}\} \geq C(\sigma, \delta, \alpha),$$

where $C(\sigma, \delta, \alpha)$ *is a positive constant only dependent on* σ, δ, α.

Remark. By definitions of the logarithmic measure and the logarithmic density, we see that if the upper logarithmic density $\overline{\log\text{dens}} H > 0$, then the logarithmic measure $\text{lm} H = +\infty$.

3. Proof of Theorem 1

Since f and f' share the finite value a CM, by Lemma 1, we can write

$$\frac{f' - a}{f - a} = e^{Q(z)} \tag{3.1}$$

where $Q(z)$ is an entire function. The case that $a = 0$ had been proved by R. Brück [1], the case that f is an entire function of finite order had been proved by L. Z. Yang [20]. Now we can suppose that $a \neq 0$ and $\sigma(f) = \infty$. Set $F = \frac{f}{a} - 1$; then F is an entire function,

$$\sigma(F) = \sigma(f) = \infty, \; \sigma_2(F) = \sigma_2(f) = \alpha$$

and F satisfies the linear differential equation

$$F' - e^{Q(z)}F = 1. \tag{3.2}$$

Because of $\sigma_2(f) = \alpha < \frac{1}{2}$, we know that for $Q(z)$, there are three cases: (1) $Q(z)$ is a constant; (2) $Q(z)$ is a polynomial with degree $\deg Q \geq 1$; (3) $Q(z)$ is a transcendental entire function with order

$$\sigma(Q) = \beta \leq \alpha < \frac{1}{2}, \; \sigma_2(e^Q) = \sigma(Q) = \beta.$$

Now we split this into three cases to prove.

Case 1. $Q(z)$ is a constant. Then Theorem 1 holds.

Case 2. $Q(z)$ is a polynomial with degree $\deg Q = n \geq 1$. We will show $\sigma_2(f) = n \geq 1$ which contradict with condition $\sigma_2(f) = \alpha < \frac{1}{2}$.

From the Wiman-Valiron theory (see [10, 12, 18]), there is a set $E_1 \subset (1, \infty)$ having logarithmic measure $\text{lm} E_1 < \infty$, we choose z satisfying $|z| = r \notin [0, 1] \bigcup E_1$ and $|F(z)| = M(r, F)$, then we have

$$\frac{F'(z)}{F(z)} = \frac{\nu(r)}{z}(1 + o(1)), \tag{3.3}$$

where $\nu(r)$ is the central index of F. Substituting (3.3) into (3.2), we obtain

$$\frac{\nu(r)}{z}(1 + o(1)) = e^{Q(z)} + \frac{1}{F(z)}. \tag{3.4}$$

Since $\sigma(F) = \sigma(f) = \infty$, and $\deg Q = n \geq 1$, $|F(z)| = M(r, F)$, for sufficiently large $|z| = r$ and any given $\varepsilon_1 (> 0)$, by (3.4), we have

$$\frac{\nu(r)}{r} \leq e^{r^{n+\varepsilon_1}}. \tag{3.5}$$

Since ε_1 is arbitrary, by (3.5) and Lemma 2, we have $\sigma_2(F) \leq n$. We assert that $\sigma_2(F) = n$. Now we assume that $\sigma_2(F) = \delta (0 \leq \delta < n)$ and prove that $\sigma_2(F) = \delta$ fails.

By Lemma 3, there is a point range $\{z_k = r_k e^{i\mu_k}\}$ such that $|f(z_k)| = M(r_k; f); \mu_k \in [0; 2¼); \lim_{k ! 1} \mu_k = \mu_0 \in [0; 2¼); r_k \notin E_1 \bigcup [0; 1]; r_k \to \infty;$ for any given " satisfying that if $\pm = 0$; then

$$0 < 3" < \min\{"_1; \frac{¼}{4n}\};$$

if $\pm > 0$; then

$$0 < 3" < \min\{\pm; "_1; n - \pm; \frac{¼}{4n}\};$$

we see that if $\pm > 0$, then we have

$$\exp\{r_k^{\pm i "}\} < º(r_k) < \exp\{r_k^{\pm + "}\}; \quad (3:6)$$

if $\pm = 0$; then for any large $M (> 0)$; we have as r_k sufficiently large

$$º(r_k) > r_k^M : \quad (3:7)$$

Let

$$Q(z) = ®_n e^{i\mu_n} z^n + b_{n i 1} z^{n i 1} + \cdots + b_0; ®_n > 0; \mu_n \in [0; 2¼):$$

By Lemma 4, there are $2n$ opened angles for above ";

$$S_j : -\frac{\mu_n}{n} + (2j - 1)\frac{¼}{2n} + " < \mu < -\frac{\mu_n}{n} + (2j + 1)\frac{¼}{2n} - "; \ (j = 0; 1; \cdots; 2n - 1): \quad (3:8)$$

For the above μ_0; there three cases: (i) $\mu_0 \in S_j$ where j is odd; (ii) $\mu_0 \in S_j$ where j is even; (iii) $\mu_0 = -\frac{\mu_n}{n} + (2j - 1)\frac{¼}{2n}$ for some j. We again divide this into three subcases.

Subcase (i). $\mu_0 \in S_j$ where j is odd. Since S_j is an opened set and $\lim_{k ! 1} \mu_k = \mu_{0;}$ there is a $K > 0$ such that $\mu_k \in S_j$ when $k > K$, by Lemma 4, we see that

$$\mathrm{Re}\{Q(r_k e^{i\mu_k})\} < -d r_k^n; \quad (3:9)$$

where $d = ®_n(1 - ")\sin(n") > 0$: For $\{z_k = r_k e^{i\mu_k}\}$; by (3.4) and $|F(z)| = M(r; F)$, we have

$$\frac{º(r_k)}{z_k}(1 + o(1)) = e^{Q(r_k e^{i\mu_k})} + o(1): \quad (3:10)$$

If $\pm > 0$, then by $3" < \pm$ and (3.6), (3.9), (3.10), we have

$$\exp\{r_k^{\pm i "}\} < º(r_k)(1 + o(1)) < r_k \exp\{-d r_k^n\} + o(r_k): \quad (3:11)$$

(3.11) is a contradiction. If $\pm = 0$, then by (3.7), (3.10), we have

$$r_k^{M i 1} < \frac{º(r_k)}{r_k}(1 + o(1)) < \exp\{-d r_k^n\} + o(1): \quad (3:12)$$

(3.12) is also a contradiction.

Subcase (ii). $\theta_0 \in S_j$ where j is even. Since S_j is an open set and $\lim_{k\to\infty}\theta_k = \theta_0$ there is $K > 0$ such that $\theta_k \in S_j$ when $k > K$. By Lemma 1, we have

$$\mathrm{Re}\{Q(r_k e^{i\theta_k})\} > d r_k^n, \tag{3.13}$$

where $d = \alpha_n(1-\varepsilon)\sin(n\varepsilon) > 0$. For $\{z_k = r_k e^{i\theta_k}\}$, by (3.4), (3.6) and (3.13), we have

$$\exp\{r_k^{\delta+\varepsilon}\} > \gamma(r_k)(1+o(1)) > r_k \exp\{d r_k^n\} - o(r_k), \tag{3.14}$$

(3.14) contradicts with $\delta + \varepsilon < n$.

Subcase (iii). $\theta_0 = -\frac{\theta_n}{n} + (2j-1)\frac{\pi}{2n}$ for some $j \in \{0, 1, \cdots, 2n-1\}$. Since $\mathrm{Re}\{Q(r_k e^{i\theta_0})\} = 0$ when r_k is sufficiently large, and a straight line $\arg z = \theta_0$ is an asymptotic line of $\{r_k e^{i\theta_k}\}$, we see that there is a $K > 0$ such that when $k > K$, we have

$$-1 < \mathrm{Re}\{Q(r_k e^{i\theta_k})\} < 1, \quad \frac{1}{e} \le |e^{Q(r_k e^{i\theta_k})}| \le e. \tag{3.15}$$

By (3.6) (or (3.7)), (3.10), (3.15), we have

$$\frac{1}{r_k}\exp\{r_k^{\delta-\varepsilon}\} - o(1) \le \frac{\gamma(r_k)}{r_k}(1+o(1)) - o(1) \le |e^{Q(r_k e^{i\theta_k})}| \le e, \tag{3.16}$$

or

$$r_k^{M-1} - o(1) \le \frac{\gamma(r_k)}{r_k}(1+o(1)) - o(1) \le |e^{Q(r_k e^{i\theta_k})}| \le e. \tag{3.17}$$

But both (3.16) and (3.17) are contradictory.

Case (3). $Q(z)$ is a transcendental entire function with order $\sigma(Q) = \beta \le \alpha < \frac{1}{2}$. By the equation (3.2), we have

$$Q(z) = \log(\frac{F'}{F} - \frac{1}{F}),$$

where $\log(\frac{F'}{F} - \frac{1}{F})$ is a principal branch of $\mathrm{Log}(\frac{F'}{F} - \frac{1}{F})$. Hence we have

$$\begin{aligned} |Q(z)| &\le |\log(\frac{F'}{F} - \frac{1}{F})| \le |\log|\frac{F'}{F} - \frac{1}{F}|| + |\arg(\frac{F'}{F} - \frac{1}{F})| \\ &\le |\log|\frac{F'}{F} - \frac{1}{F}|| + 2\pi. \end{aligned} \tag{3.18}$$

As in the proof of Case (2), we choose z satisfying $|z| = r \notin [0,1] \bigcup E_1$ and $|F(z)| = M(r, F)$, and get

$$|Q(z)| \le \log(\frac{\gamma(r)}{r}(1+o(1)) + o(1)) + 2\pi \le \log\gamma(r) + O(1), \tag{3.19}$$

where $\nu(r)$ is the central index of F. Since

$$\frac{\log\log \nu(r)}{\log r} \leq \alpha + 1$$

for sufficiently large r, by (3.19), we get

$$|Q(z)| \leq r^{\alpha+1} + O(1). \tag{3.20}$$

But by Lemma 5(or 6), we know that there exists a set $H \subset (1, \infty)$ that have logarithmic measure $\operatorname{lm} E = \infty$, such that for all z satisfying $|z| = r \in H$, we have

$$|Q(z)| \geq M(r, Q)^c, \tag{3.21}$$

where $c(0 < c < 1)$. Now for all z satisfying $|z| = r \in H \backslash (E_1 \bigcup [0, 1])$ and $|F(z)| = M(r, F)$, by (3.20) and (3.21), we get

$$\frac{M(r, Q)^c}{r^{\alpha+1}} \leq 1. \tag{3.22}$$

Since $Q(z)$ is transcendental, we see that

$$\frac{M(r, Q)^c}{r^{\alpha+1}} \to \infty,$$

which contradicts with (3.22). Theorem 1 is thus proved.

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