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BOUNDED STABLE SETS OF SKEW PRODUCT FOR MEROMORPHIC FUNCTIONS

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Abstract. Boundedness of components of the Fatou set of the skew product is studied, which is associated with finitely generated meromorphic semigroup.

1. Introduction and Main Result

For some integer $m \geq 1$, Σ_m denotes the one sided symbol's space of m digits,

$$\Sigma_m = \{1, 2, \cdots, m\} \times \cdots \times \{1, 2, \cdots, m\} \times \cdots = \prod_1^{\infty} \{1, 2, \cdots, m\}.$$

$\sigma : \Sigma_m \to \Sigma_m$ denotes the shift map, i.e. for any $w = (w_1, w_2, w_3, \cdots) \in \Sigma_m$, $\sigma w = (w_2, w_3, \cdots)$.

Let f_j $(j = 1, 2, \cdots, m, m \geq 1)$ be transcendental and meromorphic in $\mathbb{C}$. The map $\tilde{f}$ is said to be the skew product associated with the generator system $\{f_1, f_2, \cdots, f_m\}$, i.e.

$$\tilde{f} : \Sigma_m \times \mathbb{C} \to \Sigma_m \times \overline{\mathbb{C}}$$
$$(w, x) \to (\sigma w, f_{w_1}(x)),$$

where $w = (w_1, w_2, \cdots) \in \Sigma_m$. See [9] for the case of the skew product associated with rational semigroups. We define the following projection:

$$\pi \circ \tilde{f} : \Sigma_m \times \mathbb{C} \to \overline{\mathbb{C}}$$
$$(w, x) \to f_{w_1}(x).$$

Some notations and definitions are stated below.

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$\{\tilde{f}^n\}$ is said to be normal at a point $(w_0, z_0) \in \Sigma_m \times \overline{C}$, if there is a neighborhood $V \times U \subset \Sigma_m \times \overline{C}$ of (w_0, z_0) such that $\tilde{f}^n$ are defined in $V \times U$ for all n and $\{\pi \circ \tilde{f}^n\}$ is normal in $V \times U$ in the sense of Montel.

Furthermore, $\{\tilde{f}^n\}$ is said to be normal in $V \times U$ if $\{\tilde{f}^n\}$ is normal at each point $(w, z) \in V \times U$.

The Fatou set $F(\tilde{f})$ of $\tilde{f}$ is defined by the subset of $\Sigma_m \times \overline{C}$ in which $\{\tilde{f}^n\}$ is normal. The Julia set $J(\tilde{f})$ of $\tilde{f}$ is the complement of $F(\tilde{f})$, i.e.

$$J(\tilde{f}) = \Sigma_m \times \overline{C} \backslash F(\tilde{f}).$$

A component $V \times U \subset F(\tilde{f})$ is said to be bounded, if U is bounded.

If $m = 1$, the dynamical behavior of $\tilde{f}$ is the same as that of f. In this case, denote $F(\tilde{f}), J(\tilde{f})$ by $F(f), J(f)$ respectively. See [5] for reference.

Let f be transcendental and meromorphic in C. Set

$$L(r, f) = \inf_{|z|=r} |f(z)|.$$

If $f(z)$ is meromorphic in C satisfying

$$\limsup_{r \to \infty} \frac{L(r, f)}{r} = \infty,$$

then any non wandering component of $F(f)$ must be bounded (see [12], [13], [14], [15] for some extension). Obviously it is still a research topic to consider the bounded components of $F(f)$ in the research field, see also [1], [2], [8], [10], [11]. Our main result is following:

Theorem. *Let* $f_j\,(j = 1, 2, \cdots, m,\ m \geq 1)$ *be transcendental and meromorphic in* C *with the properties: given* $d > 1$, *for any positive number* $\mathcal{L} > 1$ *and for all sufficiently large* $R > 0$, *there exists* $R_j \in (R^{\frac{1}{d}}, R]$ *such that*

$$L(R_j, f_j) > \mathcal{L}R, \quad j = 1, 2, \cdots, m. \tag{1}$$

Suppose that $\tilde{f}$ *is the skew product associated with the generator system* $\{f_1, f_2, \cdots, f_m\}$. *If there is a component* $V \times U \subset F(\tilde{f})$ *such that* $\pi \circ \tilde{f}^n : V \times U \to U$ *for all* n, *and* $\pi(J(\tilde{f}))$ *has an unbounded component, then* $V \times U$ *is bounded.*

Remark. If f_j is transcendental and entire of order less than $\frac{1}{2}$ (see [4]) or with gaps (see [6]) or is transcendental meromorphic of order σ less than $\frac{1}{2}$ and $f(z)$ has the deficient number $\delta(\infty, f)$ at ∞ satisfying $\delta(\infty, f) > 1 - \cos \sigma\pi$ (see [7]), then f_j satisfies (1), for $j = 1, 2, \cdots, m$.

2. Lemmas

Let us recall some known results on hyperbolic geometry. An open set in $\overline{\mathbb{C}}$ is hyperbolic if its boundary contains at least three points. Let Ω be a hyperbolic domain and $\lambda_\Omega(z)$ denote the hyperbolic density of the hyperbolic metric on Ω. Let $\rho_\Omega(z_1, z_2)$ stand for the hyperbolic distance between z_1 and z_2 on Ω, i.e.

$$\rho_\Omega(z_1, z_2) = \inf_{\gamma \subset \Omega} \int_\gamma \lambda_\Omega(z)|dz|, \tag{2}$$

where γ is a Jordan curve joining z_1 to z_2 in Ω. If Ω is simply-connected and $d(z, \partial\Omega)$ is the Euclidean distance between $z \in \Omega$ and $\partial\Omega$, then for any $z \in \Omega$

$$\frac{1}{4d(z, \partial\Omega)} \leq \lambda_\Omega(z) \leq \frac{1}{d(z, \partial\Omega)}.$$

Let $f : U \to V$ be analytic, where U and V are hyperbolic domains. By Contraction Principle we have

$$\rho_V(f(z_1), f(z_2)) \leq \rho_U(z_1, z_2), \ \forall z_1, z_2 \in U. \tag{3}$$

In order to prove the Theorem, we need the following lemmas.

Lemma 1. *Let $\tilde{f}$ be the skew product associated with $\{f_1, f_2, \cdots, f_m\}$, where f_j are meromorphic in $\mathbb{C}$, $j = 1, 2, \cdots, m$, $m \geq 1$. Suppose $V \times U$ is a component of $F(\tilde{f})$. If $\pi \circ \tilde{f}^n(V \times U) \subset U, n = 1, 2, \cdots$ and $\pi(J(\tilde{f}))$ has an unbounded component, then for any point $(w, z_0) \in V \times U$, there exists a compact set B containing z and $\pi \circ \tilde{f}((w, z))$, $B \subset U$ such that for all sufficiently large n*

$$|\pi \circ \tilde{f}^n((w, z))| \leq c|\pi \circ \tilde{f}^{n-1}((w, z))| + c', (w, z) \in \{w\} \times B,$$

where c and c' are some constants.

Proof. Let Γ be an unbounded component of $\pi(J(\tilde{f}))$. Then $\mathbb{C} \backslash \Gamma$ is a simple connected domain and

$$\pi \circ \tilde{f}^n : \{w\} \times U \to \mathbb{C} \backslash \Gamma, n = 1, 2, \cdots.$$

Take $a \in \Gamma$. Then for any $z \in \mathbb{C} \backslash \Gamma$, we have

$$\lambda_{\mathbb{C} \backslash \Gamma}(z) \geq \frac{1}{4d(z, \Gamma)} \geq \frac{1}{4(|z| + |a|)}.$$

Let γ be a Jordan curve in U connecting z_0 and $\pi \circ \tilde{f}((w, z_0))$. Then $\pi \circ \tilde{f}(\{w\} \times \gamma)$ connects $\pi \circ \tilde{f}((w, z_0))$ and $\pi \circ \tilde{f}^2((w, z_0))$. Clearly, $\gamma \cup \pi \circ \tilde{f}(\{w\} \times \gamma)$ is compact and for any $(w,z)\in\{w\}\times\gamma$, it follows that $\pi\circ\tilde f((w,z))\in\gamma\cup\pi\circ\tilde f(\{w\}\times\gamma)$. Set

$$A=\max\{\rho_U(z,z'):z\in\gamma, z'\in\gamma\cup\pi\circ\tilde f(\{w\}\times\gamma)\}.$$

Then $A<\infty$. From (2) and (3), for sufficiently large n, we have

$$\rho_{\mathbb{C}\setminus\Gamma}(\pi\circ\tilde f^{n-1}((w,z)),\pi\circ\tilde f^{n-1}(\tilde f(w,z)))\le\rho_U(z,\pi\circ\tilde f((w,z)))\le A, z\in\gamma.$$

Note that

$$A\ge\rho_{\mathbb{C}\setminus\Gamma}(\pi\circ\tilde f^{n-1}((w,z)),\pi\circ\tilde f^{n}((w,z)))\ge\int_{|\pi\circ\tilde f^{n-1}((w,z))|}^{|\pi\circ\tilde f^{n}((w,z))|}\frac{1}{4(|z|+|a|)}|dz|.$$

By a simple calculation, we obtain

$$\frac{|\pi\circ\tilde f^{n}((w,z))|+|a|}{|\pi\circ\tilde f^{n-1}((w,z))|+|a|}\le e^{4K}, z\in\gamma.$$

Then Lemma 1 follows, with $c=e^{4A}$, $c'=(e^{4A}-1)|a|$ and $B=\gamma$.

The following lemmas from [4, pp.165].

Lemma 2. *Let D be a domain and $f_n\to f, g_n\to g$ locally and uniformly on D, f_n, g_n are analytic on D, $n=1,2,\cdots$. If $g(D)\subset D$, then $f_n\circ g_n\to f\circ g$ locally and uniformly on D, in the chordal metric.*

Lemma 3. *Let $\tilde f$ be the skew product associated with $\{f_1,f_2,\cdots,f_m\}$, where f_j are meromorphic in $\mathbb{C}$, $j=1,2,\cdots,m; m\ge 1$. Suppose $V\times U$ is a component of $F(\tilde f)$ and for a $w\in V$, $\pi\circ\tilde f^n:\{w\}\times U\to U$, $n=1,2,\cdots$. If $\{\pi\circ\tilde f^n\}$ has a nonconstant limit function on $\{w\}\times U$, then there exists a subsequence $\{\tilde f^{n_k}\}$ of $\{\tilde f^n\}$ such that*

$$\pi\circ\tilde f^{n_k}((w,z))\to z, \forall(w,z)\in\{w\}\times U, k\to\infty.$$

Proof. By the assumption in Lemma 3, any subsequence $\{t_k\}$, $\{\pi\circ\tilde f^{t_k}\}$ is normal in $\{w\}\times U$. Note that $\{\pi\circ\tilde f^n\}$ has nonconstant limit function in $\{w\}\times U$. Let $g(z)$ be a limit function of $\pi\circ\tilde f^n$ on $\{w\}\times U$. Obviously, $g:U\to U$. Then there exists a subsequence $\{l_k\}$ such that $\pi\circ\tilde f^{l_k}$ locally and uniformly converges to $g(z)$ in $\{w\}\times U$. Set

$$n_k=l_k-l_{k-1}\to\infty, k\to\infty.$$

$\{\pi\circ\tilde f^{n_k}\}$ is normal in $\{w\}\times U$. Without loss of generality, assume $\varphi(z)$ is a limit function of $\pi \circ \tilde{f}^{n_k}$ in $\{w\} \times U$. By Lemma 2, it deduces

$$\begin{aligned} \varphi \circ g(z) &= \lim_{k\to\infty} \pi \circ \tilde{f}^{n_k} \circ \pi \circ \tilde{f}^{l_k - 1}((w; z)) \\ &= \lim_{n\to\infty} [(f_{w_{l_n}} \circ \cdots \circ f_{w_1}) \circ (f_{w_{l_n - 1}} \circ \cdots \circ f_{w_1})^{-1} \\ &\quad \circ (f_{w_{l_n - 1}} \circ \cdots \circ f_{w_1})(z)] \\ &= \lim_{n\to\infty} f_{w_{l_n}} \circ \cdots \circ f_{w_1}(z) \\ &= \lim_{k\to\infty} \pi \circ \tilde{f}^{l_k}((w; z)) = g(z). \end{aligned}$$

So, $\varphi(z) \equiv z$. Lemma 3 follows.

3. Proof of Theorem

Assume $V \times U$ is unbounded. Then we shall prove by contradation that there exists a point $(w; a) \in \{w\} \times U \subset V \times U$ such that

$$(4) \qquad |\pi \circ \tilde{f}^n((w; a))| < \infty; n = 1; 2; \cdots ;$$

where $w = (w_1; w_2; \cdots ; w_n \cdots)$. In fact, if (4) is not true, then for any point $(w; z) \in \{w\} \times U$, we must have

$$(5) \qquad \pi \circ \tilde{f}^n((w; z)) \to \infty; \ n \to \infty.$$

Otherwise, suppose that there exists a point $(w; z_0) \in \{w\} \times U$ such that

$$(6) \qquad \pi \circ \tilde{f}^n((w; z_0)) \not\to \infty; \ n \to \infty.$$

From (6), there exists a subsequence $\{n_k\}_{k=1}^{\infty}$ and nonnegative constant M_0 such that

$$\lim_{k\to\infty} |\pi \circ \tilde{f}^{n_k}((w; z_0))| = M_0.$$

So, there exists $k_0 > 0$ such that for all $k > k_0$,

$$(7) \qquad |\pi \circ \tilde{f}^{n_k}((w; z_0))| < M_0 + 1.$$

Fixed $k > k_0$. Let Γ' be an unbounded component of $\pi(J(\tilde{f}))$. Then $C \backslash \Gamma'$ is a simple connected domain. From (3), we have

$$\rho_U(z_0; \pi \circ \tilde{f}((w; z_0))) \geq \rho_{C \backslash \Gamma'}(\pi \circ \tilde{f}^{n_k}((w; z_0)); \pi \circ \tilde{f}^{n_k+1}((w; z_0))).$$

Set $A_0 = \max\{\rho_U(z_0; \pi \circ \tilde{f}((w; z_0))); 1\}$. Then $A_0 < \infty$. Since

$$\lambda_{C \backslash \Gamma'}(z) \geq \frac{1}{4(|z| + |a'|)}; a' \in \Gamma';$$

and

$$\rho_{\mathbb{C}\setminus\Gamma'}(\pi\circ\tilde f^{n_k}((w,z_0)),\pi\circ\tilde f^{n_k+1}((w,z_0))) \geq \int_{|\pi\circ\tilde f^{n_k}((w,z_0))|}^{|\pi\circ\tilde f^{n_k+1}((w,z_0))|}\frac{1}{4(|z|+|a'|)}|dz|$$
$$= \frac{1}{4}\log\frac{|\pi\circ\tilde f^{n_k+1}((w,z_0))|+|a'|}{|\pi\circ\tilde f^{n_k}((w,z_0))|+|a'|}.$$

From (7), it follows that

$$|\pi\circ\tilde f^{n_k+1}((w,z_0))| \leq e^{4A_0}(|\pi\circ\tilde f^{n_k}((w,z_0))|+|a'|)$$
$$< e^{4A_0}(M_0+1+|a'|) = e^{4A_0}M_1,$$

where $M_1 = M_0+1+|a'|$. Similarly, for any integer $s>0$, we have

$$|\pi\circ\tilde f^{n_k+s}((w,z_0))| \leq se^{4sA_0}M_1. \tag{8}$$

Choose $\mathcal{L} > 2e^{4A_0}$ and sufficiently large $\tilde R > e^{4dA_0}M_1^d$. Then there exists $\tilde R_1 \in (\tilde R^{\frac{1}{d}},\tilde R]$ such that

$$|f_{w_{n_k+1}}(z)| \geq L(\tilde R_1, f_{w_{n_k+1}}) > \mathcal{L}\tilde R > e^{4A_0}M_1,\ |z| = \tilde R_1. \tag{9}$$

Choose a Jordan curve γ in U connecting $\pi\circ\tilde f^{n_k}((w,z_0))$ to a point in $\{z\in\mathbb{C}: |z|=\tilde R_1\}$. Then from (8), we have

$$f_{w_{n_k+1}}(\gamma)\cap\{z\in\mathbb{C}: |z|\leq e^{4A_0}M_1\}\neq\emptyset$$

and from (9), it follows that

$$f_{w_{n_k+1}}(\gamma)\cap\{z\in\mathbb{C}: |z|=\mathcal{L}\tilde R\}\neq\emptyset.$$

Therefore there exists $\tilde z_1\in\gamma$ satisfying

$$|f_{w_{n_k+1}}(\tilde z_1)| = \mathcal{L}\tilde R.$$

By the assumption in Theorem, there is $\tilde R_2 \in ((\mathcal{L}\tilde R)^{\frac{1}{d}}, \mathcal{L}\tilde R]$ such that

$$|f_{w_{n_k+2}}(z)| \geq L(\tilde R_2, f_{w_{n_k+2}}) > \mathcal{L}^2\tilde R > 2e^{4\times 2A_0}M_1,\ |z|=\tilde R_2. \tag{10}$$

Similarly, from (8), we have

$$f_{w_{n_k+2}}\circ f_{w_{n_k+1}}(\gamma)\cap\{z\in\mathbb{C}: |z|\leq 2e^{4\times 2A_0}M_1\}\neq\emptyset$$

and from (10), we have

$$f_{w_{n_k+2}}\circ f_{w_{n_k+1}}(\gamma)\cap\{z\in\mathbb{C}: |z|=\mathcal{L}^2\tilde R\}\neq\emptyset.$$

Hence there exists $\tilde{z}_2 \in$ ° satisfying

$$|f_{w_{n_k+2}} \circ f_{w_{n_k+1}}(\tilde{z}_2)| = \mathcal{L}^2\tilde{R}:$$

By mathematical induction, there exists $\tilde{z}_s \in$ ° satisfying

$$(11) \qquad |f_{w_{n_k+s}} \circ \cdots \circ f_{w_{n_k+1}}(\tilde{z}_s)| = \mathcal{L}^s\tilde{R}:$$

Set $A^0 = \max\{$½$_U($¼$\circ \tilde{f}^{n_k}((w; z_0)); z); z \in$ °$\}$. Then $A^0 < \infty$. Similarly, from (3), we have

$$A^0 \geq \int_{j¼±\tilde{f}^{n_k+s}((w;z_0))j}^{jf_{w_{n_k+s}}±¢¢¢±f_{w_{n_k+1}}(\tilde{z}_s)j} \frac{1}{4(|z| + |a^0|)}|dz|$$
$$= \frac{1}{4}\log\frac{|f_{w_{n_k+s}} \circ \cdots \circ f_{w_{n_k+1}}(\tilde{z}_s)| + |a^0|}{|¼ \circ \tilde{f}^{n_k+s}((w; z_0))| + |a^0|}:$$

Therefore

$$|f_{w_{n_k+s}} \circ \cdots \circ f_{w_{n_k+1}}(\tilde{z}_s)| \leq e^{4A^0}(|¼ \circ \tilde{f}^{n_k+s}((w; z_0))| + |a^0|)$$

Let $s = n_{k+1} - n_k$. From (7) and (11), it follows that

$$\mathcal{L}^{n_{k+1}\ i\ n_k}\tilde{R} \leq e^{4A^0}M_1;$$

equirvalently

$$2^{n_{k+1}\ i\ n_k}e^{4(n_{k+1}\ i\ n_k+d)A_0}M_1^d < e^{4A^0}M_1:$$

Similarly, for any integer $p \geq 1$, it follows that

$$2^{n_{k+p}\ i\ n_k}e^{4(n_{k+p}\ i\ n_k+d)A_0}M_1^d < e^{4A^0}M_1:$$

The above inequality is impossible when $p \to \infty$. This contradiction shows (5) is valid.

Next, choose a Jordan curve $°^0_1$ in U connecting z to $¼ \circ \tilde{f}((w; z))$, by Lemma 1, there are constants $L > 0$ and $L^0 > 0$ satisfying

$$(12) \qquad |¼ \circ \tilde{f}^n((w; z))| \leq L|¼ \circ \tilde{f}^{n\ i\ 1}((w; z))| + L^0; (w; z) \in \{w\} \times °^0_1:$$

Since $°^0_1$ connects z to $¼ \circ \tilde{f}((w; z))$, take a part curve $°^0_2 \subset ¼ \circ \tilde{f}(\{w\} \times °^0_1)$ lying between $¼ \circ \tilde{f}((w; z))$ and $¼ \circ \tilde{f}^2((w; z))$ and connecting $¼ \circ \tilde{f}((w; z))$ to $¼ \circ \tilde{f}^2((w; z))$, $\cdots$, similarly, take a part curve $°^0_n \subset ¼ \circ \tilde{f}^{n\ i\ 1}(\{w\} \times °^0_1)$ lying between $¼ \circ \tilde{f}^{n\ i\ 1}((w; z))$ and $¼ \circ \tilde{f}^n((w; z))$ and connecting $¼ \circ \tilde{f}^{n\ i\ 1}((w; z))$ to $¼ \circ \tilde{f}^n((w; z))$, $\cdots$, such that $°^0_n$ and $°^0_{n+1}$ have only one common end point, $n = 1; 2; \cdots$. Let $\Gamma^{00} = \cup_{n=1}^{1} °^0_n$. Then Γ^{00} is a curve approaches ∞ in U. For any point $(w; z^0) \in \Gamma^{00}$, there exists a point $(w; z^{00}) \in °^0_1$ and $n \geq 1$ such that

$$|¼ \circ \tilde{f}^{n\ i\ 1}((w; z^{00}))| = |z^0|:$$

Since for any sufficiently large $R > 0$, we have $\{z \in \mathbb{C} : |z| = R\} \cap \Gamma'' \neq \emptyset$. Then there exists infinitely many large n, and for each this kind of n, there exists a point $(w, z'_n) \in \{w\} \times \gamma'_1$ satisfying

$$R_{w_n} = |\pi \circ \tilde{f}^{n-1}((w, z'_n))|.$$

By the assumption in Theorem, it follows

$$\begin{aligned} |\pi \circ \tilde{f}^{n}((w, z'_n))| &= |f_{w_n} \circ \pi \circ \tilde{f}^{n-1}((w, z'_n))| \\ &\geq L(R_{w_n}, f_{w_n}) \\ &> (L+1)|\pi \circ \tilde{f}^{n-1}((w, z'_n))|. \end{aligned}$$

This is a contradiction to (12), because (w, z'_n) satisfying (12). Hence (4) holds.

From (4), there exists a constant $M > 0$ such that

$$|\pi \circ \tilde{f}^{n}((w, a))| < M < \infty, n = 1, 2, \cdots. \tag{13}$$

For any $K > 1$ and all sufficiently large $R > 0$, by the assumption of Theorem, there is $R_1 \leq R$ such that

$$L(R_1, \pi \circ \tilde{f}) > KR.$$

Make a Jordan curve γ in U connecting a to a point in $U \cap \{z : |z| = R\}$. Then

$$\pi \circ \tilde{f}(\{w\} \times \gamma) \cap \{z : |z| = KR\} \neq \emptyset$$

and

$$\pi \circ \tilde{f}(\{w\} \times \gamma) \cap \{z : |z| \leq M\} \neq \emptyset.$$

So there exists a point $(w, z_1) \in \{w\} \times \gamma$ satisfying

$$|\pi \circ \tilde{f}((w, z_1))| = KR.$$

By the assumption in Theorem, there is $R_2 \leq KR$ such that

$$L(R_2, f_{w_2}) > K^2 R, \ |z| = R_2.$$

Take $\gamma_1 \subset \pi \circ \tilde{f}(\{w\} \times \gamma)$, such that γ_1 connects $\pi \circ \tilde{f}((w, a))$ to a point in $U \cap \{z : |z| = KR\}$ and $\gamma_1 \subset \{z \in \mathbb{C} : |z| \leq KR\}$. Then

$$\pi \circ \tilde{f}(\{\sigma w\} \times \gamma_1) \cap \{z : |z| = K^2 R\} \neq \emptyset$$

and

$$\pi \circ \tilde{f}(\{\sigma w\} \times \gamma_1) \cap \{z : |z| \leq M\} \neq \emptyset.$$

Hence

$$\pi \circ \tilde{f}^2(\{w\} \times \gamma) \cap \{z : |z| = K^2R\} \neq \emptyset$$

and

$$\pi \circ \tilde{f}^2(\{w\} \times \gamma) \cap \{z : |z| \leq M\} \neq \emptyset.$$

There exists a point $(w, z_2) \in \{w\} \times \gamma$ satisfying

$$|\pi \circ \tilde{f}^2((w, z_2))| = K^2R.$$

Inductively, for all sufficiently large n, there exist $R_n \leq K^{n-1}R$ and a point $(w, z_n) \in \{w\} \times \gamma$ such that

$$L(R_n, f_{w_n}) > K^nR$$

and

$$|\pi \circ \tilde{f}^n((w, z_n))| = K^nR. \tag{14}$$

It remains to be considered two cases below.

Case 1. $\{\pi \circ \tilde{f}^n\}$ has only constant limit functions on $\{w\} \times U$. We can choose au unbounded connected set Γ of $\pi(J(\tilde{f}))$ such that

$$\pi \circ \tilde{f}^n((w, z)) \to q \notin \Gamma, \forall (w, z) \in \{w\} \times U, n \to \infty.$$

Then $C \setminus \Gamma$ is simple connected and

$$\pi \circ \tilde{f}^n(\{w\} \times U) \subset C \setminus \Gamma, n = 1, 2, \cdots.$$

For $a_0 \in \Gamma$ and any $z \in C \setminus \Gamma$, then

$$\lambda_{C \setminus \Gamma}(z) \geq \frac{1}{4d(z, \Gamma)} \geq \frac{1}{4(|z| + |a_0|)}.$$

Similarly, from the above proof, there is a constant A, by Contraction Principle, if follows that

$$\rho_{C \setminus \Gamma}(\pi \circ \tilde{f}^n((w, a)), \pi \circ \tilde{f}^n((w, z_n))) \leq \rho_U(a, z_n) < A.$$

So, from (13), it follows

$$|\pi \circ \tilde{f}^n((w, z_n))| \leq e^{4A}(|\pi \circ \tilde{f}^n((w, a))| + |a_0|) < e^{4A}(M + |a_0|).$$

It leads to a contradiction if we let $n \to \infty$ in the following

$$K^nR < e^{4A}(M + |a_0|) < \infty.$$

Case 2. $\{¼ \circ \tilde{f}^n\}$ has nonconstant limit function in $\{w\} \times U$. By Lemma 3, there exists a subsequence $\{¼ \circ \tilde{f}^{n_k}\}$ such that

$$¼ \circ \tilde{f}^{n_k}((w; z)) \to z; \forall (w; z) \in \{w\} \times U; k \to \infty:$$

Therefore, $\forall (w; z) \in \{w\} \times U$, for all sufficiently large k, it gets

$$|¼ \circ \tilde{f}^{n_k}((w; z))| < K|z|: \tag{15}$$

On the other hand, for all sufficiently large k, there is $(w; z) = (w; z_{n_k}) \in \{w\} \times °$ satisfying (14). And then

$$|¼ \circ \tilde{f}^{n_k}((w; z_{n_k}))| = K^{n_k} R:$$

This contradicts to (15).

Hence in any case, $V \times U$ is bounded. We complete the proof.

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