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SOLVABILITY OF A NONLINEAR FOUR–POINT BOUNDARY VALUE PROBLEM FOR A FOURTH ORDER DIFFERENTIAL EQUATION*

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Abstract. The authors consider four-point boundary value problem for a fourth order ordinary differential equations of the form

$$(E) \qquad (\phi_p(u''(t)))'' = a(t)f(u(t)), \quad t \in (0,1),$$

with one of the following boundary conditions

$$(B_1) \quad u(0) - \lambda u'(\eta) = u'(1) = 0, \quad u'''(0) = \alpha_1 u'''(\xi), \quad u''(1) = \beta_1 u''(\xi),$$

or

$$(B_2) \quad u(1) + \lambda u'(\eta) = u'(0) = 0, \quad u'''(0) = \alpha_1 u'''(\xi), \quad u''(1) = \beta_1 u''(\xi).$$

They impose growth conditions on f which guarantee existence of at least two positive solutions for the problems $(E) - (B_1)$ and $(E) - (B_2)$.

1. Introduction

In this paper, we are concerned with the existence of twin positive solutions for the fourth-order boundary value problems (BVP for short) consisting of the p-Laplacian differential equation

$$(1) \qquad (\phi_p(u''(t)))'' - a(t)f(u(t)) = 0, \quad t \in (0,1),$$

and one of the following boundary conditions

$$(2) \quad u(0) - \lambda u'(\eta) = u'(1) = 0, \quad u'''(0) = \alpha_1 u'''(\xi), \quad u''(1) = \beta_1 u''(\xi),$$

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or

(3) $\quad u(1)+\lambda u'(\eta)=u'(0)=0, \quad u'''(0)=\alpha_1 u'''(\xi), \quad u''(1)=\beta_1 u''(\xi),$

where $f: R \to [0,+\infty)$, $a:(0,1) \to [0,+\infty)$ are continuous functions. $0<\xi$, $\eta<1$, $\lambda \geq 0$, $p>1$, $\phi_p(z)=|z|^{p-2}z$, $\alpha_1<1$, $\beta_1 \in [0,1)$.

Two-point boundary value problems for differential equations are used to describe a number of physical, biological, and chemical phenomena. And for additional background and results, we refer the reader to the monograph by *Agawarl, O'Regan and Wong* [1] as well as the recent contributions by [2-5].

Recently, three point boundary value problems of the differential equations were presented and studied by many authors, see [6-9] and the references therein. Three point boundary value problems (1) – (2) or (1) – (3) have not received as much attention in the literature as Lidstone condition boundary value problem

$$\left\{\begin{array}{l} u''''(t)=a(t)f(u(t)), \quad t\in(0,1), \\ u(0)=u(1)=u''(0)=u''(1)=0, \end{array}\right. \tag{4}$$

and as second order three point boundary value problem

$$\left\{\begin{array}{l} u''(t)+a(t)f(u(t))=0, \quad t\in(0,1), \\ u(0)=0, \quad u(1)=\alpha u(\eta), \end{array}\right. \tag{5}$$

that were extensively considered, for example, in [3-5] and [7-10] respectively. The results of existence of positive solutions of BVP (1) – (2) and (1) – (3) are relatively scarce.

The motivation for the present work also originates from many recent investigations. Recently, there is an increasing interest in obtaining twin positive solutions for two point boundary value problems for both ordinary differential equations and finite difference equations, for more details, we refer the reader to [11-15]. To the best of our knowlege, existence results of multiple positive solutions for three point BVP have not been found in literature. The purpose of this paper is to establish the existence of at least two positive solutions of (1) – (2) and (1) – (3). Our arguments involve the use of the concavity and integral representation of solutions and the Avery-Henderson fixed point theorem, Theorem AH. We will impose growth conditions on f which ensure the existence of at least two positive solutions of $(1)-(2)$ and (1) – (3).

For the remainder of the paper, we assume that

(i) $0<\int_0^1 a(s)ds<+\infty$;

(ii) q satisfies $\frac{1}{p}+\frac{1}{q}=1$, and $\phi_q(z)=|z|^{q-2}z$.

We also present some background materials from the theory of cones in Banach spaces.

Definition 1. Let X be a real Banach space, a non-empty closed convex set $P \subset X$ is called a cone of X if it satisfies the following conditions:

(i) $x \in P$ and $\lambda \geq 0$ implies $\lambda x \in P$,

(ii) $x \in P$ and $-x \in P$ implies $x = 0$.

Every cone $P \subset X$ includes an ordering in X which is given by $x \leq y$ if and only if $y - x \in P$.

Definition 2. A map $\psi : P \to [0, +\infty)$ is called nonnegative continuous concave functional provided ψ is nonnegative, continuous and satisfies

$$\psi(tx + (1-t)y) \geq t\psi(x) + (1-t)\psi(y),$$

for all $x, y \in P$ and $t \in [0,1]$. Similarly, we say the map β is a nonnegative continuous convex functional on a cone P of X if

$$\beta : \quad P \to [0, +\infty)$$

is continuous and

$$\beta(tx + (1-t)y) \leq t\beta(x) + (1-t)\beta(y),$$

for all $x, y \in P$ and $t \in [0,1]$.

Definition 3. An operator is called completely continuous if it is continuous and maps bounded sets into pre-compact sets.

Let

$$P(\psi, d) = \{x \in P : \psi(x) < d\}, \quad \partial P(\psi, d) = \{x \in P : \psi(x) = d\},$$
$$\overline{P(\psi, d)} = \{x \in P : \psi(x) \leq d\}.$$

Theorem AH[15]**.** *Suppose X is a real Banach space, and P is a cone of X, γ, α are two nonnegative increasing continuous functionals, θ is a nonnegative continuous functional and $\theta(0) = 0$. There are positive numbers c and M such that*

$$\gamma(x) \leq \theta(x) \leq \alpha(x), \quad ||x|| \leq M\gamma(x) \quad for \quad x \in \overline{P(\gamma, c)}.$$

Again, assume $T : \overline{P(\gamma, c)} \to P$ is completely continuous, and there are positive numbers $0 < a < b < c$ such that

$$\theta(\mu x) \leq \mu\theta(x), \quad \mu \in (0,1], \quad x \in \partial P(\theta, b),$$

and

(i) $\gamma(Tx) > c$ *for* $x \in \partial P(\gamma, c)$;

(ii) $\theta(Tx) < b$ *for* $x \in \partial P(\theta, b)$;

(iii) $\alpha(Tx) > a$ *and* $P(\alpha, a) \neq \emptyset$ *for* $x \in \partial P(\alpha, a)$.

Then T has at least two fixed points $x_1, x_2 \in \overline{P(\gamma, c)}$ that satisfy

$$a < \alpha(x_1), \quad \theta(x_1) < b,$$

and

$$b < \theta(x_2), \quad \gamma(x_2) < c.$$

In section 2, we impose growth conditions on f which allow us to apply theorem AH in obtaining two positive solutions of BVP (1) – (2) and (1) – (3).

2. Twin Positive Solutions of (1) – (2) or (1) – (3)

In this section, we impose growth conditions on f and apply theorem AH to establish the existence of twin positive solutions (1) – (2) and (1) – (3).

In order to apply theorem AH, we must define an appropriate operator on a Banach space. To that end we first note that if $f \in C(R, R)$, then the unique solution of the second order boundary value problem

$$-y'' = f(t), \quad y'(0) = \alpha_1 y'(\xi), \quad y(1) = \beta_1 y(\xi), \tag{6}$$

is

$$y(t) = \int_0^1 G(t,s) f(s) ds, \quad t \in [0,1],$$

where $M = (1-\alpha_1)(1-\beta_1) \neq 0$, and

$$G(t,s) = \frac{1}{M} \begin{cases} 1 - \beta_1\xi - t + \beta_1 t, \quad 0 \le s \le t < \xi < 1 \;\text{ or }\; 0 \le s \le \xi \le t \le 1, \\ 1 - \beta_1\xi + (1-\beta_1)(\alpha_1 s - s - \alpha_1 t), \\ \qquad\qquad\qquad\qquad 0 \le t \le s \le \xi < 1, \\ 1 - \alpha_1 - \beta_1 s + \alpha_1\beta_1 s - t + \alpha_1 t + \beta_1 t - \alpha_1\beta_1 t, \\ \qquad\qquad\qquad\qquad 0 \le \xi \le s \le t \le 1, \\ (1-s)(1-\alpha_1), \quad 0 < \xi \le t \le s \le 1 \;\text{ or }\; 0 \le t < \xi \le s \le 1. \end{cases}$$

In fact, if $y(t)$ is a solution of (6), then we suppose that

$$y(t) = -\int_0^t (t-s) f(s) ds + At + B, \quad t \in [0,1],$$

by the boundary condition (6), we get

$$A = -\alpha_1 \int_0^{\xi} f(s)ds + \alpha_1 A,$$

and

$$-\int_0^1 (1-s)f(s)ds + A + B = -\beta_1 \int_0^{\xi} (\xi - s)f(s)ds + \beta_1 \xi A + \beta_1 B,$$

hence

$$\begin{aligned} y(t) &= -\int_0^t (t-s)f(s)ds - \frac{\alpha_1 t}{1-\alpha_1}\int_0^{\xi} f(s)ds + \frac{1}{1-\beta_1}\int_0^1 (1-s)f(s)ds \\ &\quad - \frac{1}{(1-\alpha_1)(1-\beta_1)}\int_0^{\xi} (\beta_1\xi - \beta_1 s + \alpha_1\beta_1 s - \alpha_1)f(s)ds \\ &= \int_0^1 G(t,s)f(s)ds. \end{aligned}$$

On the other hand, it is easy to know that if $\alpha_1 < 1$, $\beta_1 \in [0,1)$ then $G(t,s) \geq 0$ for $(t,s) \in [0,1] \times [0,1]$ (boundary conditions (6) was proposed in [9], however, the explicit formulation of Green's function had not been found).

If $u(t)$ is solution of problem (1) – (2), by (1) – (2), we then have

$$\phi_p(u''(t)) = -\int_0^1 G(t,s)a(s)f(u(s))ds.$$

Thus

$$u''(t) = -\phi_q\Big(\int_0^1 G(t,s)a(s)f(u(s))ds\Big). \tag{7}$$

Now, for boundary conditions (2), using (7), we have

$$-u'(t) = -\int_t^1 \phi_q\Big(\int_0^1 G(s,\tau)a(\tau)f(u(\tau))d\tau\Big)ds. \tag{8}$$

and

$$u(t) - u(0) = \int_0^t \int_s^1 \phi_q\Big(\int_0^1 G(r,\tau)a(\tau)f(u(\tau))d\tau\Big)drds, \tag{9}$$

Hence (8) – (9) imply

$$
\begin{aligned}
u(t) = &\lambda \int_\eta^1 \phi_q\Big(\int_0^1 G(s,r)a(r)f(u(r))dr\Big)ds \\
&+t\int_t^1 \phi_q\Big(\int_0^1 G(s,r)a(r)f(u(r))dr\Big)ds \\
&+\int_0^t s\phi_q\Big(\int_0^1 G(s,r)a(r)f(u(r))dr\Big)ds.
\end{aligned} \tag{10}
$$

Similarly, if $u(t)$ is a solution of (1)–(3), we have

$$
\begin{aligned}
u(t) = &\lambda \int_0^\eta \phi_q\Big(\int_0^1 G(s,r)a(r)f(u(r))dr\Big)ds \\
&+\int_0^1 \phi_q\Big(\int_0^1 G(s,r)a(r)f(u(r))dr\Big)ds \\
&-t\int_0^t \phi_q\Big(\int_0^1 G(s,r)a(r)f(u(r))dr\Big)ds \\
&-\int_t^1 s\phi_q\Big(\int_0^1 G(s,r)a(r)f(u(r))dr\Big)ds.
\end{aligned} \tag{11}
$$

Now, let the classical Banach space $X = C([0,1])$ be endowed with the norm $||x|| = \max_{0\le t\le 1} |x(t)|$, and choose the cone P_1, P_2 defined by

$$
\begin{aligned}
P_1 = \{\, u \in X :\ &u(t) \ge 0,\ u(t) \text{ is concave and increasing on } [0,1], \\
&u(0) - \lambda u'(\eta) = u'(1) = 0\},
\end{aligned} \tag{12}
$$

$$
\begin{aligned}
P_2 = \{\, u \in X :\ &u(t) \ge 0,\ u(t) \text{ is concave and decreasing on } [0,1], \\
&u(1) + \lambda u'(\eta) = u'(0) = 0\}.
\end{aligned} \tag{13}
$$

2.1. Twin Positive Solutions of (1) – (2)

We choose $r \in (\eta, 1)$ and define the nonnegative, increasing, continuous functionals γ, θ, and α by

$$
\gamma(u) = \min_{t\in[\eta,r]} u(t) = u(\eta),
$$

$$
\theta(u) = \max_{t\in[0,\eta]} u(t) = u(\eta),
$$

$$
\alpha(u) = \max_{t\in[0,r]} u(t) = u(r).
$$

We observe, for each $u \in P_1$, since $u''(t) \le 0$, one has

$$
\frac{u(\eta)-u(1)}{\eta-1} \le \frac{u(0)-u(1)}{0-1}, \quad \lambda\frac{u(\eta)-u(1)}{\eta-1} \le \lambda u'(\eta) = u(0).
$$

Then $u(\eta) \geq (1-\eta)u(0) + \eta u(1)$, and $u(0) \geq \frac{\lambda}{\eta-1}(u(\eta) - u(1))$, hence $u(\eta) \geq -\lambda(u(\eta) - u(1)) + \eta u(1)$. Thus

$$\gamma(u) = u(\eta) \geq \frac{\lambda+\eta}{\lambda+1}||u||,$$

and hence

$$||u|| \leq \frac{1+\lambda}{\eta+\lambda}\gamma(u), \quad u \in P_1.$$

Finally, we also note that

$$\theta(\mu u) = \mu\theta(u), \quad 0 \leq \mu \leq 1, \quad u \in P_1.$$

We now state the growth conditions on f so that (1) – (2) has at least two positive solutions.

Theorem 2.1. *Suppose that there are positive numbers* a, b, c such that

$$0 < a < \frac{L}{N}b < \frac{L}{N}\frac{\eta+\lambda}{1+\lambda}c,$$

and $f(w)$ *satisfies following conditions*:

$$f(w) > \phi_p\Big(\frac{c}{M}\Big), \quad c \leq w \leq \frac{1+\lambda}{\eta+\lambda}c; \tag{14}$$

$$f(w) < \phi_p\Big(\frac{b}{N}\Big), \quad 0 \leq w \leq \frac{1+\alpha}{\eta+\alpha}b; \tag{15}$$

$$f(w) > \phi_p\Big(\frac{a}{L}\Big), \quad 0 \leq w \leq a. \tag{16}$$

Then BVP (1)–(2) *has at least two positive solutions* u_1, u_2 *such that*

$$a < \max_{t\in[0,r]} u_1(t) = u_1(r) \quad with \quad u_1(\eta) = \max_{t\in[0,\eta]} u_1(t) < b, \tag{17}$$

$$b < \max_{t\in[0,\eta]} u_2(t) = u_2(\eta) \quad with \quad u_2(\eta) = \min_{t\in[\eta,r]} u_2(t) < c, \tag{18}$$

where L, M, N *are defined as follows*

$$
\begin{aligned}
M = &(\lambda+\eta)\int_\eta^1 \phi_q\Big(\int_\eta^1 G(s,r)a(r)f(u(r))dr\Big)ds \\
&+\int_0^\eta s\phi_q\Big(\int_\eta^1 G(s,r)a(r)f(u(r))dr\Big)ds, \\
N = &(\lambda+\eta)\int_\eta^1 \phi_q\Big(\int_0^1 G(s,r)a(r)f(u(r))dr\Big)ds \\
&+\int_0^\eta s\phi_q\Big(\int_0^1 G(s,r)a(r)f(u(r))dr\Big)ds, \\
L = &\lambda\int_\eta^1 \phi_q\Big(\int_0^r G(s,r)a(r)f(u(r))dr\Big)ds \\
&+r\int_r^1 \phi_q\Big(\int_0^r G(s,r)a(r)f(u(r))dr\Big)ds \\
&+\int_0^r s\phi_q\Big(\int_0^r G(s,r)a(r)f(u(r))dr\Big)ds.
\end{aligned}
$$

Proof. We begin by defining the completely continuous integral operator $T : P_1 \to X$ by

$$
\begin{aligned}
Tu(t) = &\lambda\int_\eta^1 \phi_q\Big(\int_0^1 G(s,r)a(r)f(u(r))dr\Big)ds \\
&+t\int_t^1 \phi_q\Big(\int_0^1 G(s,r)a(r)f(u(r))dr\Big)ds \\
&+\int_0^t s\phi_q\Big(\int_0^1 G(s,r)a(r)f(u(r))dr\Big)ds.
\end{aligned} \tag{19}
$$

It is well known that solutions of (1) – (2) are fixed points of T and conversely. We proceed to show that the conditions of theorem AH are satisfied.

Firstly, let $u \in \overline{P_1(\gamma,c)}$. By the nonnegativity of f and G, for $t \in [0,1]$, we have $Tu(t) \geq 0$. In addition, $(Tu)''(t) = -\phi_q(\int_0^1 G(t,s)a(s)f(u(s))ds \leq 0$, $Tu(0) - \lambda(Tu)'(\eta) = 0, (Tu)'(1) = 0$. Consequently, $Tu \in P_1$, and conclude $T : \overline{P_1(\gamma,c)} \to P_1$.

We now turn to property (i) of theorem AH. Choose $u \in \partial P_1(\gamma,c)$, then $\gamma(u) = u(\eta) = c$. Since $u \in P_1, u(t) \geq c$ for $t \in [\eta,1]$, we recall that $||u|| \leq \frac{1+\lambda}{\eta+\lambda}\gamma(u) = \frac{1+\lambda}{\eta+\lambda}c$, we have

$$c \leq u(t) \leq \frac{1+\lambda}{\eta+\lambda}c, \quad \eta \leq t \leq 1,$$

and as a consequence of assumption (14),

$$f(u(t)) > \phi_p\Big(\frac{c}{M}\Big), \quad \eta \le t \le 1.$$

Also, $Tu \in P_1$, and so

$$\begin{aligned}
\gamma(Tu) &= Tu(\eta) \\
&= \lambda \int_\eta^1 \phi_q\Big(\int_0^1 G(s,r)a(r)f(u(r))dr\Big)ds \\
&\quad +\eta \int_\eta^1 \phi_q\Big(\int_0^1 G(s,r)a(r)f(u(r))dr\Big)ds \\
&\quad + \int_0^\eta s\phi_q\Big(\int_0^1 G(s,r)a(r)f(u(r))dr\Big)ds \\
&\ge (\lambda+\eta)\int_\eta^1 \phi_q\Big(\int_\eta^1 G(s,r)a(r)f(u(r))dr\Big)ds \\
&\quad + \int_0^\eta s\phi_q\Big(\int_\eta^1 G(s,r)a(r)f(u(r))dr\Big)ds \\
&> (\lambda+\eta)\int_\eta^1 \phi_q\Big(\int_\eta^1 G(s,r)a(r)\phi_p\Big(\frac{c}{M}\Big)dr\Big)ds \\
&\quad + \int_0^\eta s\phi_q\Big(\int_\eta^1 G(s,r)a(r)\phi_p\Big(\frac{c}{M}\Big)dr\Big)ds \\
&= (\lambda+\eta)\frac{c}{M}\Big[\int_\eta^1 \phi_q\Big(\int_\eta^1 G(s,r)a(r)dr\Big)ds \\
&\quad + \int_0^\eta s\phi_q\Big(\int_\eta^1 G(s,r)a(r)dr\Big)ds\Big] \\
&= c.
\end{aligned}$$

We conclude that (i) of Theorem AH is satisfied.

We next address (ii) of theorem AH. Let us choose $u \in \partial P_1(\theta, b)$, then $\theta(u) = u(\eta) = b$, and this implies $0 \le u(t) \le b$ for $0 \le t \le \eta$. And since $u \in P_1$, we also have $b \le u(t) \le ||u|| = u(1)$ for $t \in [\eta, 1]$. Moreover,

$$||u|| \le \frac{1+\lambda}{\eta+\lambda}\gamma(u) \le \frac{1+\lambda}{\eta+\lambda}\theta(u) = \frac{1+\lambda}{\eta+\lambda}b,$$

and so

$$0 \le u(t) \le \frac{1+\lambda}{\eta+\lambda} b, \quad 0 \le t \le 1.$$

From assumption (15), we get

$$f(u(t)) < \phi_p\Big(\frac{b}{N}\Big), \quad 0 \le t \le 1.$$

Similarly, $Tu \in P_1$ and so

$$\begin{aligned}
\theta(Tu) &= Tu(\eta) \\
&= (\lambda+\eta) \int_\eta^1 \phi_q\Big(\int_0^1 G(s,r)a(r)f(u(r))dr\Big)ds \\
&\quad + \int_0^\eta s\phi_q\Big(\int_\eta^1 G(s,r)a(r)f(u(r))dr\Big)ds \\
&< (\lambda+\eta) \int_\eta^1 \phi_q\Big(\int_0^1 G(s,r)a(r)\phi_p\Big(\frac{b}{N}\Big)dr\Big)ds \\
&\quad + \int_0^\eta s\phi_q\Big(\int_0^1 G(s,r)a(r)\phi_p\Big(\frac{b}{N}\Big)dr\Big)ds \\
&= (\lambda+\eta)\frac{b}{N}\Big[\int_\eta^1 \phi_q\Big(\int_0^1 G(s,r)a(r)dr\Big)ds \\
&\quad + \int_0^\eta s\phi_q\Big(\int_0^1 G(s,r)a(r)dr\Big)ds\Big] \\
&= b.
\end{aligned}$$

In particular, (ii) of Theorem AH holds.

For the final part, we turn to (iii) of Theorem AH. For this part, if we first define $u(t) = \frac{a}{2}$ for $t \in [0,1]$. Then $\alpha(u) = \frac{a}{2} < a$, and $P_1(\alpha, a) \neq \emptyset$.

Now, let us choose $u \in \partial P_1(\alpha, a)$. Then $\alpha(u) = u(r) = a$. This implies $0 \le u(t) \le a$, for $0 \le t \le r$. From the assumption (16),

$$f(u(t)) > \phi_p(\frac{a}{L}), \quad 0 \le t \le r.$$

As before, $Tu \in P_1$ and also

$$\begin{aligned}
\alpha(Tu) &= Tu(r)\\
&= \lambda \int_\eta^1 \phi_q\Big(\int_0^1 G(s,r)a(r)f(u(r))dr\Big)ds\\
&\quad +r\int_r^1 \phi_q\Big(\int_0^1 G(s,r)a(r)f(u(r))dr\Big)ds\\
&\quad +\int_0^r s\phi_q\Big(\int_0^1 G(s,r)a(r)f(u(r))dr\Big)ds\\
&\geq \lambda \int_\eta^1 \phi_q\Big(\int_0^r G(s,r)a(r)f(u(r))dr\Big)ds\\
&\quad +r\int_r^1 s\phi_q\Big(\int_0^r G(s,r)a(r)f(u(r))dr\Big)ds\\
&\quad +\int_0^r s\phi_q\Big(\int_0^r G(s,r)a(r)f(u(r))dr\Big)ds\Big]\\
&> \lambda \int_\eta^1 \phi_q\Big(\int_0^r G(s,r)a(r)\phi_p\Big(\frac{a}{L})dr\Big)ds\\
&\quad +r\int_r^1 \phi_q\Big(\int_0^r G(s,r)a(r)\phi_p\Big(\frac{a}{L}\Big)dr\Big)ds\\
&\quad +\int_0^r s\phi_q\Big(\int_0^r G(s,r)a(r)\phi_p\Big(\frac{a}{L}\Big)dr\Big)ds\\
&= \frac{a}{L}\Big[\lambda \int_\eta^1 \phi_q\Big(\int_0^r G(s,r)a(r)dr\Big)ds\\
&\quad +r\int_r^1 \phi_q\Big(\int_0^r G(s,r)a(r)dr\Big)ds\\
&\quad +\int_0^r s\phi_q\Big(\int_0^r G(s,r)a(r)dr\Big)ds\Big]\\
&= a.
\end{aligned}$$

Therefore, (iii) of Theorem AH is satisfied. Hence there exist at least two fixed points of T which are positive solutions u_1, u_2 of the boundary value problem (1) – (2) such that

$$a < \alpha(u_1) = u_1(r) \quad \text{with} \quad u_1(\eta) = \theta(u_1) < b,$$

$$b < \theta(u_2) = u_2(\eta) \quad \text{with} \quad u_2(\eta)\gamma(u_2) < c.$$

The proof is complete.

2.2. Twin Positive Solutions of (1) – (3)

Similar to that of the section 2.1, we choose $r \in (0, \eta)$ and define the nonnegative, increasing, continuous functionals γ, θ, and α by

$$\gamma(u) = \min_{t\in[\gamma,\eta]} u(t) = u(\eta),$$
$$\theta(u) = \max_{t\in[\eta,1]} u(t) = u(\eta),$$
$$\alpha(u) = \max_{t\in[r,1]} u(t) = u(r).$$

We observe, for each $u \in P_2$,

$$||u|| \le \frac{1+\lambda}{\lambda+1-\eta}\gamma(u), \quad u \in P_2,$$

since $u''(t) \le 0$. Finally, we also note that

$$\theta(\mu u) = \mu\theta(u), \quad 0 \le \mu \le 1, \quad u \in P_2.$$

We now state the growth conditions on f so that (1) – (3) has at least two positive solutions.

Theorem 2.2. *Suppose that there are positive numbers a, b, c such that*

$$0 < a < \frac{L_1}{N_1}b < \frac{L_1}{N_1}\frac{1+\alpha-\eta}{1+\alpha}c,$$

and $f(w)$ satisfies following conditions:

$$f(w) > \phi_p\Big(\frac{c}{M_1}\Big), \quad c \le w \le \frac{1+\lambda}{1-\eta+\lambda}c; \tag{20}$$

$$f(w) < \phi_p\Big(\frac{b}{N_1}\Big), \quad 0 \le w \le \frac{1+\lambda}{1-\eta+\lambda}b; \tag{21}$$

$$f(w) > \phi_p\Big(\frac{a}{L_1}\Big), \quad 0 \le w \le a. \tag{22}$$

Then BVP (1) – (3) has at least two positive solutions u_1, u_2 such that

$$a < \max_{t\in[r,1]} u_1(t) = u_1(r), \quad with \quad u_1(\eta) = \max_{t\in[\eta,1]} u_1(t) < b, \tag{23}$$

$$b < \max_{t\in[\eta,1]} u_2(t) = u_2(\eta), \quad with \quad u_2(\eta) = \min_{t\in[r,\eta]} u_2(t) < c, \tag{24}$$

where L_1, M_1, N_1 are defined as follows

$$
\begin{aligned}
M_1 = (\lambda+1-\eta) & \int_0^{\eta} \phi_q\Big(\int_0^1 G(s,r)a(r)f(u(r))dr\Big)ds \\
& + \int_{\eta}^1 (1-s)\phi_q\Big(\int_0^1 G(s,r)a(r)f(u(r))dr\Big)ds\Big], \\
N_1 = (\lambda+1-\eta) & \int_0^{\eta} \phi_q\Big(\int_0^{\eta} G(s,r)a(r)f(u(r))dr\Big)ds \\
& + \int_{\eta}^1 (1-s)\phi_q\Big(\int_0^{\eta} G(s,r)a(r)f(u(r))dr\Big)ds\Big], \\
L_1 = \lambda & \int_0^{\eta} \phi_q\Big(\int_r^1 G(s,r)a(r)f(u(r))dr\Big)ds \\
& +(1-r)\int_0^{r} \phi_q\Big(\int_r^1 G(s,r)a(r)f(u(r))dr\Big)ds \\
& + \int_r^1 (1-s)\phi_q\Big(\int_r^1 G(s,r)a(r)f(u(r))dr\Big)ds\Big].
\end{aligned}
$$

Proof. We define a completely continuous integral operator $T : P_2 \to X$ by

$$
\begin{aligned}
Tu(t) = \lambda & \int_0^{\eta} \phi_q\Big(\int_0^1 G(s,r)a(r)f(u(r))dr\Big)ds \\
& + \int_0^1 \phi_q\Big(\int_0^1 G(s,r)a(r)f(u(r))dr\Big)ds \\
& -t\int_0^t \phi_q\Big(\int_0^1 G(s,r)a(r)f(u(r))dr\Big)ds \\
& - \int_t^1 s\phi_q\Big(\int_0^1 G(s,r)a(r)f(u(r))dr\Big)ds.
\end{aligned}
$$

i.e.

$$
\begin{aligned}
Tu(t) = \lambda & \int_0^{\eta} \phi_q\Big(\int_0^1 G(s,r)a(r)f(u(r))dr\Big)ds \\
& + \int_0^t (1-t)\phi_q\Big(\int_0^1 G(s,r)a(r)f(u(r))dr\Big)ds \\
& + \int_t^1 (1-s)\phi_q\Big(\int_0^1 G(s,r)a(r)f(u(r))dr\Big)ds.
\end{aligned}
\tag{25}
$$

The remainder of the proof is similar to that of Theorem 2.1 and hence omitted.

References

1. R. P. Agarwal, D. O'Regan, and P. J. Y. Wong, *Positive solutions of differential, difference, and integral equations*, Kluwer Academic, Dordrecht, 1999.
2. R. Y. Ma, H. Y. Wang, On the existence of positive solutions of fourth-order ordinary differential equations, *Appl. Anal.* **59** (1995), 225-231.
3. J. R. Graef, B. Yang, On a nonlinear boundary-value problem for fourth-order equations, *Appl. Anal.* **72** (1999), 439-448.
4. J. R. Graef, B. Yang, Existence and non-existence of positive solutions of fourth-order nonlinear boundary-value problems, *Appl. Anal.* **74** (2000), 201-204.
5. B. G. Zhang, L. Z. Kong, Multiple positive solutions of a class of P-Laplacian equations (in Chinese), *Annals Math.* **2A**(4) (2001), 1-10.
6. P. W. Eloe, J. Henderson, Positive solutions and nonlinear $(k, n-k)$ conjugate eigenvalue problem, *Diff. Equ. Dynam. Syst.* **6** (1998), 309-317.
7. R. Y. Ma, Positive solutions of nonlinear three point boundary value problem, *Electronic J. Diff. Equs.* **34** (1998), 1-8.
8. R. Y. Ma, Existence theorems for a second order three point boundary value problems, *J. Math. Anal. Appl.* **212** (1997), 430-442.
9. Y. R. Ma, N.Castaneta, Existence of solutions of nonlinear m-point boundary value problem, *J. Math. Anal. Appl.* **256** 2001, 556-567.
10. W. Feng, J. R. L. Webb, Solvability of a three point nonlinear boundary value problems at resonance, *Nonlinear Anal. TMA*, **30**(6) (1997), 3227-3238.
11. C. P. Gupta, Solvability of a three point nonlinear boundary value problem for a second order ordinary differential equation, *J. Math. Anal. Appl.* **168** (1992), 540-551.
12. C. P. Gupta, S. Trofimchuk, Existence of a solution to a three point boundary value problem and the spectral radius of a related linear operator, *Nonlinear Anal. TMA*, **34** (1998), 498-507.
13. P. W. Eloe, J. Henderson, and E. Kaufmann, Multiple positive solutions for difference equations, *J. Diff. Equs. Appl.* **3** (1998), 219-229.
14. F. Merdivence, Two positive solutions for boundary value problem for difference equations, *J. Diff. Equs. Appl.* **1** (1995), 262-270.
15. R. I. Avery, C. J. Chyan, Twin solutions of boundary value problems for ordinary differential equations and finite difference equations, *Comput. Math. Appl.* **42** (2001), 695-704.

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