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ON INVARIANT SUBSPACES FOR POWER-BOUNDED OPERATORS OF CLASS $C_{1\cdot}$

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Abstract. We prove that if T is a power-bounded operator of class $C_{*\cdot}$ on a Hilbert space which commutes with a nonzero quasinilpotent operator, then T has a nontrivial invariant subspace. Connections with the questions of convergence of T^n to 0 in the strong operator topology and of cyclicity of power-bounded operators of class $C_{1\cdot}$ are discussed.

A linear operator T on a Hilbert space $\mathcal{H}$ is called *power-bounded* if $\sup_{n\geq 0}\|T^n\| < \infty$. A power-bounded operator T is said to be of *class* $C_{*\cdot}$ if there exists a nonzero vector $x \in \mathcal{H}$ such that the sequence $\{\|T^n x\|\}_n$ does not converge to 0, and T is of *class* $C_{1\cdot}$ if $\{\|T^n x\|\}_n$ does not converge to 0 for every nonzero vector x. It is still an unsolved problem whether every power-bounded operator of class $C_{*\cdot}$ (in particular, $C_{*\cdot}$-contraction) has a nontrivial invariant subspace, i.e., whether there exists a (closed) subspace $\mathcal{M}$ of $\mathcal{H}$ such that $\{0\} \neq \mathcal{M} \neq \mathcal{H}$ and $T\mathcal{M} \subset \mathcal{M}$. For partial results on that problem, see, e.g., [1, Chapter XII] or [16]. In this note we prove the following theorem.

Theorem 1. *Assume that T is a power-bounded operator of class $C_{*\cdot}$ on a Hilbert space $\mathcal{H}$, which commutes with a nonzero quasinilpotent operator. Then T has a nontrivial invariant subspace.*

This theorem will follow from the following one. We recall that the operator T is called *cyclic* if it has a cyclic vector, that is, a vector x such that the sequence $\{T^n x\}_{n\geq 0}$ spans the whole space $\mathcal{H}$.

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Theorem 2. *If T is a power-bounded operator of class $C_1.$ on a Hilbert space $\mathcal{H}$ such that T commutes with a nonzero quasinilpotent operator, then T is not cyclic.*

The proof is based on the following construction of the limit isometric operator associated with T (see [11] and [21]).

Given a power-bounded operator T acting on the Hilbert space $\mathcal{H}$, fix a generalized Banach limit glim on $\ell^\infty(\mathbf{N})$ and consider the sesquilinear form w_T on $\mathcal{H}$ defined by

$$w_T(x,y) := \operatorname{glim}_{n\to\infty}\langle T^n x, T^n y\rangle, \quad x, y \in \mathcal{H}.$$

Since $\{\|T^n\|\}_n$ is bounded, it is easy to see that $\operatorname{glim}_{n\to\infty}\|T^n x\| = 0$ if and only if $\inf_{n\geq 0}\|T^n x\| = 0$, and this happens if and only if $\lim_{n\to\infty}\|T^n x\| = 0$. Let $\mathcal{H}_0(T)$ be the kernel of w_T, i.e.,

$$\mathcal{H}_0(T) := \{x \in \mathcal{H} : w_T(x,x) = 0\} = \{x \in \mathcal{H} : \lim_{n\to\infty}\|T^n x\| = 0\}.$$

Clearly, $\mathcal{H}_0(T)$ is a subspace which is invariant for any operator A in the commutant $\{T\}'$ of the operator T. Furthermore, $\mathcal{H}_0(T) \neq \mathcal{H}$ if and only if T is of class $C_{*}.$, and $\mathcal{H}_0(T) = \{0\}$ if and only if T is of class $C_1.$. Thus, Theorem 1 is an immediate consequence of Theorem 2.

Let us form the quotient space $\widehat{\mathcal{H}}_T = \mathcal{H}/\mathcal{H}_0(T)$, and let us consider the canonical mapping $\pi_T\colon \mathcal{H} \to \widehat{\mathcal{H}}_T$, $\pi_T(x) := x + \mathcal{H}_0(T) =: \hat{x}$. The sesquilinear form $\widehat{w}_T(\hat{x},\hat{y}) := w_T(x,y)$ $(x, y \in \mathcal{H})$ provides an inner product on $\widehat{\mathcal{H}}_T$, so that $\widehat{\mathcal{H}}_T$ is a pre-Hilbert space. Let $\widehat{T}$ be the operator on $\widehat{\mathcal{H}}_T$ which is defined by $\widehat{T}\hat{x} := \widehat{Tx}$. It is easy to see that $\widehat{T}$ is an isometry.

Let $\mathcal{H}_T$ be the completion of $\widehat{\mathcal{H}}_T$ and let V_T be the continuous extension of $\widehat{T}$, called the *isometric asymptote* of T in [11]. Any operator $A \in \{T\}'$ generates an operator $\widehat{A}$ on $\mathcal{H}_T$ by $\widehat{A}\hat{x} := \widehat{Ax}$ $(x \in \mathcal{H})$ (and by continuous extension from $\widehat{\mathcal{H}}_T$ to $\mathcal{H}_T$). The mapping $\gamma_T\colon A \mapsto \widehat{A}$ is a contractive algebra-homomorphism from the commutant $\{T\}'$ of T into the commutant $\{V_T\}'$ of the isometry V_T. Since γ_T is a unital algebra-homomorphism, we obtain the spectral inclusion $\sigma(\widehat{A}) \subset \sigma(A)$ $(A \in \{T\}')$. It follows that if A is quasinilpotent then so is $\widehat{A}$. It is also clear that $\widehat{A} = 0$ holds if and only if $\operatorname{ran} A \subset \mathcal{H}_0(T)$.

For a bounded linear operator V on a Hilbert space $\mathcal{K}$, let $\{V\}''$ denote the bicommutant of V. Let $\mathcal{R}(V)$ be the set of operators $f(V)$, where f runs through the set of rational functions with poles off the spectrum $\sigma(V)$, and let $\mathcal{A}(V)$ be the closure of $\mathcal{R}(V)$ in the weak operator topology. We will need the following well-known facts on these algebras.

Lemma 3. *If V is an isometry on a Hilbert space $\mathcal{K}$, then the abelian Banach algebra $\{V\}''$ is semisimple, and $\{V\}'' = \mathcal{A}(V)$.*

Proof. For the sake of completeness, we sketch the proof. The Hilbert space isometry V splits into the orthogonal sum $V = V_a \oplus U_s$, where V_a is an absolutely continuous isometry and U_s is a singular unitary operator. It is known that $\{V\}'' = \{V_a\}'' \oplus \{U_s\}''$ and $\mathcal{A}(V) = \mathcal{A}(V_a) \oplus \mathcal{A}(U_s)$; see [5] and Rudin's theorem in [8, Chapter 6]. Let μ and μ_s denote the normalized Lebesgue measure and the scalar spectral measure of U_s, respectively, on the unit circle $\mathbf{T}$, and let H^∞ be the Hardy subspace of $L^\infty(\mu)$. It can be easily verified that $\{V_a\}'' = \{\varphi(V_a) : \varphi \in H^\infty\}$ if V_a is nonunitary, $\{V_a\}'' = \{\varphi(V_a) : \varphi \in L^\infty(\mu)\}$ if V_a is unitary, and $\{U_s\}'' = \{\psi(U_s) : \psi \in L^\infty(\mu_s)\}$; see [4, Chapter IX]. Classical approximation theorems yield that $\{V\}'' = \mathcal{A}(V)$. On the other hand, the previous representation shows that every operator $A \in \{V\}''$ is subnormal, and so $\|A\|$ is equal to the spectral radius $r(A)$, which means that $\{V\}''$ does not contain nonzero quasinilpotent operators (or equivalently, the Gelfand transformation associated with $\{V\}''$ is injective). ■

Lemma 4. *The isometry V acting on the Hilbert space $\mathcal{K}$ is cyclic if and only if its commutant is abelian, that is, $\{V\}' = \{V\}''$.*

Proof. Considering the former decomposition $V = V_a \oplus U_s$, we obtain that V is cyclic if and only if both V_a and U_s are cyclic. Let us recall that a unitary operator U is cyclic if and only if U is $*$-cyclic, which means that the set $\{U^n x\}_{n=-\infty}^{\infty}$ spans the whole space with a suitable vector x; see [3]. Now, the results in [4, Chapter IX] imply the statement. ■

Proof of Theorem 2. Let us suppose that T has a cyclic vector x. Since $\|\hat{y}\| \le M\|y\|$ holds for every $y \in \mathcal{H}$, where $M = \sup\{\|T^n\|\}_{n=0}^{\infty}$, the vector $\hat{x}$ is cyclic for the limit isometry V_T. Let A be the nonzero quasinilpotent operator that commutes with T. Then $\widehat{A} = \gamma_T(A)$ commutes with V_T, hence we infer by Lemma 4 that $\widehat{A} \in \{V_T\}''$. Since $\{V_T\}''$ is semisimple by Lemma 3, we have $\widehat{A} = 0$, and so $\operatorname{ran} A \subset \mathcal{H}_0(T) = \{0\}$. Thus $A = 0$, which is a contradiction. ■

Applying the Riesz–Dunford functional calculus, Theorem 1 can be easily extended to the following statement.

Corollary 5. *Let T be a power-bounded operator of class $C_{*\cdot}$ on the Hilbert space $\mathcal{H}$. If T commutes with a nonscalar operator A having an isolated spectrum point, then T has a nontrivial invariant subspace. In particular, T has a nontrivial invariant subspace if T commutes with a nonzero, essentially quasinilpotent operator A.*

The following proposition shows how the statement of Lemma 4 can be transferred to power-bounded operators.

Proposition 6. *Let T be a power-bounded operator of class $C_{1\cdot}$ on the Hilbert space $\mathcal{H}$, and let us consider the conditions*: (a) *T is cyclic*, (b) *V_T is cyclic*, (c) $\{T\}' = \{T\}''$.

Then (a)$\Longrightarrow$(b)$\Longrightarrow$(c), *but the reverse implications are false.*

Proof. We have already seen that (a) implies (b). If V_T is cyclic then $\{V_T\}'$ is abelian by Lemma 4, which implies that $\{T\}'$ is also abelian since the mapping γ_T is one-to-one.

In [20], in terms of the Sz.-Nagy–Foias functional model of contractions, examples are given for the case when V_T is cyclic but T is noncyclic.

To show that (c) does not imply (b), let us consider the simply connected domains $\Omega_+ := \{z \in \mathbf{D} : \ \mathrm{Re}\, z > -1/2\}$ and $\Omega_- := \{z \in \mathbf{D} : \ \mathrm{Re}\, z < 1/2\}$, where $\mathbf{D}$ stands for the open unit disc. Let φ and ψ be conformal mappings of $\mathbf{D}$ onto Ω_+ and onto Ω_-, respectively. Let T_φ and T_ψ be the analytic Toeplitz operators with symbols φ and ψ, respectively, on the Hardy space H^2, that is, $T_\varphi f := \varphi f$, $T_\psi f := \psi f$ $(f \in H^2)$. We know by [18, Proposition 2] that φ and ψ are (sequential) weak-$*$ generators of the algebra H^∞, and so the operators T_φ and T_ψ have the same invariant subspaces as the operator T_χ, where $\chi(z) = z$. Since T_χ is cyclic, it follows that the operators T_φ and T_ψ are cyclic, as well.

It is clear that T_φ and T_ψ are contractions of class $C_{1\cdot}$. Furthermore, V_{T_φ} and V_{T_ψ} are unitarily equivalent to the restrictions $M_\alpha := M|\chi_\alpha L^2(\mu)$ and $M_\beta := M|\chi_\beta L^2(\mu)$, respectively, where $Mf := \chi f$ $(f \in L^2(\mu))$, $\alpha := (\Omega_+)^- \cap \mathbf{T}$ and $\beta := (\Omega_-)^- \cap \mathbf{T}$.

Let us form the orthogonal sum $T := T_\varphi \oplus T_\psi$. Since V_T is unitarily equivalent to $M_\alpha \oplus M_\beta$ and $\mu(\alpha \cap \beta) > 0$, we obtain that V_T is noncyclic. On the other hand, the conditions $\mu(\beta \setminus \alpha) > 0$ and $\mu(\alpha \setminus \beta) > 0$ imply by [6] that $\{T\}' = \{T_\varphi\}' \oplus \{T_\psi\}'$; see also [16, Theorem 18 and Corollary 15]. Taking into account that T_φ and T_ψ are cyclic, we infer that $\{T\}'$ is a semisimple abelian Banach algebra. ∎

The following example shows that Lemma 3 cannot be generalized to power-bounded operators.

Example 7. We recall that the power-bounded operator T is called of *class* C_{11} if both T and its adjoint T^* are of class $C_{1\cdot}$. The invariant subspace $\mathcal{M}$ is called *quasi-reducing* if the restriction $T|\mathcal{M}$ is of class C_{11}.

Let T be a cyclic, completely non-unitary contraction of class C_{11} on the Hilbert space $\mathcal{H}$ such that the spectrum of T is the closed unit disc $\mathbf{D}^-$, and

V_T is a cyclic bilateral shift. The existence of such operators follows from [2, Theorem 2]. For a concrete example we refer to [10, Example 12].

The lattice of the quasi-reducing invariant subspaces of T is isomorphic to the lattice of the spectral subspaces of V_T; see [9, Theorem 15] and [12, Theorem 3]. Thus, we have an abundance of quasi-reducing subspaces of T. These subspaces are exactly those which can be written in the form $(\text{ran } A)^-$, where $A \in \{T\}''$; see [9, Remark 5 and Proposition 10]. Hence, there are many nonzero operators in $\{T\}''$ which have nondense range.

On the other hand, since $\sigma(T) = \mathbf{D}^-$ and V_T is a bilateral shift, we infer by Runge's theorem and by [13, Corollary 2] that $\mathcal{A}(T) = H^\infty(T) := \{u(T) : u \in H^\infty\}$. However, for any nonzero function $u \in H^\infty$, the operator $u(T)$ is quasisimilar to $u(V_T)$ (see, e.g., [12] and [19]), and so $u(T)$ has dense range. Therefore, $\mathcal{A}(T)$ is a proper subset of $\{T\}''$.

Let T be a power-bounded operator of class $C_{1\cdot}$ on the Hilbert space $\mathcal{H}$. Let $\mathcal{A}_0(T)$ denote the norm-closure of the set $\mathcal{R}(T)$. The norm-continuity of γ_T and the condition $\sigma(T) \supset \sigma(V_T)$ imply that $\gamma_T(\mathcal{A}_0(T)) \subset \mathcal{A}_0(V_T)$. Since $\mathcal{A}_0(V_T) \subset \mathcal{A}(V_T) = \{V_T\}''$ and $\{V_T\}''$ is semisimple, we may infer (as in the proof of Theorem 2) that $\mathcal{A}_0(T)$ is semisimple. This statement was previously proved in [17].

If $\gamma_T(\{T\}'') \subset \{V_T\}''$ holds, then it follows in the same way that $\{T\}''$ is semisimple. However, a look at the operator $T = T_\varphi \oplus T_\psi$ occurring in the proof of Proposition 6 shows that the inclusion $\gamma_T(\{T\}'') \subset \{V_T\}''$ does not hold in general. Indeed, the operator $I \oplus 0$ belongs to $\{T\}''$, but $\gamma_T(I \oplus 0) = I \oplus 0$ does not belong to $\{V_T\}''$. Thus, the following problem remains open.

Question 8. *Is the abelian Banach algebra $\{T\}''$ semisimple for every power-bounded Hilbert space operator T of class $C_{1\cdot}$?*

In view of Theorem 2 and Proposition 6, the answer is affirmative if T is cyclic.

Remark 9. We note that if V_T is of finite multiplicity, then every quasinilpotent operator B in the commutant of V_T is nilpotent. Indeed, considering the functional model of V_T (as in [15]), we obtain that B is an operator of multiplication by a function Ψ defined on the unit circle $\mathbf{T}$ and taking on operator values $\Psi(z)$ acting on Hilbert spaces $\mathcal{H}(z)$ with $\dim \mathcal{H}(z) \leq m$ $(z \in \mathbf{T})$, where m is the multiplicity of V_T. Since $\|B^n\| = \text{ess sup}\{\|\Psi(z)^n\| : z \in \mathbf{T}\}$ holds for every n, we infer by the spectral radius formula that $\Psi(z)$ is quasinilpotent, and so $\Psi(z)^m = 0$ is valid for a.e. $z \in \mathbf{T}$. Therefore, $B^m = 0$ is also true.

As a consequence, we obtain that if the power-bounded operator T of class $C_{1\cdot}$ is of finite multiplicity, then the quasinilpotent operators in the commutant of T are nilpotent. So, if T is of finite multiplicity then the problem above can be reduced to the question whether every nilpotent operator A in the bicommutant of T is necessarily zero.

The following result on the stability of the semigroup $\{T^n\}_{n\geq 0}$ is related to Theorem 1 and has an analogous proof.

Theorem 10. *Suppose that T is a cyclic power-bounded operator on a Hilbert space $\mathcal{H}$ such that T commutes with a quasinilpotent operator A. Then $\{T^n\}_{n\geq 0}$ is stable on the range of A, that is, $\lim_{n\to\infty}\|T^n x\| = 0$ holds for every $x \in (\mathrm{ran}\ A)^-$.*

In connection with Theorem 10, let us also note the following related fact contained in [22].

Theorem 11. *Let T be a power-bounded operator which commutes with a compact operator K with dense range. Then $\{T^n\}_{n\geq 0}$ is stable if and only if T does not have a unimodular eigenvalue.*

We note that most of the previous results can be extended without any difficulty to operators T such that the norm-sequence $\{\|T^n\|\}_{n\geq 0}$ is regular in the sense of [14].

Studying these problems in the general Banach space setting, we encounter the obstacle that Lemma 3 fails, since $\{V\}''$ is not necessarily semisimple if V is an isometry on an arbitrary Banach space, see [7].

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