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SOCLE SERIES OF A COMMUTATIVE ARTINIAN RING

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Abstract. Let R be a commutative artinian ring, and $f(x) \in R[x]$ be a nonconstant monic polynomial. The main purpose of this paper is to determine the socle series of $R[x]/\langle f(x)\rangle$ in terms of the socle series of R. As an application of the results proved, it is proved that R is a QF-ring if and only if $R[x]/\langle f(x)\rangle$ is a QF-ring. As another application , a necessary and sufficient condition for a local artinian ring R having a semisimple ideal B, with R/B a PIR, to be a split extension of a PIR by a semisimple module, is given.

1. Introduction

Let R be a local, commutative artinian ring. By Cohen [3], R has a coefficient subring T, and $R = T[a_1, a_2, \ldots a_n]$ for some $a_i \in R$. If $n \neq 1$, i.e., if R is not a simple extension of its coefficient subring, not much information about the ideal structure of R can be obtained from the ideal structure of T. To apply induction on n, we need to investigate the relationship between the ideal structure of a local artinian ring R and a local artinian ring that is a simple extension of R. For this purpose, we consider $S = R[x]/\langle g(x)^t + u\rangle$ for some monic polynomial $g(x) \in R[x]$, which is irreducible modulo the radical J of R, where t is a positive integer, and $u \in J[x]$ is of degree $< t \deg g(x)$. In Section 1, Theorem 1.5 shows that the composition lengths of socle (S) and socle (R) are the same. As an application of this result, it is proved in Theorem 1.6 that for any nonconstant monic polynomial $f(x) \in R[x]$, R is a QF-ring if and only if $R[x]/\langle f(x)\rangle$ is a QF- ring. It is also proved that any artinian ring is a homomorphic image of a QF-ring. In Section 2, $S = R[x]/\langle g(x)^t\rangle$ is studied. Each member $\text{soc}^k(S)$ of the socle series of S is determined in terms of the members $\text{soc}^i(R)$ of the socle series of R. Theorem 2.8 gives the composition

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length of each factor $\mathrm{soc}^k(S)/\mathrm{soc}^{k-1}(S)$ in terms of the composition lengths of the factors $\mathrm{soc}^i(R)/\mathrm{soc}^{i-1}(R)$. This information can be useful in the classification of artinian rings. In Theorem 2.10, it is shown that any local artinian ring S with square of its radical zero, is determined within isomorphisms by its residue field, characteristic and composition length. A local artinian ring R is called a weak principal ideal ring (in short, a WPI-ring) if it contains a semisimple ideal B such that R/B is a PIR. Theorem 2.11 gives a necessary and sufficient condition for a WPI-ring to be a split extension of a PIR by a semisimple module.

All rings considered here are commutative. For any ring R, $\mathrm{soc}^0(R) = \{0\}$ and for $i > 0, \mathrm{soc}^i(R)/\mathrm{soc}^{i-1}(R) = \mathrm{soc}(R/\mathrm{soc}^{i-1}(R))$. For any module M_R of finite composition length, $d_R(M)$ denotes its composition length. An artinian self-injective ring is called a QF-ring. A local artinian ring R is a QF-ring if and only if socle(R) is simple; see Faith [4, p. 217, Exercise 5]. For any $g(x) \in R[x]$, cont $(g(x))$ denotes the content of $g(x)$, i.e., the ideal of R generated by the coefficients of $g(x)$.

1. Socle

Throughout, R is a local commutative artinian ring, unless otherwise stated, and $g(x) \in R[x]$ is a monic polynomial of degree m, which is irreducible modulo $J = J(R)$. Let t be a fixed positive integer, and $u \in J[x]$ be such that deg $u < mt$. Set $S = R[x]/\langle g(x)^t + u\rangle$. This S is a local ring with $J(S) = \langle J, g(x)\rangle/\langle g(x)^t + u\rangle$.

Lemma 1.1. *Let R be any ring and $f(x) \in R[x]$ be a nonconstant monic polynomial.*

(i) *For any ideal A of R, if $f(x)b(x) \in A[x]$ for some $b(x) \in R[x]$, then $b(x) \in A[x]$.*

(ii) *For any $h(x) \in R[x]$, if $h(x) = f(x)w(x) + b(x)$ for some $b(x), w(x) \in R[x]$, with* deg $b(x) <$ deg $f(x)$, *then $b(x), w(x) \in A[x]$, where A is the content of $h(x)$.*

(iii) *Given $h(x) \in R[x]$, there exists $k(x) \in R[x]$ with* $\mathrm{cont}(k(x)) \subseteq \mathrm{cont}(h(x))$ *such that for any $a \in R$, $f(x)$ divides $ah(x)$ if and only if $ah(x) = ak(x)f(x)$.*

Proof. Obvious. ∎

Lemma 1.2.

(a) *Given a $z_1 \in \mathrm{soc}(R)[x]$, and $1 \le i \le t$, if $z_1 g(x)^{t-i} \in \langle g(x)^t + u\rangle$, then $z_1 = z_1' g(x)$ for some $z_1' \in \mathrm{soc}(R)[x]$*

(b) *Given $a \in \mathrm{soc}(R)$, if $ag(x)^{t-i} \in \langle g(x)^t + u\rangle$ for some $1 \le i \le t$, then $a = 0$.*

Proof. Let $z_1 g(x)^{t-i} \in \langle g(x)^t + u \rangle$. Then $z_1 g(x)^{t-i} = (g(x)^t + u)w$ for some $w \in R[x]$. By Lemma 1.1 (i), $w \in \mathrm{soc}(R)[x]$. So $uw = 0$, $z_1 = g(x)^i w = z_1' g(x)$, with $z_1' = wg(x)^{i-1} \in \mathrm{soc}(R)[x]$. This proves (a). Further, (b) is immediate from (a). ■

For any $f(x) \in R[x]$, $\bar{f}(x)$ denotes its natural image in S.

Lemma 1.3. *Let B be an ideal of R contained in* $\mathrm{soc}(R)$, *and* $0 \neq z \in \mathrm{soc}(R)$ *such that* $zR \cap B = 0$. *Then in* S, $\bar{z}\bar{g}(x)^{t-1}S \neq 0$, *and*

$$\bar{z}\bar{g}(x)^{t-1}S \cap \bar{B}\bar{g}(x)^{t-1}S = 0.$$

Proof. Observe that $\overline{\mathrm{soc}(R)}\bar{g}(x)^{t-1}S \subseteq \mathrm{soc}(S)$. By Lemma 1.2(b), $\bar{z}\bar{g}(x)^{t-1} \neq 0$. So $\bar{z}\bar{g}(x)^{t-1}S$ is a minimal ideal of S. Suppose $\bar{z}\bar{g}(x)^{t-1}S \cap \bar{B}\bar{g}(x)^{t-1}S \neq 0$. Then $\bar{z}\bar{g}(x)^{t-1} \in \bar{B}\bar{g}(x)^{t-1}S$, so

$$zg(x)^{t-1} = bg(x)^{t-1} + (g(x)^t + u)w$$

for some $b \in B[x]$ and $w \in R[x]$. Thus, modulo B, $zg(x)^{t-1} = (g(x)^t + u)w$. By comparing the degrees on both sides, it follows that $w \in B[x]$, and hence $z \in B$. This is a contradiction, which proves the result. ■

Corollary 1.4. *If* $\mathrm{soc}(R) = \bigoplus_1^s A_i$ *for some minimal ideals* A_i, *then the following hold.*

(i) $\overline{\mathrm{Soc}(R)}\bar{g}(x)^{t-1}S = \bigoplus\limits_1^s \bar{A}_i\bar{g}(x)^{t-1}S$,

(ii) $d_R(\mathrm{soc}(R)) \leq d_S(\mathrm{soc}(S))$.

Theorem 1.5. $\mathrm{Soc}(S) = \overline{\mathrm{soc}(R)}\bar{g}(x)^{t-1}S$ *and* $d_S(\mathrm{soc}(S)) = d_R(\mathrm{soc}(R))$.

Proof. Now $\overline{\mathrm{soc}(R)}\bar{g}(x)^{t-1}S \subseteq \mathrm{soc}(S)$. Let $\bar{\lambda}(x) \in \mathrm{soc}(S)$. Then $\bar{\lambda}(x)\bar{g}(x) = 0$. So in $R, \lambda(x)g(x) = (g(x)^t + u)w$ for some $w \in R[x]$. Then $g(x)(\lambda(x) - g(x)^{t-1}w) = uw \in J[x]$. By Lemma 1.1, $\lambda(x) = g(x)^{t-1}w + w_1$ for some $w_1 \in J[x]$ such that $g(x)w_1 = uw$. Now $w = g(x)w' + w''$ for some $w', w'' \in R[x]$, with deg $w'' <$ deg $g(x)$. Then $uw' \in J[x]$, and $\lambda(x) = (g(x)^t + u)w' + g(x)^{t-1}w'' + (w_1 - uw')$ with $w_2 = w_1 - uw' \in J[x]$. So in S, $\bar{\lambda}(x) = \bar{g}(x)^{t-1}\overline{w}'' + \overline{w}_2$. Also $g(x)w_2 = uw''$. Thus, without loss of generality, we take

$$\lambda(x) = g(x)^{t-1}w + w_1,$$

with deg $w <$ deg $g(x)$ and $g(x)w_1 = uw$. Consider $a \in J$. Then $\bar{\lambda}(x)\bar{a} = 0$. For some $w_a \in R[x]$,

$$(g(x)^{t-1}w + w_1)a = (g(x)^t + u)w_a.$$

Then $w_a \in J[x]$. As $g(x)w_1 = uw$, we get $(g(x)^t + u)wa = (g(x)^t + u)g(x)w_a$, and $wa = g(x)w_a$. By comparing the degrees of both sides, we get $wa = 0$. So $w \in \text{soc}(R)[x]$. Then $g(x)w_1 = uw = 0$, which gives $w_1 = 0$. Hence $\lambda(x) = g(x)^{t-1}w$. This proves that $\text{soc}(S) = \overline{\text{soc}(R)}\bar{g}(x)^{t-1}S$. Now the second part is immediate from Corollary 1.4. ■

A local artinian ring R is QF if and only if $\text{soc}(R)$ is simple; see Faith [4, p. 217]. Exercise 6 on page 217 in [4] is a particular case of the following.

Theorem 1.6. *Let R be a commutative artinian ring, and $f(x) \in R[x]$ be a monic, nonconstant polynomial. Then R is a QF-ring if and only if $S = R[x]/\langle f(x)\rangle$ is a QF-ring. Any artinian ring is a homomorphic image of a QF-ring.*

Proof. As R is artinian, S is also artinian. Without loss of generality, we suppose that R is a local ring. As S is a direct sum of local rings, by Azumaya [2, Lemma 3], $f(x) = \prod_1^k f_i(x)$, with $f_i(x)$ monic, such that

$$S = \bigoplus_{1}^{k} R[x]/\langle f_i(x)\rangle$$

with each $S_i = R[x]/\langle f_i(x)\rangle$ a local ring. Then $f_i(x) = g_i(x)^{t_i} + u_i$ for some monic polynomial $g_i(x) \in R[x]$ irreducible modulo J, and $u_i \in J[x]$ with deg $u_i <$ deg $f_i(x)$. So without loss of generality, we take S to be a local ring. By Theorem 1.5, $d_R(\text{soc}(R)) = d_S(\text{soc}(S))$. This gives that R is QF if and only if S is QF.

To prove the last part, without loss of generality, we consider a local artinian ring S. Now $S = R[a_1, a_2, ..., a_n]$, where R is a coefficient subring of S. As R is a local artinian principal ideal ring, it is a QF-ring. By applying induction on n and by using the first part, it follows that S is a homomorphic image of a QF-ring. ■

Lemma 1.7. *Let R be any local, artinian ring, $f(x) \in R[x]$ be a monic polynomial of degree $m \geq 1$, and $u(x) \in J(R)[x]$. Then there exists a monic polynomial $h(x) \in R[x]$ of degree m, such that $\langle f(x) + u(x)\rangle = \langle h(x)\rangle$.*

Proof. Set $S = R[x]/\langle f(x) + u(x)\rangle$. Now $A = \langle J, f(x) + u(x)\rangle = \langle J, f(x)\rangle$. Then $S/SJ \cong R[x]/A$ as R-modules. Further, the R-module $R[x]/A$ is generated by the m cosets $x^i + A$, $0 \leq i \leq m-1$. As J is nilpotent, S_R is finitely generated. By [2, Theorem 6], S_R is generated by $\{\bar{x}_i : 0 \leq i \leq m-1\}$, where $\bar{x}_i = x_i + \langle f(x) + u(x)\rangle$. Thus in S, $\bar{x}^m = \sum_{i=0}^{m-1} a_i\bar{x}^i$ for some $a_i \in R$. Then $h(x) = x^m - \sum_{i=0}^{m-1} a_i x^i \in \langle f(x) + u(x)\rangle$. As R is a local ring, and

$f(x)$, $h(x)$ both are monic polynomials of the same degree, it follows that $h(x) = (f(x) + u(x))(1 + v(x))$ for some $v(x) \in J[x]$. As $1 + v(x)$ is a unit in $R[x]$, $\langle f(x) + u(x)\rangle = \langle h(x)\rangle$. ■

In view of the above lemma, Theorem 1.6 gives the following:

Theorem 1.8. *Let R be any local, commutative ring, $f(x) \in R[x]$ be a monic, nonconstant polynomial, and $u(x) \in J[x]$. Then R is QF if and only if $R[x]/\langle f(x) + u(x)\rangle$ is QF.*

2. Socle Series

Throughout this section, R is a local, commutative artinian ring, and $g(x) \in R[x]$ is a monic polynomial which is irreducible modulo $J = J(R)$. For a fixed positive integer t,

$$S = R[x]/\langle g(x)^t\rangle.$$

Proposition 2.1. *For $1 \le i \le t$,*

$$\mathrm{soc}^i(S) = \langle \overline{\mathrm{soc}^1(R)}\bar{g}(x)^{t-i},\ \overline{\mathrm{soc}^2(R)}\bar{g}(x)^{t-i+1}, \dots, \overline{\mathrm{soc}^i(R)}\bar{g}(x)^{t-1}\rangle.$$

Proof. We apply induction on i. By Theorem 1.5, the result holds for $i = 1$. Let it hold for some $i = k < t$. Let $\bar{\lambda}(x) \in \mathrm{soc}^{k+1}(S)$. Then $\bar{\lambda}(x)\bar{g}(x) \in \mathrm{soc}^k(S)$, which gives $\lambda(x)g(x) = \sum_{j=1}^{k} z_j g(x)^{t-k+j-1} + g(x)^t v$ for some $z_j \in \mathrm{soc}^j(R)[x]$ and $v \in R[x]$. Thus

$$\lambda(x) = \sum_{j=1}^{k} z_j g(x)^{t-k+j-2} + g(x)^{t-1} v.$$

By dividing v by $g(x)$ in $R[x]$, we get

$$\bar{\lambda}(x) = \sum_{j=1}^{k} \bar{z}_j \bar{g}(x)^{t-k+j-2} + \bar{g}(x)^{t-1}\bar{w}$$

for some $w \in R[x]$ with deg $w <$ deg $g(x)$. Consider $a \in J$. Then $\bar{\lambda}(x)\bar{a} \in \mathrm{soc}^k(S)$. So $\sum_{j=1}^{k} z_j a g(x)^{t-k+j-2} + g(x)^{t-1} a w = \sum_{j=1}^{k} u_j g(x)^{t-k+j-1} + g(x)^t w'$ for some $u_j \in \mathrm{soc}^j(R)[x]$ and $w' \in R[x]$. As $z_j a \in \mathrm{soc}^{j-1}(R)[x] \subseteq \mathrm{soc}^k(R)[x]$, we get $g(x)^{t-1} a w \in \mathrm{soc}^k(R)[x] + \langle g(x)^t\rangle$. Then $g(x)^{t-k-1} v_1 = g(x)^t v_2 + g(x)^{t-1} a w$ for some $v_1 \in \mathrm{soc}^k(R)[x]$ and $v_2 \in R[x]$. By using Lemma 1.1(ii), we get $aw \in \mathrm{soc}^k(R)[x]$. Thus $w \in \mathrm{soc}^{k+1}(R)[x]$. Consequently,

$$\bar{\lambda}(x) = \sum_{j=1}^{k+1} \bar{b}_j \bar{g}(x)^{t-(k+1)+j-1}$$

with $b_j = z_j$ for $1 \leq j \leq k$ and $b_{k+1} = w$. This completes the proof. ■

Proposition 2.2. *For any* $i \geq 0$,

$$\mathrm{soc}^{t+i}(S) = \langle \overline{\mathrm{soc}^{i+1}(R)}, \overline{\mathrm{soc}^{i+2}(R)}\bar{g}(x), \ldots, \overline{\mathrm{soc}^{i+t}(R)}\bar{g}(x)^{t-1} \rangle$$

Proof. We apply induction on i. By Proposition 2.1, the result holds for $i = 0$. Let it be true for some $i = k$. Consider $\bar{\lambda}(x) \in \mathrm{soc}^{t+k+1}(S)$. Then $\bar{\lambda}(x)\bar{g}(x) \in \mathrm{soc}^{t+k}(S)$, which gives

$$\lambda(x)g(x) = \sum_{j=1}^{t} z_{k+j}g(x)^{j-1} + g(x)^t w$$

for some $z_{k+j} \in \mathrm{soc}^{k+j}(R)[x]$ and $w \in R[x]$. This gives $z_{k+1} = z'_{k+1}g(x)$ for some $z'_{k+1} \in \mathrm{soc}^{k+1}(R)[x]$. Thus

$$\lambda(x) = z'_{k+1} + \sum_{j=2}^{t} z_{k+j}g(x)^{j-2} + g(x)^{t-1}w.$$

So in S,

$$\bar{\lambda}(x) = \bar{z}'_{k+1} + \sum_{j=2}^{t} \bar{z}_{k+j}\bar{g}(x)^{j-2} + \bar{g}(x)^{t-1}\bar{v}$$

for some $v \in R[x]$ with $\deg\ v < \deg g(x)$. Consider any $a \in J$. Then $\bar{\lambda}(x)\bar{a} \in \mathrm{soc}^{t+k}(S)$, which gives

$$z'_{k+1}a + \sum_{j=2}^{t} z_{k+j}ag(x)^{j-2} + g(x)^{t-1}av = \sum_{j=1}^{t} u_{k+j}g(x)^{j-1} + g(x)^t w'$$

for some $u_{k+j} \in \mathrm{soc}^{k+j}(R)[x]$ and $w' \in R[x]$. Consequently, $g(x)^{t-1}av = w_1 + g(x)^t w'$ with $w_1 \in \mathrm{soc}^{t+k}(R)[x]$. By Lemma 1.1(ii), $av \in \mathrm{soc}^{t+k}(R)[x]$, and so $v \in \mathrm{soc}^{t+k+1}(R)[x]$. Also, $z'_{k+1} + z_{k+2} \in \mathrm{soc}^{k+2}(R)[x]$. Set

$$u_{k+2} = z'_{k+1} + z_{k+2}, u_{k+1+j} = z_{k+1+j}$$

for $2 \leq j \leq t-1$ and $u_{k+1+t} = v$. Then $\bar{\lambda}(x) = \sum_{j=1}^{t} \bar{u}_{k+1+j}\bar{g}(x)^{j-1}$. This completes the proof. ■

Propositions 2.1 and 2.2 can be combined to give the following

Theorem 2.3. *Let R be any local, commutative, artinian ring, and $g(x) \in R[x]$ be a monic polynomial, which is irreducible modulo $J = J(R)$. For some positive integer t, let $S = R[x]/\langle g(x)^t \rangle$.*

(I) *For* $1 \le s \le t$,

$$\operatorname{soc}^s(S) = \sum_{j=1}^{s} \overline{\operatorname{soc}^j(R)}[x]\bar{g}(x)^{t-s+j-1}.$$

(II) *For* $s \ge t$

$$\operatorname{soc}^s(S) = \sum_{j=s-t+1}^{s} \overline{\operatorname{soc}^j(R)}[x]\bar{g}(x)^{t-s+j-1}.$$

(III) *For any* $j \ge 1, 0 \le k \le t-1$ *and* $z_j \in \operatorname{soc}^j(R)[x], \bar{z}_j\bar{g}(x)^k \in \operatorname{soc}^{t-k+j-1}(S)$.

In fact, (I) *and* (II) *in the above theorem can be put together to say that for any* $s \ge 0$, $\operatorname{soc}^s(S) = \sum \operatorname{soc}^j(R)[x]\bar{g}(x)^{t-s+j-1}$, *where the summation runs over* $j \ge 1$, $t-s+j-1 \ge 0$.

Lemma 2.4. *For* $1 \le j \le t-1$, *if* $\overline{z_1}\bar{g}(x)^{t-j} \in \operatorname{soc}^{j-1}(S)$ *for some* $z_1 \in \operatorname{soc}^1(R)[x]$, *then* $z_1 \in g(x)\operatorname{soc}^1(R)[x]$.

Proof. If $j = 1$, the result follows from Lemma 1.2. Let $j > 1$. By Theorem 2.3(I),

$$z_1 g(x)^{t-j} = \sum_{k=1}^{j-1} u_k g(x)^{t-j+k} + wg(x)^t$$

for some $u_k \in \operatorname{soc}^k(R)[x]$ and $w \in R[x]$. Thus $z_1 g(x)^{t-j} = w' g(x)^{t-j+1}$ for some $w' \in R[x]$. Consequently, $z_1 = g(x)w'$, and by Lemma 1.1(i), $w' \in \operatorname{soc}^1(R)[x]$. ■

Lemma 2.5. *For* $1 \le i \le t$, *if*

$$\bar{\lambda}(x) = \sum_{k=1}^{i} \bar{z}_k\bar{g}(x)^{t-i+k-1} \in \operatorname{soc}^{i-1}(S)$$

for some $z_k \in \operatorname{soc}^k(R)[x]$, *then*

$$z_k \in g(x)\operatorname{soc}^k(R)[x] + \operatorname{soc}^{k-1}(R)[x].$$

Proof. We apply induction on i. For $i = 1, \bar{\lambda}(x) = \bar{z}_1 g(x)^{t-1} \in \operatorname{soc}^0(S)$. By Lemma 1.2, $z_1 = z_1' g(x)$ for some $z_1' \in \operatorname{soc}(R)[x]$. So the result holds for $i = 1$. Let $i > 1$ and let the result hold for $i - 1$. Let $a \in J$. Then $z_1 a = 0$, and

$$\bar{\lambda}(x)a = \sum_{k=2}^{i} \bar{z}_k a g(x)^{t-i+k-1} \in \operatorname{soc}^{i-2}(S)$$

with $z_k a \in \text{soc}^{k-1}(R)[x]$. By the induction hypothesis, $z_k a \in g(x)\text{soc}^{k-1}(R)[x] + \text{soc}^{k-2}(R)[x]$. Consider $\bar{R} = R/\text{soc}^{k-2}(R)$. Over $\bar{R}$, $\bar{z}_k\bar{a}$ is divisible by $\bar{g}(x)$. By Lemma 1.1(iii), there exists $h(x) \in \text{soc}^k(R)[x]$ such that $[z_k - h(x)g(x)]a \in \text{soc}^{k-2}(R)[x]$ for every $a \in J$. Consequently, $z_k - h(x)g(x) \in \text{soc}^{k-1}(R)[x]$. So $z_k \in g(x)\text{soc}^k(R)[x] + \text{soc}^{k-1}(R)[x]$ for $2 \leq k \leq i$. Then by Theorem 2.3(I), $\sum_{k=2}^{i} z_k g(x)^{t-i+k-1} \in \text{soc}^{i-1}(S)$. Thus $z_1 g(x)^{t-i} \in \text{soc}^{i-1}(S)$. Apply Lemma 2.4. This completes the proof. ■

Similarly by using Theorem 2.3(II), we get the following

Lemma 2.6. *For any* $i \geq 0$, *if*

$$\bar{\lambda}(x) = \sum_{j=1}^{t} \bar{z}_{i+j}\bar{g}(x)^{j-1} \in \text{soc}^{t+i-1}(S)$$

for some $z_{i+j} \in \text{soc}^{i+j}(R)[x]$, *then*

$$z_{i+j} \in g(x)\text{soc}^{i+j}(R)[x] + \text{soc}^{i+j-1}(R)[x].$$

Lemma 2.7. *For any* $i > 0$, *let* $M_i = \text{soc}^i(R)[x]/(g(x)\text{soc}^i(R)[x] + \text{soc}^{i-1}(R)[x])$, *and* $\sigma : \text{soc}^i(R)[x] \to M_i$ *be the natural homomorphism. Then the following hold.*

(i) *For any* $a \in \text{soc}^i(R)\backslash\text{soc}^{i-1}(R)$, $\sigma(a) \neq 0$.

(ii) M_i *is an* $S/J(S)$*-module.*

(iii) *Let* $\eta : \text{soc}^i(R) \to \text{soc}^i(R)/\text{soc}^{i-1}(R)$ *be the natural homomorphism. Let* A *be any ideal of* R *contained in* $\text{soc}^i(R)$, *and let* $a \in \text{soc}^i(R)\backslash\text{soc}^{i-1}(R)$ *be such that* $\eta(A) \cap \eta(aR) = 0$. *Then* $\sigma(A)S \cap \sigma(a)S = 0$.

(iv) $d_S(M_i) = d_R(\text{soc}^i(R)/\text{soc}^{i-1}(R))$.

Proof. Suppose, $\sigma(a) = 0$ for some $a \in \text{soc}^i(R)\backslash\text{soc}^{i-1}(R)$. Then $a = g(x)z + u$ for some $z \in \text{soc}^i(R)$ and $u \in \text{soc}^{i-1}(R)[x]$. By Lemma 1.1(ii), $a \in \text{soc}^{i-1}(R)$. This is a contradiction. Hence $\sigma(a) \neq 0$. As $M_i J = 0$, $M_i g(x) = 0$, $J(S) = \langle g(x), J\rangle$, it is immediate that M_i is an $S/J(S)$-module. For (iii), let $\sigma(A)S \cap \sigma(a)S \neq 0$. By (i) and (ii), $\sigma(a)S$ is a simple S-module. So $\sigma(a) \in \sigma(A)S$. Then for some $f(x) \in A[x]$, $a - f(x) \in g(x)\text{soc}^i(R)[x] + \text{soc}^{i-1}(R)[x]$. By using Lemma 1.1(ii), we may take $\deg f(x) < \deg g(x)$. Now $a - f(x) = g(x)w + u$ for some $u \in \text{soc}^{i-1}(R)[x]$. By using Lemma 1.1(ii), we get $a - f(x) \in \text{soc}^{i-1}(R)[x]$. If b is the constant term of $f(x)$, $a - b \in \text{soc}^{i-1}(R)$. This gives $\eta(a) \in \eta(A)$, which is a contradiction. This proves (iii). We write $\text{soc}^i(R)/\text{soc}^{i-1}(R) = \bigoplus_{j=1}^{u} \eta(a_j)R$

for some $\eta(a_j) \neq 0$. As $M_i = \sigma(\text{soc}^i(R))S$, by using (i) and (iii), we get $M_i = \bigoplus_{j=1}^{u} \sigma(a_j)S$. This proves (iv). ■

Theorem 2.8. *Let R be a local commutative artinian ring, and $g(x) \in R[x]$ be a monic polynomial, irreducible modulo $J(R)$. For some positive integer t, let $S = R[x]/\langle g(x)^t \rangle$. Then the following hold.*

(I) *For* $1 \leq s \leq t$,

$$\text{soc}^s(S)/\text{soc}^{s-1}(S) \cong \bigoplus_{k=1}^{s} \text{soc}^k(R)[x]/(g(x)\text{soc}^k(R)[x] + \text{soc}^{k-1}(R)[x])$$

and

$$d_S(\text{soc}^s(S)/\text{soc}^{s-1}(S)) = \sum_{k=1}^{s} d_R(\text{soc}^k(R)/\text{soc}^{k-1}(R)).$$

(II) *For* $i > 0$,

$$\text{soc}^{t+i}(S)/\text{soc}^{t+i-1}(S) \cong \bigoplus_{j=1}^{t} \text{soc}^{i+j}(R)[x]/(g(x)\text{soc}^{i+j}(R)[x] + \text{soc}^{i+j-1}(R)[x])$$

and

$$d_S(\text{soc}^{t+i}(S)/\text{soc}^{t+i-1}(S)) = \sum_{j=1}^{t} d_R(\text{soc}^{i+j}(R)/\text{soc}^{i+j-1}(R))$$

Proof. Consider $1 \leq s \leq t$. Define

$$\sigma : \bigoplus_{k=1}^{s} \text{soc}^k(R)[x] \to \text{soc}^s(S)/\text{soc}^{s-1}(S)$$

such that

$$\sigma(\bigoplus_{k=1}^{s} z_k) = \sum_{k=1}^{s} \bar{z}_k \bar{g}(x)^{t-s+k-1} + \text{soc}^{s-1}(S), \ z_k \in \text{soc}^k(R)[x].$$

By Theorem 2.3 σ is an $R[x]$-epimorphism. By Lemma 2.5, ker $\sigma \subseteq \bigoplus_{k=1}^{s}(g(x)\text{soc}^k (R)[x] \bigoplus \text{soc}^{k-1}(R)[x])$. However by Theorem 2.3, $\bigoplus_{k=1}^{s}(g(x)\text{soc}^k(R)[x] + \text{soc}^{k-1} (R)[x])$ is contained in ker σ. This proves the first part of (I). The second part of (I) follows from lemma 2.7(iv). Similarly, (II) can be proved. ■

Remark 2.9. In the above theorem, let n be the index of nilpotency of $J(R)$. For $1 \leq s \leq \min(t, n)$, (I) gives

$$d_S(\mathrm{soc}^s(S)/\mathrm{soc}^{s-1}(S)) = \sum_{j=1}^{s} d_R(\mathrm{soc}^j(R)/\mathrm{soc}^{j-1}(R)).$$

Let $t \geq n$. As $\mathrm{soc}^j(R) = R$ for every $j \geq n$, we get the following. For $n \leq s \leq t$,

$$d_S(\mathrm{soc}^s(S)/\mathrm{soc}^{s-1}(S)) = \sum_{j=1}^{n} d_R(\mathrm{soc}^j(R)/\mathrm{soc}^{j-1}(R)).$$

For $0 < i \leq n-1$, $d_S(\mathrm{soc}^{t+i}(S)/\mathrm{soc}^{t+i-1}(S)) = \sum_{k=1}^{n-i} d_R(\mathrm{soc}^{i+k}(R)/\mathrm{soc}^{i+k-1}(R))$. Let $t < n$. Then, for $0 < i \leq n - t$,

$$d_S(\mathrm{soc}^{t+i}(S)/\mathrm{soc}^{t+i-1}(S)) = \sum_{j=1}^{t} d_R(\mathrm{soc}^{i+j}(R)/\mathrm{soc}^{i+j-1}(R)).$$

For $0 < i \leq t-1$, $d_S(\mathrm{soc}^{n+i}(S)/\mathrm{soc}^{n+i-1}(S)) = \sum_{j=1}^{t-i} d_R\ (\mathrm{soc}^{n-t+i+j}(R)/\mathrm{soc}^{n-t+i+j-1}(R))$. In any case, $\mathrm{soc}^{t+n-2}(S) = J(S)$, and $\mathrm{soc}^{t+n-1}(S) = S$.

Let S be any ring and M be an S-module. Then $R = S \times M$ becomes a ring, in which $(r, x) + (s, y) = (r + s, x + y)$ and $(r, x)(s, y) = (rs, yr + xs)$. This R is an extension of S, called a split extension of S by the module M. If S is a local ring, so is R. Clearly, $J(S)^2 = 0$ if and only if $J(R)^2 = 0$. If a ring R has a subring S and an ideal M such that $S \cap M = 0$ and R is canonically isomorphic to the split extension of S by M, we write $R = S \triangleright M$. If a local artinian ring R contains a nonzero semisimple ideal B such that R/B is a principal ideal ring, then R is called a WPI-ring. As an application of the results on socle series, we determine when a WPI-ring is a split extension of a PIR by a semisimple module. We start with the following.

Theorem 2.10. *Let S be a local artinian ring such that $J(S) \neq 0$ but $J(S)^2 = 0$, and let R be a coefficient subring of S. Then S is a split extension of R. Any two local rings S and S' with squares of their radicals zero are isomorphic if and only if they have isomorphic residue fields, same characteristic and same composition length.*

Proof. To start with, we take $S = R[a] \neq R$, where R is a local subring of S such that $\bar{S} = S/J(S) = \bar{R}$. There exists $c \in R$ such that $(a - c)^2 = 0$. Consider $T = R[x]/\langle (x - c)^2 \rangle$. We get an R-epimorphism $\sigma : T \to S$ such that $\sigma(\bar{x}) = a$. If R is a field, then T is isomorphic to S and it is the split extension of R by the simple R-module $V = \{r(x - c) + \langle (x - c)^2 \rangle : r \in R\}$. Assume that R is not a field. By Remark 2.9, the index of nilpotency of $J(T)$ is three. Further, $J(T) = \mathrm{soc}^2(T) = \langle \overline{\mathrm{soc}(R)}, \overline{\mathrm{soc}^2(R)}\,\overline{(x-c)} \rangle = \overline{\mathrm{soc}(R)[x]} + \overline{R[x](x-c)}$. This gives $J(T)^2 = \overline{\mathrm{soc}(R)(x-c)} = \mathrm{soc}(T)$. Let $A = \ker \sigma$. As the index of nilpotency of $J(T/A)$ is two, $\mathrm{soc}(T) \subseteq A$. Any simple T-module or a simple S-module is a simple R-module. Thus, by Theorem 2.8(I), $d_R(J(T)/\mathrm{soc}(T)) = d_T(J(T)) = d_R(\mathrm{soc}(R)) + d_R(\mathrm{soc}^2(R)/\mathrm{soc}(R)) = d_R(J(R)) + d_R(R/J(R)) = d_R(R)$. Consequently, $d_R(T/\mathrm{soc}(T)) = 1 + d_R(R) \leq d_R(S) = d_R(T/A)$. This gives $A = \mathrm{soc}(T)$. Now R has a natural embedding in T/A. As an abelian group, $T/A = R \oplus V$, where $V = \{\overline{r(x-c)^2} + A : r \in R\}$ is a simple R-module. Clearly T/A is a split extension of R by V. As S is a finite extension of its coefficient ring, it now follows that S is a split extension of its coefficient subring. This proves the first part. Let two local artinian rings S and S' with squares of their radicals zero have isomorphic residue fields, same characteristic and same composition lengths. Let R and R' be their coefficient rings. As R and R' have isomorphic residue fields and have same characteristic, being image of the same v-ring, as given in [3], they are isomorphic. Then the fact that S and S' are split extensions and have same composition length, gives that S and S' are isomorphic. ■

The above theorem has some similarity with Theorem 1 in [1] for finite rings.

Theorem 2.11. *Let R be a WPI-ring with M as its maximal ideal, and having a semisimple ideal B such that R/B is a PIR. Then R is a split extension of a PIR by a semisimple module if and only if*

(i) $M^2 = 0$, *or*

(ii) *char* $R/M = 0$, *or*

(iii) *char* $R/M = p > 0$ *and* $p \in M^i$ *for some* $i > 1$, *or*

(iv) *char* $R/M = p > 0$ *and* $p \in M \backslash (M^2 + B)$.

Proof. Let n be the index of nilpotency of M. Now $\mathrm{soc}(R) = M^{n-1} + B$ and for any $y \in M \backslash (M^2 + B)$, $M = yR + M^2 + B = yR + B$, and $M^i = y^i R$ for $i \geq 2$. Thus $M = yR \oplus C$ for some $C \subseteq B$. By Theorem 2.10, the result holds for $n = 2$. Let $n > 2$. Let R satisfy one of the conditions (ii), (iii) and (iv). Consider $T = R/M^2$. By Theorem 2.10, $T = T' \triangleright W$, where T' is a coefficient subring of T and W is a semisimple ideal of T.

Case I. Char T is either zero or a prime number. Then T' is a field. Now $T' = F/M^2$ for some subring F of R containing M^2. Further, $W = M/M^2$. Consider $R_1 = F + yR$. Then $R = R_1 + C$. Now C is a semisimple R_1-module and R_1 is a PIR with $J(R_1) = yR = yR_1$ and $R_1 \cap C = 0$. Hence $R = R_1 \triangleright C$.

Case II. For some prime number p, char T is p^2. Then $T' = L/M^2$ for some subring L containing M^2 with $J(T') = (pL + M^2)/M^2$ and $W = K/M^2$ for some ideal $K \subseteq M$. Clearly, $p \in M \backslash M^2$. By (iii) and (iv), $p \notin M^2 + C$, and so $M = pR + M^2 + C$ and $M^2 = p^2 R$. But every simple R-module is a simple L-module. This yields $pR = pL + p^2 R = J(L) = pL + J(L)^2$. Hence L is a principal ideal ring and $R = L \lhd C$.

Let R be a split extension of a PI subring S by a semisimple ideal D. Then $J(S) = bS = bR$ for some $b \in S$ and $J(S)^i = M^i$ for $i \geq 2$. Let $n > 2$. Suppose char $R/M = p > 0$. As S is a PIR, it yields that $pS = pR = J(S)^i$ for some $i \geq 1$. However, $J(S)^j = M^j$ for $j \geq 2$. If $i \geq 2$, (iii) holds. Let $pS = J(S)$. As $J(S) \nsubseteq M^2 + D$, $p \in M \backslash (M^2 + D)$. However, soc$(R) = M^{n-1} \oplus D = M^{n-1} + B$ gives soc$^{n-2}(R) = M^2 + D = M^2 + B$. So R satisfies (iv). ■

We now give an example of a WPI-ring that is not a split extension of a PIR by any semisimple module.

Example. Let p be any prime number and $S = Z/(p^3)[x]$. Let A be the ideal of S generated by the polynomials $(x^2 - p)p$ and $(x^2 - p)x$. Let $R = S/A$. This R is a local ring with $J(R)$ generated by $\bar{x} = x + A$ and $\bar{p} = p + A$. As $px = x^2 x - x(x^2 - p) \equiv x^2 x$ (mod A) and $p^2 = px^2 - p(x^2 - p) \equiv px^2$ (mod A), $J(R)^2 = \langle \bar{x}^2 \rangle$. Suppose $x^2 - p + A \in \langle \bar{x} \rangle$. Then $\bar{p} \in \langle \bar{x} \rangle$, $p = xf(x) + (x^2 - p)xg(x) + (x^2 - p)ph(x)$. Dividing $f(x)$ by $x^2 - p$, we take $f(x) = ax + b$ for some a, $b \in Z/(p^3)$. Then $x(ax + b) - p$ is divisible by $x^2 - p$. Consequently, $b = 0$ and $ap = p$. This yields that a is a unit and that $xf(x) - p = a(x^2 - p)$. Then $-a = xg(x) + ph(x)$. By putting $x = 0$, we get $-a = ph(0)$. This is impossible, as a is a unit. Also, $(x^2 - p) + A \in$ soc(R). We get $J(R) = \langle \bar{x} > \oplus C$, where $C = \langle x^2 - p + A \rangle$. R/C is a PIR, $\bar{p} \in J(R) \backslash J(R)^2$ and $\bar{p} \in J(R)^2 \oplus C$. By Theorem 2.11, R is not a split extension of PIR by any semisimple module.

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