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HAHN-BANACH-KANTOROVICH TYPE THEOREMS WITH THE RANGE SPACE NOT NECESSARILY (O)-COMPLETE

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Abstract. In the classical Hahn-Banach-Kantorovich theorem, the range space Y is Dedekind complete. In this paper, by extending the arguments of the original Hahn-Banach-Kantorovich theorem and using an idea of Y. A. Abramovich and A. W. Wickstead, we can weaken the order theoretic assumption on Y and obtain more general results in the settings of Banach lattices as well as ordered linear spaces.

1. INTRODUCTION

In the operator version of the Hahn-Banach-Kantorovich theorem, the range space Y is assumed to be Dedekind complete. This assumption can be considerably relaxed by using a weaker interpolation property, the so-called Cantor property on Y. Some generalizations of this type were given by H. B. Cohen [3], J. Lindenstrauss [9] and G. Buskes [2]. In particular, Y. A. Abramovich and A. W. Wickstead [1] provided us the following

Theorem 1 [1]. *Let X and Y be Banach lattices such that X is separable and Y has the Cantor property. Let $P : X \to Y_+$ be a continuous seminorm. If G is a linear subspace of X and $T : G \to Y$ is a continuous linear operator satisfying $T(v) \le P(v)$ for all v in G, then there exists a continuous extension S of T to the whole of X such that $S(x) \le P(x)$ for all x in X.*

In this paper, we obtain two new results along the line. The first one states that any positive linear operator from a majorizing subspace of a separable Banach lattice

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into a Banach lattice with the Cantor property can be extended. The second one states that any (o)-continuous linear operator from a subspace of an ordered linear space with (os)-property into an ordered linear space with the strong (σ)-interpolation property dominated by an (o)-continuous seminorm can also be extended.

2. Preliminaries

As far as the linear-order-theoretical terminology is concerned, we mostly follow Cristescu's book [4]. In particular, an ordered linear space X is said to have the (os)-*property* if there exists a countable subset D of X such that for each x in X there is a sequence $\{x_n\}_n$ in D with $x_n \xrightarrow{o} x$. A linear subspace G of X is a *majorizing subspace* if for every x in X there exists a v in G with $x \leq v$. Consequently, there also exists a u in G such that $u \leq x$.

Definition. Let Y be an ordered linear space. Y is said to have the *Cantor property* (or the (σ)-*interpolation property* or the *countable property*) if for every increasing sequence $\{x_n\}_n$ and every decreasing sequence $\{z_m\}_m$ in Y with $x_n \leq z_m, \forall\, n, m \in \mathbb{N}$, there is a y in Y such that $x_n \leq y \leq z_m, \forall\, n, m \in \mathbb{N}$. Y is said to have the *strong* (σ)-*interpolation property* if for every pair of sequences $\{x_n\}_n$ and $\{z_m\}_m$ in Y with $x_n \leq z_m, \forall\, n, m \in \mathbb{N}$, there is a y in Y such that $x_n \leq y \leq z_m, \forall\, n, m \in \mathbb{N}$. In case Y is a vector lattice, these two notions coincide.

G. Seever [10] showed that for a completely regular space K, $C(K)$ has the Cantor property if and only if K is an F-space, i.e., every pair of disjoint open (F_σ)-sets in K has disjoint closures. C. B. Huijsmans and B. De Pagter [8] showed that an Archimedean vector lattice Y has the Cantor property if and only if Y is uniformly complete and normal. In general, for a vector lattice we have: Dedekind completeness implies Dedekind (σ)-completeness implies Cantor property implies order completeness implies uniform completeness (see, e.g., [12, p. 696]).

In case Y is a Banach lattice, A. W. Wickstead [11] proved that the following are all equivalent: (1) Y has the Cantor property; (2) the space of all regular operators from convergent sequences into Y has the strong (σ)-interpolation property; (3) the space of all regular operators from convergent sequences into Y has the Riesz decomposition property. More recently, N. Daneţ [6] showed that they are also equivalent to: $(3')$ the space of all regular operators from any separable Banach lattice into Y has the Riesz decomposition property.

3. Main Results

We start with a Kantorovich-type theorem concerning the extension of a positive linear operator. Note that every positive linear operator from a majorizing subspace of a Banach lattice into a Banach lattice is continuous.

Theorem 2. *Let X be a separable Banach lattice, G a majorizing subspace of X, and Y a Banach lattice with the Cantor property. If $T : G \to Y$ is a positive linear operator, then there exists a positive linear operator $S : X \to Y$ such that $S(v) = T(v)$, $\forall v \in G$.*

Proof. Let $x_0 \in X \setminus G$ and G_1 the linear hull of $G \cup \{x_0\}$. We will extend T to G_1. Because G is a majorizing subspace of X, we can choose u, v from G such that $u \leq x_0 \leq v$. Since the operator T is positive, we have

$$T(u) \leq T(v). \tag{1}$$

Let W be the nonempty set of all such u, v in G. Since X is separable, there exists a countable dense subset D of W. In particular, the inequality (1) holds for any u, v in D with $u \leq x_0 \leq v$. By the Cantor property of Y we can find a y_0 in Y satisfying

$$T(u) \leq y_0 \leq T(v), \text{ for all } u, v \in D, \ u \leq x_0 \leq v.$$

Since T is continuous, the last double inequality remains true for all u, v in G with $u \leq x_0 \leq v$. Now, letting $T_1(x_0) = y_0$ we obtain a desired extension of T, namely, $T_1 : G_1 \to Y$, defined by

$$T_1(v + \lambda x_0) = T(v) + \lambda y_0.$$

Obviously, G_1 is again a majorizing subspace of X. Moreover, $T_1 : G_1 \to Y$ is positive. Indeed, let $v + \lambda x_0 \geq 0$ with $\lambda \neq 0$. If $\lambda > 0$ then $x_0 \geq -(1/\lambda)v$, which implies $y_0 \geq T(-(1/\lambda)v) = -(1/\lambda)T(v)$. Therefore, $T_1(x_0) \geq -(1/\lambda)T(v)$, and thus $T_1(v + \lambda x_0) \geq 0$. If $\lambda < 0$, we get the same result.

Finally, a routine application of Zorn's lemma will finish the proof. ∎

Recall that an *axial element* is an e in X_+ such that for each x in X there exists $\lambda > 0$ satisfying $x \leq \lambda e$.

Corollary 3. *Let X and Y be Banach lattices such that X is separable and contains an axial element e and Y has the Cantor property. Then for each y_0 in Y_+ there exists a positive linear operator $U : X \to Y$ with $U(e) = y_0$.*

Proof. Because e is an axial element of X, the linear hull $G = \mathrm{Sp}(e)$ is a majorizing subspace of X. We define $T : G \to Y$ by $T(\lambda e) = \lambda y_0$ and then apply Theorem 2. ∎

Before stating another corollary of Theorem 2, we remark that any linear subspace G of an ordered linear space X containing an element in the interior $\mathrm{Int}X_+$

of the positive cone X_+ of X is majorizing. Moreover, any positive linear operator from X into an ordered linear space Y vanishing in a majorizing subspace is necessarily zero.

Corollary 4. *Let X be a separable Banach lattice with* $\mathrm{Int}X_+ \neq \emptyset$, *and Y a Banach lattice with the Cantor property. Then for any linear subspace G of X disjoint from* $\mathrm{Int}X_+$, *there exists a nonzero positive linear operator $U : X \to Y$ with $U\,|_G = 0$.*

Proof. We choose an element x_0 from $\mathrm{Int}X_+$ and denote by G_0 the linear hull of $G \cup \{x_0\}$. It follows that G_0 is a majorizing subspace of X. Define $T_0 : G_0 \to Y$ by $T_0(v + \lambda x_0) = \lambda y_0$ for some fixed element y_0 in Y_+.

Let us prove that T_0 is positive. Let $v \in G$ and $\lambda \neq 0$ such that $v + \lambda x_0 \geq 0$. Suppose that $\lambda < 0$. Then $-\lambda x_0 \in \mathrm{Int}X_+$ and hence $v = v + \lambda x_0 + (-\lambda x_0) \in \mathrm{Int}X_+$. This conflicts with the hypothesis that $G \cap \mathrm{Int}X_+ = \emptyset$. So $\lambda > 0$ and hence $T_0(v + \lambda x_0) = \lambda y_0 \geq 0$. By Theorem 2, we can extend T_0 to a positive linear operator $U : X \to Y$. Obviously, $U\,|_G = 0$. ∎

The following results supplement Theorem 1. The first appears without proof in [7].

Theorem 5. *Suppose X and Y are ordered linear spaces, G is a linear subspace of X with the* (os)*-property, and Y has the strong (σ)-interpolation property. Let $T : G \to Y$ be an* (o)*-continuous linear operator and $P : X \to Y_+$ an* (o)*-continuous seminorm such that $T(v) \leq P(v)$ for all v in G. Then for any x_0 in $X \setminus G$ we can extend T to an* (o)*-continuous linear operator $T_1 : G_1 = \mathrm{Sp}(G \cup \{x_0\}) \to Y$ such that $T_1(z) \leq P(z)$ for all z in G_1.*

Proof. Because G has the (os)-property, there exists a countable subset D of G such that, for each v in G, there is a sequence $(v_n)_{n\in\mathbb{N}}$ in D with $v_n \xrightarrow{o} v$. If $u, v \in G$, then

$$\begin{aligned} T(u) - T(v) &= T(u - v) \leq P(u - v) \\ &= P((u + x_0) - (v + x_0)) \leq P(u + x_0) + P(v + x_0). \end{aligned}$$

So

$$-P(v + x_0) - T(v) \leq P(u + x_0) - T(u), \text{ for all } u, v \in G. \tag{2}$$

In particular, the inequality holds for all u, v in D. Using the strong (σ)-interpolation property of Y, we find a y_0 in Y such that

$$-P(v + x_0) - T(v) \leq y_0 \leq P(u + x_0) - T(u), \text{ for all } u, v \in D. \tag{3}$$

But T and P are (o)-continuous and hence the inequalities (3) hold for all u, v in G. Now, by letting

$$T_1(v + \lambda x_0) = T(v) + \lambda y_0,$$

we obtain a linear extension of T to G_1.

It remains to show that $T_1(v + \lambda x_0) \leq P(v + \lambda x_0)$ for all v in G and λ in $\mathbb{R}$, or, equivalently,

$$T(v) + \lambda y_0 \leq P(v + \lambda x_0) \text{ for all } v \in G \text{ and } \lambda \in \mathbb{R}. \tag{4}$$

If $\lambda = 0$, the inequality (4) is valid because $T_1 = T \leq P$ on G. If $\lambda > 0$, using the right inequality in (3), for $(1/\lambda)v$ instead of u, we obtain

$$y_0 \leq P\left(\frac{1}{\lambda}v + x_0\right) - T\left(\frac{1}{\lambda}v\right) = \frac{1}{\lambda}[P(v + \lambda x_0) - T(v)].$$

Therefore,

$$T(v) + \lambda y_0 \leq P(v + \lambda x_0).$$

If $\lambda < 0$, we use the left inequality in (3) to establish (4) instead.

Being dominated by the (o)-continuous seminorm P, the extension T_1 of T is (o)-continuous as well. ■

Corollary 6. *Suppose in Theorem* 5, *in addition*, *every linear subspace of* X *has the* (os)-*property. Then there exists an* (o)-*continuous linear operator* $S : X \to Y$ *such that* $S(v) = T(v)$ *for all* v *in* G, *and* $S(x) \leq P(x)$ *for all* x *in* X.

Proof. It follows from Theorem 5 and an application of Zorn's lemma. ■

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