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EXISTENCE OF STRONG SOLUTIONS TO SOME QUASILINEAR ELLIPTIC PROBLEMS ON BOUNDED SMOOTH DOMAINS

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Abstract. We consider the following quasilinear elliptic problems in a bounded smooth domain Z of $\mathbb{R}^N$, $N \geq 3$:

$$\begin{cases} Lu = \displaystyle\sum_{i,j=1}^{N} a_{ij}(x,u)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b_i(x,u)\frac{\partial u}{\partial x_i} + c(x,u)u = f(x) & \text{in } Z, \\ u = 0 & \text{on } \partial Z, \end{cases}$$

where $f(x) \in L^p(Z)$ and all the coefficients a_{ij}, b_i, c are Carathédory functions. Suppose that $a_{ij} \in C^{0,1}(\bar{Z} \times \mathbb{R})$, a_{ij}, $\partial a_{ij}/\partial x_i$, $\partial a_{ij}/\partial r$, b_i, $c \in L^\infty(Z \times \mathbb{R})$, $c \leq 0$ for $i, j = 1, ...N$ and the oscillations of $a_{ij} = a_{ij}(x,r)$ with respect to r are sufficiently small. A global estimate for a solution $u \in W^{2,p}(Z) \cap W_0^{1,p}(Z)$ is established and the existence of a strong solution $u \in W^{2,p}(Z) \cap W_0^{1,p}(Z)$ is proved for $p > N$.

Furthermore, we replace $f(x)$ by $f(x,r,\xi)$ which is defined on $Z \times \mathbb{R} \times \mathbb{R}^N$ and is a $Carath\acute{e}dory$ function. Assume that

$$|f(x,r,\xi)| \leq C_0 + h(|r|)|\xi|^\theta, \qquad 0 \leq \theta < 2,$$

where C_0 is a nonnegative constant, $h(|r|)$ is a locally bounded function, and $-c \geq \alpha_0 > 0$ for some constant α_0. We prove the existence of solution $u \in W^{2,p}(Z) \cap W_0^{1,p}(Z)$ for the equation $Lu = f(x,u,\nabla u)$.

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1. Introduction

Let Ω be a bounded $C^{1,1}$ domain in $\mathbb{R}^N$, $N \geq 3$, and L be the following elliptic operator in the general form:

$$Lu = \sum_{i,j=1}^{N} a_{ij}(x,u)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b_i(x,u)\frac{\partial u}{\partial x_i} + c(x,u)u, \qquad x \in \Omega$$

We study the existence of strong solutions to the following problems:

$$\begin{cases} Lu = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $f \in L^p(\Omega)$, and

$$\begin{cases} Lu = f(x,u,\nabla u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $f(x,r,\xi)$ has less than quadratic growth in ξ. All the coefficient functions a_{ij}, b_i, c and the function $f(x,r,\xi) : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ are Carathédory functions, that is, the function $x \mapsto f(x,r,\xi)$ is measurable for all $(r,\xi) \in \mathbb{R} \times \mathbb{R}^N$ and the function $(r,\xi) \mapsto f(x,r,\xi)$ is continuous for a.e $x \in \Omega$.

The basic idea is to consider a mapping F defined on $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ by letting $u = F(v)$ be the unique solution in $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ to the linear Dirichlet problem:

$$\begin{cases} L_v u = \sum_{i,j=1}^{N} a_{ij}(x,v)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b_i(x,v)\frac{\partial u}{\partial x_i} + c(x,v)u = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

The unique solvability of problem (1.2) is guaranteed by the linear existence result [1, p. 241] under appropriate coefficient conditions. We notice that F is well-defined for $p > N/2$. We shall then obtain solutions of problem (1.1) by finding fixed points of F.

The regularity theorem of Agmon-Douglis-Nirenberg [2] asserts that

$$\|u\|_{W^{2,p}} \leq C(\|u\|_{L^p} + \|L_v u\|_{L^p}), \tag{1.3}$$

where C is a constant dependent on the moduli of continuity of the coefficients $a_{ij}(x, v(x))$ on $\bar{\Omega}$, etc. If $a_{ij}(x,r) = a_{ij}(x)$, then the constant C in (1.3) is independent of v and by [1, p. 243], there exists a constant C independent of v such that

$$\|u\|_{W^{2,p}} \leq C\|L_v u\|_{L^p} = C\|f\|_{L^p}. \tag{1.4}$$

According to the uniqueness of problem (1.2), F is a continuous mapping in the topology of $W^{1,p}(\Omega)$ (Lemma 2.2.1). From (1.4), $\|u\|_{W^{2,p}} \leq K$ for some constant $K > 0$. Let

$$\mathcal{K} = \{v \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) | \quad \|v\|_{W^{2,p}(\Omega)} \leq K\}.$$

By the Sobolev imbedding theorem, $\mathcal{K}$ is a compact convex set in $W^{1,p}(\Omega)$. Applying the Schauder fixed point theorem, we then obtain a solution to problem (1.1).

In the general case $a_{ij} = a_{ij}(x, r)$, the essence of our consideration is to establish estimate (1.3) for which the constant C is independent of v. If $\Omega = B$ is a ball in $\mathbb{R}^N$, it has been shown in [3, Proposition 3.1.2] that

$$\|u\|_{W^{2,p}(B)} \leq C(\|u\|_{L^p(B)} + \|L_v u\|_{L^p(B)}), \tag{1.5}$$

where C is independent of v. In Section **2**, we intend to transform the coordinates in a bounded smooth domain Z into a ball B. By imposing stronger conditions on $a_{ij} \in C^{0,1}(\bar{Z} \times \mathbb{R})$ so that the oscillations with respect to r are sufficiently small, we have the same estimate of (1.5) in Proposition **2.1.1**. Together with the maximum principle of A. D. Aleksandrove [1, p. 220],

$$\sup_Z |u| \leq C\|f\|_{L^N(Z)},$$

where C is a nonnegative constant, we show that u is $W^{2,p}(Z)$ bounded. By the same argument as above, the existence of strong solutions to problem (1.1) is proved in Proposition **2.2.2**.

Based on the preceding results, in Section **3**, we further study the existence of strong solutions to the following quasilinear elliptic problem:

$$\begin{cases} Lu = f(x, u, \nabla u) & \text{in } Z, \\ u = 0 & \text{on } \partial Z. \end{cases} \tag{1.6}$$

Suppose that

$$-c \geq \alpha_0 > 0, \quad \text{for some constant } \alpha_0,$$

and $f(x, r, \xi)$ is a Carathédory function which satisfies

$$|f(x, r, \xi)| \leq C_0 + h(|r|)|\xi|^\theta,$$

where C_0 is a nonnegative constant, h is a locally bounded function and $0 \leq \theta < 2$. Then problem (1.6) has a strong solution $u \in W^{2,p}(Z) \cap W_0^{1,p}(Z)$ provided that the oscillations of a_{ij} with respect to r are sufficiently small. The result will be shown in Theorem **3.1**. To prove the theorem, we consider the approximation of problem (1.6). Denote the corresponding solutions by (u_n) (derived in Lemma 3.2). We first obtain a L^∞ bound of subsequence of (u_n) (Lemma 3.3), still relabeled as (u_n), and then establish a $W^{2,p}$ bound of (u_n) (Lemma 3.4). Finally, we pass the limit to verify that the limit u of (u_n) is a $W^{2,p}(Z) \cap W_0^{1,p}(Z)$ solution of problem (1.6).

The following notations are used in this paper. We denote by Ω, $\partial\Omega$, B, Z, and ∇u the open set in $\mathbb{R}^N$, the boundary of Ω, the ball in $\mathbb{R}^N$, the bounded smooth domain in $\mathbb{R}^N$, and the gradient of u, respectively. We define $C^{k,\alpha}(\bar{\Omega})$ to be the space of functions in $C^k(\bar{\Omega})$ consisting of function whose kth order partial derivatives are uniformly Hölder continuous with exponent α in Ω, $0 < \alpha \leq 1$, and C_0^∞ to be the space of functions in $C^\infty(\Omega)$ with compact support in Ω. Let $W^{m,p}(\Omega)$:=$\{u \in L^P(\Omega) \mid$ weak derivatives $D^\alpha u \in L^P(\Omega)$ for all $|\alpha| \leq m\}$ and $W_0^{m,p}$ be the closure of $C_0^\infty(\Omega)$ in $W^{m,p}(\Omega)$. We denote by $D^2u = [D_{ij}u]$ the Hessian matrix of second derivatives $D_{ij}u$ $(= \partial^2 u/\partial x_i \partial x_j)$, $i, j = 1, 2, ..., N$.

2. The Existence of Strong Solutions in Bounded Smooth Domains

Let Z be a bounded domain in $\mathbb{R}^N$ which is $C^{1,1}$ diffeomorphic to a ball B in $\mathbb{R}^N$, ψ be a $C^{1,1}$ diffeomorphism from $\bar{Z}$ onto a ball $\bar{B}$ in $\mathbb{R}^N$ and L be a second-order elliptic operator of the following form:

$$(2.0) \qquad Lu = \sum_{i,j=1}^{N} a_{ij}(x,u)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b_i(x,u)\frac{\partial u}{\partial x_i} + c(x,u)u \qquad x \in Z.$$

In this section, we consider the Dirichlet problem for $Lu = f(x)$ with $f \in L^p(Z)$. A global $W^{2,p}$ estimate for u $\in W^{2,p}(Z) \cap W_0^{1,p}(Z)$ is also established and is used to prove the existence of a strong solution u $\in W^{2,p}(Z) \cap W_0^{1,p}(Z)$.

2.1. Global Estimate

An operator L in (2.0) is said to be uniformly elliptic in Ω if there exists a constant $\lambda > 0$ such that

$$(2.1.1) \qquad \sum_{i,j=1}^{N} a_{ij}(x,r)\xi_i\xi_j \geq \lambda|\xi|^2 \qquad \text{for } (r,\xi) \in \mathbb{R} \times \mathbb{R}^N \text{ and } a.e.\ x \in \Omega.$$

For a fixed point $x \in \mathbb{R}^N$, we denote by osc $a_{ij}(x,r)$ the oscillation of a_{ij} with respect to r in $\mathbb{R}$, that is, osc $a_{ij}(x,r) = \sup\{|a_{ij}(x,r_1) - a_{ij}(x,r_2)| \mid r_1, r_2 \in \mathbb{R}\}$, and let

$$\text{osc } a(x,r) = \max_{1 \leq i,j \leq N} \text{osc } a_{ij}(x,r).$$

For $v \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$, let

$$L_v u = \sum_{i,j=1}^{N} a_{ij}(x,v)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b_i(x,v)\frac{\partial u}{\partial x_i} + c(x,v)u.$$

Recall the Marcinkiewicz Interpolation and Calderon-Zygmund theorems. The L^p estimate for a solution $u \in W_0^{2,p}(\Omega)$ of Poisson's equation in a domain Ω [1, p. 235] is given by

$$\|D^2 u\|_{L^p(\Omega)} \leq K\|\Delta u\|_{L^p(\Omega)}, \tag{2.1.2}$$

where $K = K(N, P)$ is a nonnegative constant. Notice that if Ω is a unit ball B, the global estimate of the $W^{2,p}(B)$ norm on u is given by [3, Proposition 3.1.2]

$$\|u\|_{W^{2,p}(B)} \leq C(\|u\|_{L^p(B)} + \|L_v u\|_{L^P(B)}), \tag{2.1.3}$$

where C is a constant (independent of v) dependent on $N, P, \lambda, \Lambda, \partial B$, B and the moduli of continuity of the coefficients $a_{ij}(x,r)$ with respect to x on $\bar{B}$, $|a_{ij}|, |b_i|, |c| \leq \Lambda$ and osc $a(x,r) < \lambda/4K$ $\forall x \in B$, osc $a(x,r) < \lambda/8N^2K \forall x \in \partial B$, K is a constant by (2.1.2). We start to establish a similar $W^{2,p}(Z)$ estimate as (2.1.3) for a bounded smooth domain Z of $\mathbb{R}^N$. A global $W^{2,p}(Z)$ estimate can be derived by using the diffeomorphism to transform the coordinates to B and then applying the $W^{2,p}(B)$ estimate. Therefore, we have the following proposition.

Proposition 2.1.1. *Let Z be a bounded smooth domain in $\mathbb{R}^N$ and the coefficients of L satisfies*

$$a_{ij} \in C^{0,1}(\bar{Z} \times \mathbb{R}),\ b_i, c \in L^\infty(Z \times \mathbb{R}),\ |a_{ij}|,\ |b_i|,\ |c| \leq \Lambda, \tag{2.1.4}$$

where Λ is a positive constant, $i, j = 1, ..., N$. Assume that there exists a $C^{1,1}$ diffeomorphism ψ from $\bar{Z}$ onto unit ball $\bar{B}$ in $\mathbb{R}^N$, $\psi(\partial Z) = \partial B$,

$$G = \begin{bmatrix} \frac{\partial \psi_1}{\partial x_1} & \cdots & \frac{\partial \psi_1}{\partial x_N} \\ \vdots & \ddots & \vdots \\ \frac{\partial \psi_N}{\partial x_1} & \cdots & \frac{\partial \psi_N}{\partial x_N} \end{bmatrix},$$

$$\text{osc } a(x,r) \leq \frac{\lambda}{4(\frac{\beta}{\alpha})K} \qquad \forall x \in Z, \tag{2.1.5}$$

$$\text{osc } a(x,r) \leq \frac{\lambda}{8N^2(\frac{\beta}{\alpha})K} \qquad \forall x \in \partial Z, \tag{2.1.6}$$

where

$$\xi(GG^T)\xi^T \geq \alpha|\xi|^2 \quad \textit{for some constant } \alpha > 0 \text{ ([4, P.539])}, \tag{2.17}$$

$$\beta = \max_{x \in \bar{Z}, 1 \leq i,j \leq N} \sum_{r,s}^{N} \left|\frac{\partial \psi_i(x)}{\partial x_r} \frac{\partial \psi_j(x)}{\partial x_s}\right| > 0, \ \textit{and } K \textit{ is a constant by } (2.1.2).$$

Then if $u \in W^{2,p}(Z) \cap W_0^{1,p}(Z)$ *and* $L_v u \in L^p(Z)$, *with* $1 < p < \infty$, *we have the estimate*

$$\|u\|_{W^{2,p}(Z)} \leq C(\|L_v u\|_{L^P(Z)} + \|u\|_{L^p(Z)}), \tag{2.1.8}$$

where C *is constant* (*independent of* v) *dependent on* $N, P, \lambda, \Lambda, \partial Z, Z, \psi$ *and the moduli of continuity of the coefficients* $a_{ij}(x,r)$ *with respect to* x *on* $\bar{Z}$.

Proof. $\psi = (\psi_1, ..., \psi_N)$ is $C^{1,1}$ diffeomorphism from $\bar{Z}$ onto $\bar{B}$. Let $y = \psi(x)$ for $x \in Z$, $\tilde{u}(y) = u(x)$, $\tilde{v}(y) = v(x)$ and $\tilde{L}_{\tilde{v}}\tilde{u}(y) = L_v u(x)$, where

$$\tilde{L}_{\tilde{v}}\tilde{u}(y) = \sum_{i,j=1}^{N} \tilde{a}_{ij}(y, \tilde{v}(y)) \frac{\partial^2 \tilde{u}}{\partial y_i \partial y_j} + \sum_{i=1}^{N} \tilde{b_i}(y, \tilde{v}(y)) \frac{\partial \tilde{u}}{\partial y_i} + c(y, \tilde{v}(y))\tilde{u} \quad \text{in } B,$$

$$\tilde{a}_{ij}(y, \tilde{v}(y)) = \sum_{r,s=1}^{N} \frac{\partial \psi_i}{\partial x_r} \frac{\partial \psi_j}{\partial x_s} a_{rs}(x, u(x)),$$

$$\tilde{b_i}(y, \tilde{v}(y)) = \sum_{r,s=1}^{N} \frac{\partial^2 \psi_i}{\partial x_r \partial x_s} a_{rs} + \sum_{r=1}^{N} \frac{\partial \psi_i}{\partial x_r} b_r(x, u(x)), \text{ and } \tilde{c}(y, \tilde{v}(y)) = c(x, u(x)).$$

It is readily seen that $\tilde{a}_{ij} \in C^{0,1}(\bar{B} \times \mathbb{R})$, $\tilde{b_i}$, $\tilde{c} \in L^\infty(B \times \mathbb{R})$. For all $\xi = (\xi_1, ..., \xi_N) \in \mathbb{R}^N$, we have

$$\mathbf{1^0} \quad \begin{aligned} \sum \tilde{a}\xi_i\xi_j &= \xi\tilde{a}\xi^T \\ &= (\xi G)a(\xi G)^T \\ &\geq \lambda|\xi G|^2 \\ &= \lambda(\xi G)(\xi G)^T \\ &= \lambda\xi G G^T \xi^T \\ &\geq \lambda\alpha|\xi|^2 = \tilde{\lambda}|\xi|^2 \quad \textit{by } (2.1.7), \textit{ where } \tilde{\lambda} = \alpha\lambda, \end{aligned}$$

$$\mathbf{2^0} \quad y \in B: \quad \begin{aligned} \text{osc } \tilde{a}(y,r) &= \max_{1 \leq i,j \leq N} \text{ osc } \tilde{a}_{ij}(y,r) \\ &\leq \max_{1 \leq i,j \leq N} \sum_{r,s} \left|\frac{\partial \psi_i(x)}{\partial x_r} \frac{\partial \psi_j(x)}{\partial x_s}\right| \text{ osc } a_{rs}(x,r) \\ &\leq \beta \frac{\lambda}{4(\frac{\beta}{\alpha})K} = \frac{\alpha\lambda}{4K} = \frac{\tilde{\lambda}}{4K} \quad \text{by (2.1.5)}, \end{aligned}$$

$$y \in \partial B: \quad \text{osc } \tilde{a}(y,r) \leq \frac{\tilde{\lambda}}{8N^2K} \quad \text{by (2.1.6)},$$

$$\mathbf{3^0} \quad |\tilde{a}_{ij}| \le \beta\Lambda \quad \forall i,j,\ |\tilde{c}| \le \Lambda,$$

$$|\tilde{b_i}| = \Big|\sum_{r,s=1}^{N} \frac{\partial^2 \psi_i}{\partial x_r \partial x_s} a_{rs} + \sum_{r=1}^{N} \frac{\partial \psi_i}{\partial x_r} b_r(x,u(x))\Big|$$
$$\le \beta_1 \Lambda \quad \forall i,$$

where

$$\max_{x\in\bar{Z},1\le i\le N} \Big|\sum_{r,s}^{N} \frac{\partial^2\psi_i}{\partial x_r\partial x_s}\Big| + \Big|\sum_{r,s}^{N} \frac{\partial \psi_i}{\partial x_r}\Big| = \beta_1 \text{ for a constant } \beta_1 > 0. \tag{2.19}$$

Hence we get $|\tilde{a}_{ij}|, |\tilde{b_i}|, |\tilde{c}| \le \tilde{\Lambda} = \max\{1,\beta_1,\beta\}\Lambda$, osc $\tilde{a}(y,r) \le \frac{\tilde{\lambda}}{4K}$ $\forall y \in B$ and osc $\tilde{a}(y,r) \le \frac{\tilde{\lambda}}{8N^2K}$ $\forall y \in \partial B$. Since the coefficient of $\tilde{L}$ satisfies the assumption of [3, Prop. 3.1.2], we have the global estimate of $W^{2,p}$ on $\tilde{u}$ by (2.1.3),

$$\|\tilde{u}\|_{W^{2,p}(B)} \le C(\|\tilde{u}\|_{L^p(B)} + \|\tilde{L}_{\tilde{v}}\tilde{u}\|_{L^P(B)}),$$

where $C = C(N,p,\tilde{\lambda},\tilde{\Lambda},\psi)$ and C is independent of v. Since G is a nonsingular bounded operator for all $x \in \bar{Z}$, we have

$$\int_B |\tilde{u}(y)|^p dy = \int_Z |u(x)|^p |J\psi(x)| dx$$
$$\le \max_{x\in\bar{Z}} |\det G| \int_Z |u(x)|^p dx,$$

where $J\psi(x) = \det G$, where implies that $\|\tilde{u}\|_{L^p(B)} \le \sigma\|u\|_{L^p(Z)}$, where $\sigma = (\max_{x\in\bar{Z}} |\det G|)^{1/p} > 0$. Similarly, we obtain

$$\|\tilde{L}_{\tilde{v}}\tilde{u}\|_{L^P(B)} \le \sigma\|L_v u\|_{L^P(Z)},$$

$$\int_Z |u(x)|^p dx = \int_B |\tilde{u}(y)|^p |J\psi^{-1}(y)| dy$$
$$\le \max_{y\in\bar{B}} |J\psi^{-1}(y)| \int_B |\tilde{u}(u)|^p dy$$

implies that $\|u\|_{L^p(Z)} \le \rho\|\tilde{u}\|_{L^p(B)}$, where $\rho = (\min_{x\in\bar{Z}} |\det G|)^{-1/p} > 0$,

$$
\begin{aligned}
&\left\|\frac{\partial u}{\partial x_i}\right\|_{L^p(Z)} = \left\|\sum_r^N \frac{\partial \tilde{u}}{\partial y_r}\frac{\partial y_r}{\partial x_i}\right\|_{L^p(Z)} \\
&\le (\int_Z |\frac{\partial \tilde{u}(\psi(x))}{\partial y_1}\frac{\partial y_1}{\partial x_i}|^p dx)^{\frac{1}{p}} + ... + (\int_Z |\frac{\partial \tilde{u}(\psi(x))}{\partial y_N}\frac{\partial y_N}{\partial x_i}|^p dx)^{\frac{1}{p}} \\
&\le \max_{x\in\bar{Z},1\le r\le N} |\frac{\partial y_r}{\partial x_i}|[(\int_Z |\frac{\partial \tilde{u}(\psi(x))}{\partial y_1}|^p dx\Big)^{\frac{1}{p}} + ... + (\int_Z |\frac{\partial \tilde{u}(\psi(x))}{\partial y_N}|^p dx)^{\frac{1}{p}}] \\
&\le \max_{x\in\bar{Z},1\le r\le N} |\frac{\partial y_r}{\partial x_i}|[(\int_B |\frac{\partial \tilde{u}(y)}{\partial y_1}|^p |J\psi^{-1}(y)|dy)^{\frac{1}{p}} + ... + (\int_Z |\frac{\partial \tilde{u}(y)}{\partial y_N}|^p |J\psi^{-1}(y)|dy)^{\frac{1}{p}}] \\
&\le \max_{x\in\bar{Z},1\le r\le N} |\frac{\partial y_r}{\partial x_i}|\rho \sum_r^N \left\|\frac{\partial \tilde{u}}{\partial y_r}\right\|_{L^p(B)} \\
&\le \beta_1\rho \sum_r^N \left\|\frac{\partial \tilde{u}}{\partial y_r}\right\|_{L^p(B)} \qquad \text{by (2.1.9),}
\end{aligned}
$$

which implies that

$$
\left\|\frac{\partial u}{\partial x_i}\right\|_{L^p(Z)} \le \beta_1\rho \sum_r^N \left\|\frac{\partial \tilde{u}}{\partial y_r}\right\|_{L^p(B)}
$$

for all i, and

$$
\begin{aligned}
\left\|\frac{\partial^2 u}{\partial x_i \partial x_j}\right\|_{L^p(Z)} &= \left\|\sum_{r,s}^N \frac{\partial^2 \tilde{u}}{\partial y_r \partial y_s}\frac{\partial y_r}{\partial x_i}\frac{\partial y_s}{\partial x_j} + \sum_r^N \frac{\partial \tilde{u}}{\partial y_r}\frac{\partial^2 y_r}{\partial x_i \partial x_j}\right\|_{L^p(Z)} \\
&\le \left\|\sum_{r,s}^N \frac{\partial^2 \tilde{u}}{\partial y_r \partial y_s}\frac{\partial y_r}{\partial x_i}\frac{\partial y_s}{\partial x_j}\right\|_{L^p(Z)} + \left\|\sum_r^N \frac{\partial \tilde{u}}{\partial y_r}\frac{\partial^2 y_r}{\partial x_i \partial x_j}\right\|_{L^p(Z)} \\
&\le \max_{x\in\bar{Z},1\le r,s\le N} |\frac{\partial y_r}{\partial x_i}\frac{\partial y_s}{\partial x_j}| \sum_{r,s}^N \left\|\frac{\partial^2 \tilde{u}}{\partial y_r \partial y_s}\right\|_{L^p(Z)} \\
&\quad + \max_{x\in\bar{Z},1\le r\le N} |\frac{\partial^2 y_r}{\partial x_i \partial x_j}| \sum_{r,s}^N \left\|\frac{\partial \tilde{u}}{\partial y_r}\right\|_{L^p(Z)} \\
&\le \beta\rho \sum_{r,s}^N \left\|\frac{\partial^2 \tilde{u}}{\partial y_r \partial y_s}\right\|_{L^p(B)} + \beta_1\rho \sum_{r,s}^N \left\|\frac{\partial \tilde{u}}{\partial y_r}\right\|_{L^p(B)},
\end{aligned}
$$

which implies that

$$
\left\|\frac{\partial^2 u}{\partial x_i \partial x_j}\right\|_{L^p(Z)} \le \beta\rho \sum_{r,s}^N \left\|\frac{\partial^2 \tilde{u}}{\partial y_r \partial y_s}\right\|_{L^p(B)} + \beta_1\rho \sum_{r,s}^N \left\|\frac{\partial \tilde{u}}{\partial y_r}\right\|_{L^p(B)}
$$

for all i and j. To summarize, we can obtain that

$$\|u\|_{W^{2,p}(Z)} \leq \eta \|\tilde{u}\|_{W^{2,p}(B)},$$

where η is a nonnegative constant dependent of ψ. Thus, returning to our original coordinate Z, we have got our estimates,

$$\|u\|_{W^{2,p}(Z)} \leq C(\|u\|_{L^p(Z)} + \|L_v u\|_{L^P(Z)}),$$

where $C = C(N, p, \lambda, \Lambda, Z, \partial Z, \psi)$. ■

2.2. Existence Results

The results of the preceding section will now be applied to establish the existence of solutions of the following quasilinear elliptic problem:

$$(2.2.1)\quad \begin{cases} Lu = \displaystyle\sum_{i,j=1}^{N} a_{ij}(x,u)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b_i(x,u)\frac{\partial u}{\partial x_i} + c(x,u)u = f(x) & \text{in } Z, \\ u = 0 & \text{on } \partial Z. \end{cases}$$

where $f \in L^p(Z)$, $p \geq N$. For the moment, we suppose $a_{ij} \in C^{0,1}(\bar{Z} \times \mathbb{R})$, a_{ij}, $\partial a_{ij}/\partial x_i$, $\partial a_{ij}/\partial r$, b_i, c are bounded Carathédory functions, with $c \leq 0$. By the existence and uniqueness theorem of the strong solution for the Dirichlet problem [1, p. 241], there exists a unique solution $u \in W^{2,p}(Z) \cap W_0^{1,p}(Z)$ to the equation $L_v u = f(x)$ for each $v \in W_0^{1,p}$. Consider the mapping F which assigns $v \in W^{2,p}(Z) \cap W_0^{1,p}(Z)$ to the solution $u \in W^{2,p}(Z) \cap W_0^{1,p}(Z)$ satisfying the following equation

$$(2.2.2)\qquad L_v u = \sum_{i,j=1}^{N} a_{ij}(x,v)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b_i(x,v)\frac{\partial u}{\partial x_i} + c(x,v)u = f(x) \quad x \in Z,$$

i.e., $F : v \in W^{2,p}(Z) \cap W_0^{1,p}(Z) \mapsto F(v) = u \in W^{2,p}(Z) \cap W_0^{1,p}(Z)$ (F is well-defined provided $p > N/2$). From the following theorem, we can obtain the L^∞ estimate for the solution $u = F(v)$ to equation (2.2.2).

Weak Maximum Principle of A. D. Aleksandrov [1, p. 220]:
Consider

$$Lu = \sum_{i,j=1}^{N} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b_i(x)\frac{\partial u}{\partial x_i} + c(x)u = f(x),$$

where L is elliptic in the domain Ω, and the coefficient matrix $A = [a_{ij}]$ is positive definite everywhere in Ω. For such operators, we will let D denote the determinant of A and set $D^* = D^{1/n}$ so that D^* is the geometric mean of the eigenvalues of A such that $0 < \omega \le D^* \le \gamma$, where ω and γ are the minimum and maximum eigenvalues of A respectively. If $|b|/D^*$, $f/D^* \in L^N(\Omega)$, $c \le 0$ in Ω, $u \in C^0(\bar{\Omega}) \cap W^{2,N}_{\text{loc}}(\Omega)$, and $Lu \ge f$ in bounded domain Ω, then

$$\sup_{\Omega} u \le \sup_{\partial\Omega} u^+ + C \left\| \frac{f}{D^*} \right\|_{L^N(\Omega)},$$

where C is a constant dependent on N, diam Ω, and $\|b/D^*\|_{L^N(\Omega)}$.

For the equation (2.2.2), u is zero on the boundary of Z. Since a_{ij} is bounded, $D^* = D^{1/N}$ is a bounded function and $0 < \lambda \le D^*$, where λ is an ellipticity constant in (2.1.1). For $p \ge N$, we then have $f \in L^N(Z)$ and

$$\sup_{Z} |u| \le C\|f\|_{L^N(Z)}, \tag{2.2.3}$$

where C is a constant dependent on N, λ, Λ, and diam Z (the maximum principle is valid for $p \ge N$). With the aid of (2.1.8), we have the following inequality

$$\|u\|_{W^{2,p}} \le C\|f\|_{L^p(Z)} \quad \text{for all } u = F(v), v \in W^{2,p}(Z) \cap W^{1,p}_0(Z). \tag{2.2.4}$$

We proceed to show that there exists a fixed point u of F; u then is a solution of the problem (2.2.1) by the Schauder Fixed Point Theorem. It suffices to show that $F : \mathcal{K} \to \mathcal{K}$ is continuous and $\mathcal{K}$ is a compact convex set in a Banach space. We have the following lemma.

Lemma 2.2.1. *Let $p \ge N$. Under the hypotheses of Proposition* 2.1.1, *the mapping $F : W^{2,p}(Z) \cap W^{1,p}_0(Z) \to W^{2,p}(Z) \cap W^{1,p}_0(Z)$ is continuous in the topology of $W^{1,p}(Z)$.*

Proof: If $\{v_n\} \subset W^{2,p}(Z) \cap W^{1,p}_0(Z)$ and $v_n \to v$ in $W^{1,p}(Z)$, then there exists a subsequence, denoted by v_n, such that $v_n \to v$ a.e., and $\nabla v_n \to \nabla v$ a.e. Let $u_n = F(v_n)$ and $u = F(v)$. We will show that $u_n \to u$ in $W^{1,p}(Z)$. Since $f \in L^p(Z)$, and $p \ge N$, by (2.2.4), $\{u_n\}$ is bounded in $W^{2,p}(Z)$. Also since $W^{2,p}(Z) \hookrightarrow W^{1,p}(Z)$ is a compact imbedding, there exists a subsequence (we relabel as $\{u_n\}$) such that $u_n \to w$ in $W^{1,p}(Z)$ with $w \in W^{1,p}(Z)$, $u_n \to w$ a.e., and $\nabla u_n \to \nabla w$ a.e. We claim that w is a weak solution of the following equation

$$\sum_{i,j=1}^{N} a_{ij}(v) \frac{\partial^2 w}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b_i(x,v) \frac{\partial w}{\partial x_i} + c(x,v)w = f(x). \tag{2.2.5}$$

It suffices to show that

$$\begin{aligned} &\int_Z \sum_{i,j=1}^{N} a_{ij}(v) \frac{\partial w}{\partial x_j} \frac{\partial \phi}{\partial x_i} + \int_Z \sum_{i=1}^{N} [\sum_{j=1}^{N} (\frac{\partial a_{ji}(v)}{\partial x_j} + \frac{\partial a_{ji}(v)}{\partial r} \frac{\partial v}{\partial x_j}) - b_i(v)] \frac{\partial w}{\partial x_i} \phi \\ &\quad + \int_Z (-c(x,v)) w\phi = \int_Z -f\phi \qquad \text{for all } \phi \in C^\infty_0(Z). \end{aligned} \tag{2.2.6}$$

Let $\phi \in C_0^\infty(Z)$. Since $u_n = F(v_n)$, we have

$$\int_Z \sum_{i,j=1}^{N} a_{ij}(v_n)\frac{\partial u_n}{\partial x_j}\frac{\partial \phi}{\partial x_i} + \int_Z \sum_{i=1}^{N}[\sum_{j=1}^{N}(\frac{\partial a_{ji}(v_n)}{\partial x_j} + \frac{\partial a_{ji}(v_n)}{\partial r}\frac{\partial v_n}{\partial x_j}) - b_i(v_n)]\frac{\partial u_n}{\partial x_i}\phi$$
$$+ \int_Z (-c(v_n))u_n\phi = \int_Z -f\phi.$$

Since a_{ij}, $\partial a_{ij}/\partial x_i$, $\partial a_{ij}/\partial r$, b_i, c are bounded Carathédory functions, $u_n \to w$ a.e., $\nabla u_n \to \nabla w$ a.e., by Lebesgue's Dominated Convergence Theorem we have

$$\int_Z \sum_{i,j=1}^{N} a_{ij}(v_n)\frac{\partial u_n}{\partial x_j}\frac{\partial \phi}{\partial x_i} + \int_Z \sum_{i=1}^{N}[\sum_{j=1}^{N}(\frac{\partial a_{ji}(v_n)}{\partial x_j} + \frac{\partial a_{ji}(v_n)}{\partial r}\frac{\partial v_n}{\partial x_j}) - b_i(v_n)]\frac{\partial u_n}{\partial x_i}\phi$$
$$+ \int_Z (-c(v))w\phi = \int_Z (-f)\phi \qquad \text{for all } \phi \in C_0^\infty(Z).$$

Hence (2.2.5) holds. It follows from the uniqueness of the solution to equation (2.2.2) that we have $u = w$ and $u_n \to u$ in $W^{1,p}(Z)$. Therefore, the proof is completed. ∎

Proposition 2.2.2. *Let Z be a bounded smooth domain in $\mathbb{R}^N$ satisfying the assumption of Proposition* 2.1.1. *Suppose $a_{ij} \in C^{0,1}(\bar{Z} \times \mathbb{R})$, a_{ij}, $\partial a_{ij}/\partial x_i$, $\partial a_{ij}/\partial r$, b_i, $c \in L^\infty(Z \times \mathbb{R})$, $c \leq 0$ with $i, j = 1, ...N$. Then, for $p \geq N$, there exist a solution $u \in W^{2,p}(Z) \cap W_0^{1,p}(Z)$ to problem* (2.2.1).

Proof. Consider $u = F(v)$ for $v \in W^{2,p}(Z) \cap W_0^{1,p}(Z)$. According to (2.2.4), we can obtain a nonnegative constant K, such that

$$\|u\|_{W^{2,p}(Z)} \leq K \quad \text{for } v \in W^{2,p}(Z) \cap W_0^{1,p}(Z).$$

Let

$$\mathcal{K} = \{v \in W^{2,p}(Z) \cap W_0^{1,p}(Z) | \quad \|v\|_{W^{2,p}(Z)} \leq K\}.$$

Then F is continuous from $\mathcal{K}$ into itself in the topology of $W^{1,p}$ by Lemma 2.2.1. Since $\mathcal{K}$ is bounded in $W^{2,p}(Z)$ and $W^{2,p} \hookrightarrow W^{1,p}$ is a compact imbedding, $\mathcal{K}$ is a precompact set in $W^{1,p}(Z)$. We claim that $\mathcal{K}$ is closed in $W^{1,p}(Z)$. To see this, let $\{u_n\} \subset \mathcal{K}$ be such that $u_n \to u$ in $W^{1,p}(Z)$. Since $\{u_n\}$ is bounded in $W^{2,p}$ and $W^{2,p}$ is a reflexive space, there exists a subsequence weakly convergent to $w \in W^{2,p}$. It can be shown that $w = u$. With the aid of $\|u\|_{W^{2,p}} \leq \underline{\lim}_n \|u_n\|_{W^{2,p}} \leq K$, we obtain that $\mathcal{K}$ is closed in $W^{1,p}$. Hence $\mathcal{K}$ is a compact and convex set in $W^{1,p}$ which is a Banach space. It follows readily from the Schauder Fixed Point Theorem that there exists a solution $u \in W^{2,p}(Z) \cap W_0^{1,p}(Z)$ of problem (2.2.1) in $\mathcal{K}$. ∎

Remark 2.2.3. It follows from the proof of Proposition 2.2.2 that the solutions of equation (2.2.1) are bounded in $W^{2,p}(Z)$.

3. An Application to the Existence of Strong Solutions to ome Quasilinear Elliptic Problems

In this section, we consider the following quasilinear elliptic problem:

$$(3.1)\quad \begin{cases} \sum_{i,j=1}^{N} a_{ij}(x,u)\dfrac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b_i(x,u)\dfrac{\partial u}{\partial x_i} + c(x,u)u = f(x,u,\nabla u) & \text{in } Z, \\ u = 0 & \text{on } \partial Z, \end{cases}$$

where Z is a smooth domain in $\mathbb{R}^N$, $a_{ij} \in C^{0,1}(\bar{Z} \times \mathbb{R})$, a_{ij}, $\partial a_{ij}/\partial x_i$, $\partial a_{ij}/\partial r$, b_i, c, $f(x,r,\xi)$ are Carathéodory functions and $\sum_{i,j=1}^{N} a_{ij}\xi_i\xi_j \geq \lambda|\xi|^2$ with a nonnegative constant λ. The results of Section **2** are used to prove the following theorem.

Theorem 3.1. *Let Z be a bounded smooth domain in $\mathbb{R}^N$ satisfying the assumption of Proposition* 2.1.2. *Suppose $a_{ij} \in C^{0,1}(\bar{Z} \times \mathbb{R})$, a_{ij}, $\partial a_{ij}/\partial x_i$, $\partial a_{ij}/\partial r$, b_i, $c \in L^\infty(Z \times \mathbb{R})$ with $i,j = 1,...N$, $-c \geq \alpha_0 > 0$ for some constant α_0 and*

$$(3.2)\qquad |f(x,r,\xi)| \leq C_0 + h(|r|)|\xi|^\theta \qquad 0 \leq \theta < 2,$$

where C_0 is a nonnegative constant and $h(|r|)$ is a locally bounded function. Then there exists a solution $u \in W^{2,p}(Z) \cap W_0^{1,p}(Z)$ to problem (3.1).

The proof of Theorem 3.1 is done in the following steps:

(1) Approach equation (3.1) by truncation, and then prove the existence of approximating solutions $\{u_n\}$.
(2) Establish L^∞ bound for the subsequence of $\{u_n\}$.
(3) Establish $W^{2,p}$ bound for the subsequence of $\{u_n\}$.
(4) Pass the approximating problem to the limit.
(5) Verify that the limit u of the subsequence of approximating solutions $\{u_n\}$ in $W_0^{1,p}$ belongs to $W^{2,p} \cap W_0^{1,p}$.

Lemma 3.2. *Suppose that $f(x,r,\xi)$ has an L^∞ bound. Then for $1 \leq p < \infty$ there exists a solution $u \in W^{2,p}(Z) \cap W_0^{1,p}(Z)$ to problem* (3.1) *under the assumption of Theorem* 3.1.

Proof: For each $v \in W^{1,p}(Z)$, $f(x,v,\nabla v) \in L^\infty(Z) \subset L^p(Z)$, the existence and uniqueness theorem [1, p. 241] asserts that there exists a unique $u \in W^{2,p}(Z) \cap W_0^{1,p}(Z)$ to the equation

$$L_v u = \sum_{i,j=1}^{N} a_{ij}(x,v)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b_i(x,v)\frac{\partial u}{\partial x_i} + c(x,v)u = f(x,v,\nabla v).$$

Moreover, by Proposition 2.1.1, we have the global estimate

$$\|u\|_{W^{2,p}(Z)} \leq C(\|u\|_{L^p(Z)} + \|f(x, v, \nabla v)\|_{L^p(Z)}),$$

with a constant $C > 0$ (independent of v). Without loss of generality, we assume $p \geq N$. Since $f \in L^\infty$, from the Maximum Principle of A. D. Aleksandrov [1, p. 220], we obtain $\|u\|_{W^{2,p}(Z)} \leq M$ for some constant $M > 0$. Following the same process in subsection 2.2, let T be the map which associates $v \in W^{2,p}(Z) \cap W_0^{1,p}(Z)$ to the solution $u \in W^{2,p}(Z) \cap W_0^{1,p}(Z)$ satisfying $L_v u = f(x, v, \nabla v)$. Notice that since $f(x, r, \xi)$ is a bounded Carathédory function, we have $f(x, v_n, \nabla v_n) \to f(x, v, \nabla v)$ in $L^1(Z)$ if $v_n \to v$ in $W^{1,p}$ and $v_n \to v$, $\nabla v_n \to \nabla v$ a.e. By a similar argument as in the proof of Lemma 2.2.1, we can show that T: $W^{2,p}(Z) \cap W_0^{1,p}(Z) \to W^{2,p}(Z) \cap W_0^{1,p}(Z)$ is continuous in the topology $W^{1,p}(Z)$. The existence of the solution $u \in W^{2,p}(Z) \cap W_0^{1,p}(Z)$ to problem (3.1) then follows from Proposition 2.2.2. ■

Let's now consider the approximating problem of problem (3.1):

$$(3.3) \quad \begin{cases} \displaystyle\sum_{i,j=1}^{N} a_{ij}(x, u) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b_i(x, u) \frac{\partial u}{\partial x_i} + c(x, u) u = f_n(x, u, \nabla u) & \text{in } Z, \\ u = 0 & \text{on } \partial Z, \end{cases}$$

where $f_n(x, r, \xi)$ is the truncation of f by $\pm n$, i.e.,

$$f_n(x, r, \xi) = \begin{cases} n & \text{if } f(x, r, \xi) \geq n, \\ f(x, r, \xi) & \text{if } |f(x, r, \xi)| \leq n, \\ -n & \text{if } f(x, r, \xi) \leq -n. \end{cases}$$

Clearly, $f_n(x, r, \xi) \in L^\infty(Z) \subset L^p(Z)$ for all n. According to Lemma 3.2, for each $1 \leq p < \infty$, there exists a solution $u_n \in W^{2,p}(Z) \cap W_0^{1,p}(Z)$ to the approximating problem (3.3). Without loss of generality, we assume $p > N \geq 3$.

Lemma 3.3. *Under the assumption of Theorem* 3.1, *there exists a subsequence of the approximating solution* $\{u_n\}$ *to problem* (3.1) *which is* L^∞ *bounded.*

Proof: Since $a_{ij} \in C^{0,1}(\bar{Z} \times \mathbb{R})$, the problem in (3.1) can be written in the following divergence form:

$$(3.4) \qquad -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i} a_{ij}(x, u) \frac{\partial u}{\partial x_j} + \tilde{f}(x, u, \nabla u) = 0,$$

where

$$\tilde{f}(x,u,\nabla u) = \sum_{i=1}^{N}\{\sum_{j=1}^{N}[\frac{\partial a_{ji}(x,u)}{\partial x_j} + \frac{\partial a_{ji}(x,u)}{\partial r}\frac{\partial u}{\partial x_j}] - b_i(x,u)\}\frac{\partial u}{\partial x_i}$$
$$-c(x,u)u + f(x,u,\nabla u).$$

Since a_{ij}, $\partial a_{ij}/\partial x_i$, $\partial a_{ij}/\partial r$, b_i, $c \in L^\infty(Z\times\mathbb{R})$ with $i,j = 1,...N$, there exists a constant $\Lambda > 0$ such that a_{ij}, $\partial a_{ij}/\partial x_i$, $\partial a_{ij}/\partial r$, b_i, $c \leq \Lambda$. Thus,

$$|\sum_{i,j=1}^{N}[\frac{\partial a_{ji}(x,u)}{\partial x_j}\frac{\partial u}{\partial x_i} + \frac{\partial a_{ji}(x,u)}{\partial r}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}] - \sum_{i=1}^{N} b_i(x,u)\frac{\partial u}{\partial x_i} - c(x,u)u|$$
$$\leq \frac{\Lambda N}{2}(N + |\nabla u|^2) + \Lambda(|\nabla u|^2) + \frac{\Lambda}{2}(N + |\nabla u|^2) + \Lambda|u|$$
$$\leq M|\nabla u|^2 + \Lambda|u| + C',$$

for some nonnegative constants $M = (\Lambda/2)(N+3/2)$, $C' = (\Lambda N/2)(N+1)$. Together with the hypothesis of (3.2), we have

$$|\tilde{f}(x,r,\xi)| \leq C_0 + h(|r|)(1+|\xi|^2) + M|\xi|^2 + \Lambda|r| + C'$$
$$\leq b(|r|)(1+|\xi|^2),$$

where b is an increasing function from $\mathbb{R}^+$ into $\mathbb{R}^+$. Let $\phi = -C_0/\alpha_0$ and $\psi = C_0/\alpha_0$. It's clear that ϕ and ψ are the sub- and supper-solution of problem (3.1), respectively. Thus, it follows from [5, Proposition 3.6] that there is a subsequence of the approximating sequence of solutions $\{u_n\}$ to problem (3.1) (we relabel as (u_n)) with $\phi \leq u_n \leq \psi$ in Z. Hence (u_n) are $L^\infty(Z)$ bounded. ■

Theorem (Interpolation Inequality of Gagliardo-Nirenberg).

Let $\Omega \subset \mathbb{R}^N$ be an open bounded regular set and $u \in L^r \cap W^{2,p}(\Omega)$ with $1 \leq p \leq \infty$ and $1 \leq r \leq \infty$. Then $u \in W^{1,q}(\Omega)$ where q is the harmonic average of p and r, that is $1/q = ((1/2)+(1/p))/2$ and

(3.5) $$\|\nabla u\|_{L^q} \leq C\|u\|_{W^{2,p}}^{\frac{1}{2}}\|u\|_{L^r}^{\frac{1}{2}}.$$

In particular $r = \infty$ and then $q = 2p$. We have $u \in W^{1,2p}(\Omega)$ and

(3.6) $$\|\nabla u\|_{L^{2p}} \leq C\|u\|_{W^{2,p}}^{\frac{1}{2}}\|u\|_{L^\infty}^{\frac{1}{2}}.$$

Lemma 3.4. *Under the assumptions of Theorem* 3.1, *there exists a subsequence of the approximating solution $\{u_n\}$ in $W^{2,p}(Z)\cap W_0^{1,p}(Z)$ to problem* (3.1) *which is $W^{2,p}$ bounded.*

Proof. By Lemma 3.3, there exists a sequence $\{u_n\}$ which is L^∞ bounded. Since $h(|r|)$ is locally bounded, we have $|h(u_n)| \le M$ for some constant $M > 0$. According to (3.2), we have $|f_n(x, u_n, \nabla u_n)| \le C_0 + h(|u_n|)|\nabla u_n|^\theta$, $0 \le \theta < 2$. Hence there exists a constant $C_1 > 0$ such that

$$|f_n(x, r, \xi)| \le C_1(1 + |\nabla u_n|^\theta). \tag{3.7}$$

Since $\theta < 2$, there exists a constant $C_\epsilon > 0$ for all $\epsilon > 0$ such that

$$|\nabla u_n|^\theta \le C_\epsilon + \epsilon|\nabla u_n|^2. \tag{3.8}$$

Thus $|f_n(x, u_n, \nabla u_n)| \le M_1 + \epsilon C_1|\nabla u_n|^2$ for a constant $M_1 > 0$. With the help of the global estimate (2.1.9), we have

$$\begin{aligned}\|u_n\|_{W^{2,p}(Z)} &\le C(\|u_n\|_{L^p} + \|f_n(x, u_n, \nabla u_n)\|_{L^p}) \\ &\le M_2 + \epsilon C_2\|\nabla u_n\|^2_{L^{2P}}\end{aligned}$$

for some constant M_2, $C_2 > 0$. Since $u_n \in L^\infty(Z) \cap W^{2,p}(Z)$, from the interpolation of Gagliardo-Nirenberg Theorem, we obtain

$$\begin{aligned}\|u_n\|_{W^{2,p}(Z)} &\le M_2 + \epsilon C_2\|u_n\|_{W^{2,P}(Z)}\|u_n\|_{L^\infty(Z)} \\ &\le M_2 + \epsilon C_3\|u_n\|_{W^{2,P}(Z)},\end{aligned}$$

where M_2, C_3 are nonnegative constants. Hence, by choosing $C_3\epsilon = 1/2$, we obtain $\|u_n\|_{W^{2,p}(Z)} \le M_3$ for some constant $M_3 > 0$. Therefore, $\{u_n\}$ are $W^{2,p}$ bounded. ∎

By Lemma 3.4, we get a sequence of approximating solutions to problem (3.1) which is $W^{2,p}$ bounded. It follows from the compactness of the imbedding $W^{2,p} \hookrightarrow W^{1,p}$ that there exists a norm convergent subsequence in $W^{1,p}$. We extract a subsequence, which is denoted again by $\{u_n\}$ such that

$$u_n \to u \quad \text{a.e.}, \quad \nabla u_n \to \nabla u \quad \text{a.e.}, \quad \text{and} \quad u_n \to u \quad \text{in } W^{1,p}.$$

In what follows, we show that u is a solution of problem (3.1). By passing to the limit, we obtain

$$\begin{aligned}&\int \sum_{i,j=1}^N a_{ij}(u_n)\frac{\partial u_n}{\partial x_j}\frac{\partial \phi}{\partial x_i} + \int \sum_{i,j=1}^N \Big[\frac{\partial a_{ji}(u_n)}{\partial x_j} + \frac{\partial a_{ji}(u_n)}{\partial r}\frac{\partial u_n}{\partial x_j}\Big]\frac{\partial u_n}{\partial x_i}\phi \\ &- \int \sum_{i=1}^N b_i(u_n)\frac{\partial u_n}{\partial x_i}\phi - \int c(u_n)u_n\phi = \int -f_n(x, u_n, \nabla u_n)\phi \\ \to &\int \sum_{i,j=1}^N a_{ij}(u)\frac{\partial u}{\partial x_j}\frac{\partial \phi}{\partial x_i} + \int \sum_{i,j=1}^N \Big[\frac{\partial a_{ji}(u)}{\partial x_j} + \frac{\partial a_{ji}(u)}{\partial r}\frac{\partial u}{\partial x_j}\Big]\frac{\partial u}{\partial x_i}\phi \\ &- \int \sum_{i=1}^N b_i(u)\frac{\partial u}{\partial x_i}\phi - \int c(u)u\phi \quad \forall \phi \in C_0^\infty(Z).\end{aligned}$$

The next lemma shows that $f_n(x, u_n, \nabla u_n) \to f(x, u, \nabla u)$ in $L^1(Z)$. Therefore, u is a $W^{1,p}(Z)$ solution to the problem (3.1).

Theorem (Vitali Convergence Theorem).

Let $1 \leq p \leq \infty$ and (Ω, Σ, μ) be a measurable space. Let $\{f_n\}$ be a sequence of functions in L^p converging almost everywhere to a function f. Then f is in L^p and $\|f_n - f\|_p$ converges to zero if and only if

(1) $\lim_{\mu(E)\to 0} \int_E |f_n|^p d\mu = 0$ *uniformly* $\forall n$;

(2) *for each $\epsilon > 0$ there exists a set E_ϵ such that $\mu(E_\epsilon) < \infty$ and $\int_{\Omega - E_\epsilon} |f_n|^p d\mu < \epsilon$ for $n = 1, 2, \ldots$.*

Lemma 3.5. $f_n(x, u_n, \nabla u_n) \to f(x, u, \nabla u)$ *in* $L^1(Z)$.

Proof. Since f is a Carathédory function, $u_n \to u$ a.e., and $\nabla u_n \to \nabla u$ a.e. we have $f_n(x, u_n, \nabla u_n) \to f(x, u, \nabla u)$ a.e. According to (3.7), we have

$$\begin{aligned} |f_n(x, u_n, \nabla u_n)| &\leq C_1(1 + |\nabla u_n|^\theta) \\ &\leq C_1(2 + |\nabla u_n|^2). \end{aligned}$$

Since $\{u_n\}$ is H^1 bounded with $p > N \geq 3$, $\{f_n\}$ is a sequence of functions in $L^1(Z)$. Now, by Vitali Convergence Theorem, we conclude that $f_n(x, u_n, \nabla u_n) \to f(x, u, \nabla u)$ in $L^1(Z)$. ■

Lemma 3.6. *Under the assumptions of Theorem* 3.1, *the limit u of the approximating solutions $\{u_n\}$ to problem* (3.1) *belongs to $W^{2,p}(Z) \cap W_0^{1,p}(Z)$.*

Proof. By Lemma 3.4, there exists a constant $M > 0$ such that $\|u_n\|_{W^{2,p}(Z)} \leq M$ for all n. Let

$$\mathcal{K} = \{v \in W^{2,p}(Z) \cap W_0^{1,p}(Z) | \quad \|v\|_{W^{2,p}(Z)} \leq M\}.$$

By the same argument as in the proof of Proposition 2.2.2, it follows that $\mathcal{K}$ is closed in $W^{1,p}$. Thus the limit u of (u_n) belongs to $W^{2,p}(Z) \cap W_0^{1,p}(Z)$. ■

Therefore, the existence of solutions in $W^{2,p}(Z) \cap W_0^{1,p}(Z)$ asserted in Theorem 3.1 now follows readily from **Lemmas 3.2-3.6**.

Lemma 3.7. *If $f(x, r, \xi)$ has a quadratic growth in ξ, that is $\theta = 2$ in* (3.2), *then there exists an H^1 bound for the approximating solutions $\{u_n\}$ to problem* (3.1).

Proof. The differential equation in (3.1) can be written in the following divergence form:

$$-\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i} a_{ij}(x,u) \frac{\partial u}{\partial x_j} - c(x,u)u = g(x,u,\nabla u),$$

where

$$g(x,u,\nabla u) = -f(x,u,\nabla u) + \sum_{i=1}^{N} b_i(u) \frac{\partial u}{\partial x_i} - \sum_{i,j=1}^{N} \frac{\partial a_{ij}}{\partial x_i} \frac{\partial u}{\partial x_j} - \sum_{i,j=1}^{N} \frac{\partial a_{ij}}{\partial r} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j}.$$

Since $f(x,r,\xi)$ satisfies (3.2), we have

$$|g(x,r,\xi)| \leq C + E(|r|)|\xi|^2,$$

where C is a nonnegative constant and E is a locally bounded function in $\mathbb{R}^+$. Following from the proof of [6, Theorem 2.1], the approximating solution $\{u_n\}$ is H^1 bounded. ■

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