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COUNTEREXAMPLES IN ERGODIC THEORY OF EQUICONTINUOUS SEMIGROUPS OF OPERATORS

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Abstract. The paper gives counterexamples in abstract ergodic theory of an equicontinuous semigroup $\mathcal{S}$ of linear operators on a locally convex space X. In particular, it is shown that the orbit of an element $x \in X$ may contain a unique fixed point of $\mathcal{S}$ without x being necessarily ergodic.

1. Introduction and Preliminaries

Let $\mathcal{S}$ be a semigroup of continuous linear operators on a locally convex space X, and let co $\mathcal{S}$ be the set of all convex combinations of elements of $\mathcal{S}$. Further, we define

$$\mathcal{F}(\mathcal{S}) = \bigcap_{A \in \mathcal{S}} (I - A)^{-1}(0); \tag{1.1}$$

the elements of the set $\mathcal{F}(\mathcal{S})$ are the *fixed points* of $\mathcal{S}$. We observe that co $\mathcal{S}$ is a semigroup containing $\mathcal{S}$ as a subsemigroup. For any $x \in X$ and any $\mathcal{H} \subset \operatorname{co} \mathcal{S}$, we set

$$\mathcal{H}x = \bigcup_{A \in \mathcal{H}} Ax, \quad \mathcal{K}(x) = \overline{\operatorname{co}}\,(\mathcal{S}x), \quad \mathcal{K}(x, \mathcal{H}) = \bigcap_{A \in \mathcal{H}} \mathcal{K}(Ax). \tag{1.2}$$

$\mathcal{K}(x)$ is called the *orbit* of x under $\mathcal{S}$ and $\mathcal{K}(x, \mathcal{H})$ the *joint orbit* of x under $\mathcal{H}$. (Alternatively, $\mathcal{K}(x)$ is the closure of (co $\mathcal{S})x$.)

Definition 1.1. Let $\mathcal{S}$ be an equicontinuous semigroup of linear operators on X. We say that a point $x \in X$ is *ergodic* under $\mathcal{S}$ if the joint orbit $\mathcal{K}(x, \operatorname{co}\mathcal{S})$ consists of a single point. By $\mathcal{E}(\mathcal{S})$ we denote the set of all ergodic points of $\mathcal{S}$.

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There is an interesting relation between ergodicity of an element and the Alaoglu–Birkhoff convergence [1].

An *Alaoglu–Birkhoff net* (*AB-net*) $\{x_\alpha\}$ is a map $\alpha \mapsto x_\alpha$ of a transitively ordered index set Δ into a Hausdorff topological space Z. We say that a net $\{x_\alpha\}$ *converges in the sense of Alaoglu–Birkhoff* (*AB-converges*) to $a \in Z$ if for each neighbourhood $N(a)$ of a and each $\alpha \in \Delta$ there exists $\alpha_0 \geq \alpha$ in Δ such that $x_\beta \in N(a)$ for all $\beta \geq \alpha_0$. The point $b \in Z$ is a *cluster point* of the AB-net $\{x_\alpha\}$ if, for each neighbourhood $N(b)$ of b and each $\alpha \in \Delta$, there exists $\beta \geq \alpha$ in Δ with $x_\beta \in N(b)$. The AB-convergence was introduced in [1, pp. 293-295] and further studied in subsequent works such as [2, 3, 5].

Returning to our operator semigroup $\mathcal{S}$ on a locally convex space X, we consider AB-nets of the following type: For a given $x \in X$, $\{x_A\}$ in this paper will always denote the net $A \mapsto Ax$ with the index set co $\mathcal{S}$ transitively ordered by stipulating that

$$A \leq B \text{ if there exists } C \in \text{co } \mathcal{S} \text{ such that } CA = B.$$

We write $x_A \to a$ if the net $\{x_A\}$ AB-converges to $a \in X$ in the locally convex topology of X, and $x_A \rightharpoonup a$ if it AB-converges in the weak topology of X.

We then have the following criteria for ergodicity in which convergence means the AB-convergence.

Theorem 1.2. *If $\mathcal{S}$ is an equicontinuous semigroup $\mathcal{S}$ of linear operators on a locally convex space X, the following conditions are equivalent* :

(i) *x is ergodic with $\mathcal{K}(x, \text{co}\,\mathcal{S}) = \{a\}$.*

(ii) $\mathcal{K}(x, \text{co}\,\mathcal{S}) \cap \mathcal{F}(\mathcal{S}) = \{a\}$.

(iii) $a \in \mathcal{K}(x, \text{co}\,\mathcal{S}) \cap \mathcal{F}(\mathcal{S})$.

(iv) $x_A \to a$.

(v) $x_A \rightharpoonup a$.

(vi) $\{x_A\}$ *clusters weakly at a fixed point of $\mathcal{S}$.*

Proof. (i)$\Longrightarrow$(ii). From $\mathcal{K}(a) \subset \mathcal{K}(x, \text{co}\,\mathcal{S})$ follows $\mathcal{K}(a) = \{a\}$, and hence a is a fixed point of $\mathcal{S}$.

(ii)$\Longrightarrow$(iii) is clear.

(iii)$\Longrightarrow$(iv). For a given 0-neighbourhood U in X choose a 0-neighbourhood V such that $\text{co}\,\mathcal{S}(V) \subset U$ (equicontinuity). If $A \in \text{co}\,\mathcal{S}$, find $C \in \text{co}\,\mathcal{S}$ satisfying $CAx - a \in V$ ($a \in \mathcal{K}(Ax)$). Then $A_0 := CA \geq A$, and for each $B = DA_0 \geq A_0$ with $D \in \text{co}\,\mathcal{S}$,

$$x_B - a = D(CAx - a) \in D(V) \subset U.$$

This proves $x_A \to a$.

(iv)$\Longrightarrow$(v) is obvious.

(v)$\Longrightarrow$(vi). We need to prove that $a \in \mathcal{F}(\mathcal{S})$. To this end, we use properties of the AB-convergence found in [1, pp. 293-295]. Let $T \in \mathcal{S}$. Then $Tx_A \rightharpoonup Ta$ (weak AB-continuity of T). The AB-net $\{x_{TA} : A \in \operatorname{co}\mathcal{S}\}$ is a subnet of $\{x_A : A \in \operatorname{co}\mathcal{S}\}$, and $Tx_A = x_{TA} \rightharpoonup a$. Hence $Ta = a$ by the uniqueness of limits in Hausdorff spaces.

(vi)$\Longrightarrow$(i). Let $a \in \mathcal{F}(\mathcal{S})$ be a weak cluster point of x_A. We show that $a \in \mathcal{K}(x, \operatorname{co}\mathcal{S})$. Let $A \in \operatorname{co}\mathcal{S}$ and let $N(a)$ be a weak neighbourhood of a. Then there exists $B = CA \geq A$ such that $x_B = CAx \in N(a)$, so that a is in the weak closure of $\operatorname{co}\mathcal{S}(Ax) \subset \mathcal{K}(Ax)$. Since $\mathcal{K}(Ax)$ is a closed convex set, $a \in \mathcal{K}(Ax)$ for each $A \in \operatorname{co}\mathcal{S}$. In particular, $a \in \mathcal{K}(x)$. Suppose that $b \in \mathcal{K}(x, \operatorname{co}\mathcal{S})$. If U is a convex 0-neighbourhood in X, choose a 0-neighbourhood V such that $\operatorname{co}\mathcal{S}(V) \subset (1/2)U$. There are A, $B \in \operatorname{co}\mathcal{S}$ such that $a - Ax \in V$ and $BAx - b \in (1/2)U$. Then

$$a - b = B(a - Ax) + (BAx - b) \in B(V) + \tfrac{1}{2}U \subset U,$$

which proves that $a = b$. Hence $\mathcal{K}(x, \operatorname{co}\mathcal{S}) = \{a\}$. ■

In the following section we will see that some conditions of ergodicity under an equicontinuous semigroup given by Alaoglu and Birkhoff in [1] may fail. Fortunately, the main results of [1], in particular [1, Theorem 6], are unaffected by this failure:

Theorem 1.3 (Alaoglu–Birkhoff). *Let X be a uniformly convex Banach space whose dual X^* is strictly convex. If $\mathcal{S}$ is a contraction semigroup on X, then $\mathcal{E}(\mathcal{S}) = X$.*

This theorem is reproduced in Krengel's monograph [4] as Theorem 1.10.

2. Counterexamples

In this section, $\mathcal{S}$ denotes an equicontinuous semigroup of linear operators on a locally convex space X.

We include the following theorem to motivate Example 2.2.

Theorem 2.1. *Let $\mathcal{S}$ be an equicontinuous semigroup of linear operators on a locally convex space X, and let Γ be a convex subset of X invariant under $\mathcal{S}$ such that*

(2.1) $$|\mathcal{F}(\mathcal{S}) \cap \mathcal{K}(x)| = 1 \text{ for each } x \in \Gamma.$$

Then $\Gamma \subset \mathcal{E}(\mathcal{S})$.

Proof. Since Γ is convex and invariant under $\mathcal{S}$, it is also invariant under $\operatorname{co}\mathcal{S}$. For any $x \in \Gamma$ and $A \in \operatorname{co}\mathcal{S}$, $\mathcal{K}(Ax)$ contains a (unique) fixed point a by hypothesis. By (2.1), we have $\mathcal{K}(Ax) \cap \mathcal{F}(\mathcal{S}) = \mathcal{K}(x) \cap \mathcal{F}(\mathcal{S}) = \{a\}$ for all $A \in \operatorname{co}\mathcal{S}$. Then $\mathcal{K}(x, \operatorname{co}\mathcal{S}) \cap \mathcal{F}(\mathcal{S}) = \{a\}$ and the conclusion follows from Theorem 1.2. ■

Example 2.2. The condition

$$|\mathcal{F}(\mathcal{S}) \cap \mathcal{K}(x)| = 1 \tag{2.2}$$

is necessary but not sufficient for the ergodicity of $x \in X$.

Let x be ergodic with $\mathcal{K}(x, \operatorname{co}\mathcal{S}) = \{b\}$ and let $a \in \mathcal{F}(\mathcal{S}) \cap \mathcal{K}(x)$. Then an argument similar to the one used in the proof of (vi)$\Longrightarrow$(i) in Theorem 1.2 shows that $a = b$, which means that (2.2) is necessary for the ergodicity of x.

To show that the condition is not sufficient, consider the Banach space ℓ^1 and the semigroup $\mathcal{S}$ of bounded linear operators on ℓ^1 generated by

$$V(\xi_1, \xi_2, \xi_3, \dots) = (\tfrac{1}{2}\xi_1, \xi_2, \xi_3, \dots) \quad \text{and} \quad T(\xi_1, \xi_2, \xi_3, \dots) = (0, \xi_1, \xi_2, \dots).$$

We observe that $VT = T$ and that each element of $\mathcal{S}$ is of the form T^iV^j, $i+j \geq 1$. Let $x = (1, 0, 0, \dots) \in \ell^1$. Since $V^jx = 2^{-j}x \to 0$ in norm as $j \to \infty$, the orbit $\mathcal{K}(x)$ contains 0. Moreover, 0 is the only fixed point of $\mathcal{S}$ since $I - T$ is injective. From the general form of operators in $\mathcal{S}$ we deduce that any operator $A \geq T$ in $\operatorname{co}\mathcal{S}$ can be expressed as a convex combination $A = \sum_{i=1}^n \lambda_i T^i$. Then

$$\|Ax\| = \|\sum_{i=1}^n \lambda_i e_{i+1}\| = \sum_{i=1}^n \lambda_i = 1,$$

where $e_i = (\delta_{ik})_{k=1}^\infty$. This shows that $x_A \not\to 0$. Hence x is not ergodic by Theorem 1.2.

The preceding construction is a counterexample to [1, Theorem 2], which claims that $x \in X$ is ergodic if and only if $|\mathcal{F}(\mathcal{S}) \cap \mathcal{K}(x)| = 1$.

The next result shows that condition (vi) of Theorem 1.2 cannot be weakened.

Example 2.3. The condition that $\{x_A\}$ has a weak cluster point is necessary but not sufficient for x to be ergodic: Let $\mathcal{S}$ be the semigroup of contractions on the Banach space ℓ^1 generated by

$$V(\xi_1, \xi_2, \xi_3, \dots) = \Big(\sum_{i=1}^\infty \xi_i, 0, 0, \dots\Big) \quad \text{and} \quad T(\xi_1, \xi_2, \xi_3, \dots) = (0, \xi_1, \xi_2, \dots).$$

Observe that $VT = V$ and $V^2 = V$. If $x = (1, 0, 0, \dots) \in \ell^1$, then $VAx = x$ for all $A \in \operatorname{co}\mathcal{S}$. Hence the net $\{x_A : A \in \operatorname{co}\mathcal{S}\}$ contains a constant subnet

$\{x_{VA} : A \in \operatorname{co}\mathcal{S}\}$ convergent to x, but x is not a fixed point of $\mathcal{S}$. There x is not ergodic by Theorem 1.2. The necessity is obvious.

Our example also shows that the condition $\mathcal{K}(x, \operatorname{co}\mathcal{S}) \neq \emptyset$ is necessary but not sufficient for x to be ergodic.

Example 2.4. The condition $|\mathcal{K}(x,\mathcal{S}) \cap \mathcal{F}(\mathcal{S})| \geq 1$ is necessary but not sufficient for x to be ergodic: Let Π be the permutation group on the set $\mathbb{N}$ of all positive integers. For any $\sigma \in \Pi$, we define the operator $T_\sigma : \ell^\infty \to \ell^\infty$ by

$$T_\sigma(\xi_1, \xi_2, \ldots) = (\xi_{\sigma(1)}, \xi_{\sigma(2)}, \ldots).$$

All operators of the form T_σ ($\sigma \in \Pi$) form an operator group $\mathcal{G}$ of linear isometries on ℓ^∞ which is group isomorphic to Π. Let $x = (1, -1, 1, -1, 1, -1, \ldots)$. We show that

$$0 \in \mathcal{K}(x, \mathcal{G}) \quad \text{and} \quad 0 \notin \mathcal{K}(x, \operatorname{co}\mathcal{G}).$$

Let $\omega \in \Pi$ be the permutation that interchanges each odd number with its successor. Then $(1/2)(I+T_\omega) \in \operatorname{co}\mathcal{S}$, and $((1/2)(I+T_\omega)A^{-1})Ax = 0$ for any $A \in \mathcal{G}$. Hence $0 \in \mathcal{K}(Ax)$ for each $A \in \mathcal{G}$. There exist permutations σ_1, σ_2, $\sigma_3 \in \Pi$ such that

$$\sigma_1(2n) = 3n, \quad \sigma_2(2n) = 3n - 1, \quad \sigma_3(2n) = 3n - 2 \quad \text{for } n \in \mathbb{N}.$$

Then

$$T_{\sigma_1}x = (1, 1, -1, 1, 1, -1, 1, 1, -1, \ldots),$$
$$T_{\sigma_2}x = (1, -1, 1, 1, -1, 1, 1, -1, 1, 1, \ldots),$$
$$T_{\sigma_3}x = (-1, 1, 1, -1, 1, 1, -1, 1, 1, 1, \ldots).$$

Set $V = (1/3)(T_{\sigma_1} + T_{\sigma_2} + T_{\sigma_3}) \in \operatorname{co}\mathcal{S}$. Then $Vx = ((1/3), (1/3), (1/3), \ldots)$, and $AVx = Vx$ for all $A \in \operatorname{co}\mathcal{G}$, so that the orbit $\mathcal{K}(Vx)$ does not contain 0. Hence $\mathcal{K}(x, \operatorname{co}\mathcal{S}) \cap \mathcal{F}(\mathcal{S}) = \emptyset$.

The preceding construction is a counterexample to [1, Lemma 7.1], which claims that x is ergodic if and only if $\mathcal{K}(x,\mathcal{S})$ contains a fixed point of $\mathcal{S}$.

Remark 2.5. What Alaoglu and Birkhoff actually proved in [1] is the following: *Let $\mathcal{H}$ be a subset of* $\operatorname{co}\mathcal{S}$ *which contains $\mathcal{S}$ and all operators $A \in \operatorname{co}\mathcal{S}$ such that $A \geq T$ for some $T \neq I$ in $\mathcal{S}$. Then x is ergodic if and only if $\mathcal{K}(x,\mathcal{H})$ contains a fixed point.* (See the first six lines of the proof of [1, Lemma 7.1]). Note that $\operatorname{co}\mathcal{S}$ has the property required for $\mathcal{H}$ if $\mathcal{S}$ contains a pair T, T^{-1}, where $T \neq I$.

We observe that the group $\mathcal{G}$ constructed in Example 2.4 illustrates the phenomenon of multiple fixed points discussed in [1, § 13]: The orbit $\mathcal{K}(x)$ contains all fixed points of the form $(\alpha, \alpha, \alpha, \ldots)$, $0 \leq \alpha \leq 1$.

Example 2.6. Example 2.4 can be modified to show that even the stronger condition $|\mathcal{K}(x,\mathcal{S}) \cap \mathcal{F}(\mathcal{S})| = 1$ does not ensure the ergodicity of x: Let $\mathcal{G}_1$ be the smallest group of linear operators on ℓ^∞ containing the group $\mathcal{G}$ defined in Example 2.4, and the operator

$$P(\xi_1, \xi_2, \xi_3, \dots) = (-\xi_1, \xi_2, \xi_3, \dots).$$

Then $\mathcal{F}(\mathcal{G}_1) = \{0\}$. If x and V have the same meaning as in Example 2.4 and if $A \in \operatorname{co}\mathcal{G}_1$, then all coordinates of AVx from a certain index on are equal to $1/3$. As above, $0 \in \mathcal{K}_1(x,\mathcal{G}_1)$, but $0 \notin \mathcal{K}_1(Vx)$ (subscript 1 refers to orbits under $\mathcal{G}_1$).

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