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EXACT PROFILE VALUES OF SOME GRAPH COMPOSITIONS

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Abstract. It is known that the determination of the profile for arbitrary graphs is NP-complete. The *composition* of two graphs G and H is the graph with vertex set $V(G) \times V(H)$ and (u_1, v_1) is adjacent to (u_2, v_2) if either u_1 is adjacent to u_2 in G or $u_1 = u_2$ and v_1 is adjacent to v_2 in H. The exact values of the profile of the composition of a path with other graphs, a cycle with other graphs, a complete graph with other graphs and a complete bipartite graph with other graphs are established.

1. Introduction and Terminology

For a graph G, $V(G)$ denotes the set of vertices of G and $E(G)$ denotes the set of edges of G.

Let $G = (V, E)$ be a graph on n vertices. A one-to-one onto mapping $f : V \longrightarrow \{1, 2, \ldots, n\}$ is called a *proper numbering* of G. For a proper numbering f, the *profile width* $w_f(v)$ *of a vertex* v in a graph G is the number

$$w_f(v) = \max_{x \in N[v]} (f(v) - f(x)),$$

where $N[v] = \{x \in V : x = v \text{ or } xv \in E\}$ is the closed neighborhood of v. Since $v \in N[v]$, we have $w_f(v) = 0$ if $f(v) \le f(x)$ for all $x \in N[v]$. The *profile* $P_f(G)$ *of a proper numbering f of G* is defined by

$$P_f(G) = \sum_{v \in V} w_f(v)$$

and the *profile* $P(G)$ of G is the number

$$P(G) = \min\{P_f(G) : f \text{ is a proper numbering of } G\}.$$

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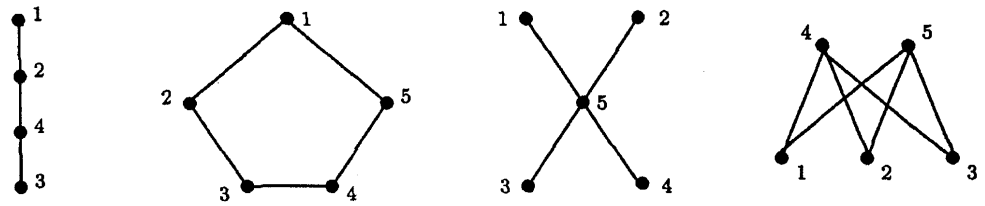

FIG. 1. Profile numberings for P_4, C_5, $K_{1,4}$ and $K_{2,3}$.

A proper numbering f is called a *profile numbering* of G if $P_f(G) = P(G)$. See Figure 1 for some examples.

Lin and Yuan [3] have shown that the profile minimization problem of an arbitrary graph is equivalent to the interval graph completion problem, which was shown to be NP-complete by Garey and Johnson [1]. Since minimizing the profile of a graph has some important applications, a large number of approximation algorithms have been developed, published and used. But the exact value of profile is known for only a few classes of graphs. See Lai and Williams [2].

Definition 1. The *composition* $G[H]$ of a graph G with a graph H is the graph with vertex set $V(G) \times V(H)$ such that (u_1, v_1) is adjacent to (u_2, v_2) if either u_1 is adjacent to u_2 in G or if $u_1 = u_2$ and v_1 is adjacent to v_2 in H.

Figure 2 shows $P_3[P_4]$. For a composition graph $G[H] = (V, E)$, with graph G of order m and H of order n, we represent the vertex set as $V = \{v_{i,j} : 1 \leq i \leq m, 1 \leq j \leq n\}$, where column j is denoted by Q_j $(1 \leq j \leq n)$, which represents a copy of G, and row i $(1 \leq i \leq m)$ is denoted by R_i, which represents a copy of H.

The following result from [4] is used extensively for the work done in this paper.

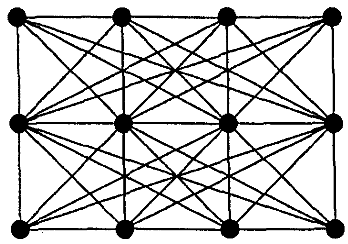

FIG. 2. $P_3[P_4]$

Proposition 1 (*Lin and Yuan* [4]). *Let G be a graph of order n. For any proper numbering f of G,*

$$P_f(G) = \sum_{i=1}^{n} |N(S_i)|, \ \textit{where } S_t = \{v : v \in V(G) \textit{ and } f(v) \le t\}.$$

2. Paths with Other Graphs

In this section, we establish the profile of the composition of paths with other graphs.

Theorem 1. *Let $G = P_m[H]$, where H is a graph with n vertices. Then*

$$P(G) = \begin{cases} m-1 & \textit{for } n = 1, \\ P(H) & \textit{for } m = 1, \\ P(H) + \dfrac{3n^2 - n}{2} & \textit{for } m = 2 \textit{ and } n > 1, \\ 2P(H) + \dfrac{mn(3n-1)}{2} - 2n^2 + n & \textit{for } m \ge 3 \textit{ and } n > 1. \end{cases}$$

Proof. For $n = 1$, $G = P_m$; so $P(G) = m - 1$. For $m = 1$, $G = P_1[H] = H$; so $P(G) = P(H)$. For $m = 2$, we note that every vertex in R_2 is adjacent to every vertex in R_1 so once we number a vertex in both rows, all of the unnumbered vertices in the rows are in $N_f(S)$. By Proposition 1, we know that the profile numbering of G must completely number one row before numbering any vertex in the other row. This will minimize $\sum_{k=1}^{mn} |N(S_k)|$, which in turn minimizes $P_f(G)$. Without loss of generality, assume that we first completely number R_1 and then R_2. We want to number the vertices in R_1 in the order of a profile numbering (say, f) of H. Since every vertex in R_2 is adjacent to the vertex $f^{-1}(1)$, it does not matter how we number the vertices in R_2. Hence, $P(G) = P(H) + \sum_{i=0}^{n-1}(n+i) = P(H) + (3n^2 - n)/2$.

For $m \ge 3$, we first show that $P(G) \le 2P(H) + mn(3n-1)/2 - 2n^2 + n$. Assume that g is a profile numbering of H. Consider a numbering f such that

$$f(v_{i,j}) = \begin{cases} g(v_j) + (i-1)n & \text{for } 1 \le i \le m-2, 1 \le j \le n, \\ g(v_j) + (m-1)n & \text{for } i = m-1, 1 \le j \le n, \\ g(v_j) + (m-2)n & \text{for } i = m, 1 \le j \le n. \end{cases}$$

Then

$$\begin{aligned} P_f(G) &= P(H) + (m-3)\sum_{i=0}^{n-1}(n+i) + P(H) + \sum_{i=0}^{n-1}(2n+i) \\ &= 2P(H) + \frac{mn(3n-1)}{2} - 2n^2 + n. \end{aligned}$$

Let h be a profile numbering of G. Now, assume that $P_h(G) < 2P(H) + mn(3n-1)/2 - 2n^2 + n$. For the same reason as in the case with $m = 2$, h must completely number a row before starting another row. Furthermore, h must begin with one of the end rows (say, R_1); otherwise $P_h(G) \geq P_f(G) + n$. We claim that h must then number the rows in the order $R_1, R_2, \ldots, R_{m-2}, R_m, R_{m-1}$.

We prove the claim by contradiction. Assume that this pattern is not followed. And, let the first violation of this pattern be for h at R_p to R_q, where $q = p + r$, $r > 1$ and $p < m - 2$. Then if $q < m$, then

$$\begin{aligned}\sum_{i=p+1}^{q+1} \sum_{j=1}^{n} w_h(v_{i,j}) &\geq P(H) + (5n^2 - n) + \frac{(r-2)(3n^2-n)}{2} \\ &> \frac{(r+1)(3n^2-n)}{2} \\ &= \sum_{i=p+1}^{q+1} \sum_{j=1}^{n} w_f(v_{i,j}).\end{aligned}$$

If $q = m$, then

$$\begin{aligned}\sum_{i=p+1}^{q} \sum_{j=1}^{n} w_h(v_{i,j}) &\geq P(H) + (5n^2 - n) + \frac{(r-3)(3n^2-n)}{2} \\ &> P(H) + \frac{5n^2-n}{2} + \frac{(r-2)(3n^2-n)}{2} \\ &= \sum_{i=p+1}^{q} \sum_{j=1}^{n} w_f(v_{i,j}).\end{aligned}$$

So, the claim is proved and h must number the rows in the same order as f.

Since within each row f numbers the same way as g, which is a profile numbering of H, it follows that

$$\begin{aligned}\sum_{i=1}^{m} \sum_{j=1}^{n} w_f(v_{i,j}) &= 2P(H) + \frac{mn(3n-1)}{2} - 2n^2 + n \\ &< \sum_{j=1}^{n} w_h(v_{1,j}) + \sum_{j=1}^{n} w_h(v_{m,j}) + \frac{mn(3n-1)}{2} - 2n^2 + n \\ &= P_h(G),\end{aligned}$$

which implies that $P_f(G) < P_h(G)$, a contradiction. ■

Figure 1 shows a profile numbering for the graphs P_4, C_5, $K_{1,4}$, and $K_{2,3}$. In general, $P(P_n) = n-1$, $P(C_n) = 2n-3$, $P(K_n) = n(n-1)/2$, $P(K_{1,n}) = n$, and for $r \leq n$, $P(K_{r,n}) = rn + r(r-1)/2$ (see Lin and Yuan [4]). These observations lead directly to the following corollary.

Corollary 1. *For* $m \geq 3$,

$$P(G) = \begin{cases} \dfrac{mn(3n-1)}{2} - 2n^2 + 3n - 2 & for\ G = P_m[P_n], \\ \dfrac{mn(3n-1)}{2} - 2n^2 + 5n - 6 & for\ G = P_m[C_n], \\ \dfrac{mn(3n-1)}{2} - n^2 & for\ G = P_m[K_n]. \end{cases}$$

3. Cycles and Complete Graphs with Other Graphs

In this section, we establish the profile of the composition of cycles with other graphs and the profile of the composition of complete graphs with other graphs.

Theorem 2. *Let* $G = C_m[H]$, *where* H *is a graph with* n *vertices. Then*

$$P(G) = P(H) + \frac{n(5mn - 7n - m + 1)}{2}.$$

Proof. Let g be a profile numbering of H. Define a numbering $f(v_{i,j}) = (i-1)n + g(v_j)$ for $1 \leq i \leq m$ and $1 \leq j \leq n$. Then using an argument similar to that in the proof of Theorem 1, we see that a profile numbering is produced, namely,

$$\begin{aligned} P_f(G) &= P(H) + \frac{(m-2)n(3n-1)}{2} + \frac{n(2mn - n - 1)}{2} \\ &= P(H) + \frac{n(5mn - 7n - m + 1)}{2}. \end{aligned}$$

∎

Corollary 2. *For* $m \geq 3$ *and* $n \geq 3$,

$$P(G) = \begin{cases} \dfrac{n(5mn - 7n - m + 3)}{2} - 1 & for\ G = C_m[P_n], \\ \dfrac{n(5mn - 7n - m + 5)}{2} - 3 & for\ G = C_m[C_n], \\ \dfrac{n(5mn - 6n - m)}{2} & for\ G = C_m[K_n]. \end{cases}$$

Theorem 3. *For* $m \geq 1$, *let* $G = K_m[H]$, *where* H *is a graph with* n *vertices. Then*

$$P(G) = P(H) + \frac{(mn + n - 1)(mn - n)}{2}.$$

Proof. In the composition of a complete graph with another graph, every vertex in R_i is adjacent to all other vertices which are in rows other than R_i. Hence once we number a vertex 1, we know

$$\sum_{i=n+1}^{mn} w_f(v_i) = \frac{(mn+n-1)(mn-n)}{2}.$$

Since $\min \sum_{i=1}^{n} w_f(v_i) = P(H)$, we then have $P(G) \geq P(H)+(mn+n-1)(mn-n)/2$, and $P(H)+(mn+n-1)(mn-n)/2$ is achievable by numbering R_1 with $1,\dots,n$; R_2 with $n+1,\dots,2n$ etc. and within each row following a profile numbering of H. So, $P(G) = P(H)+(mn+n-1)(mn-n)/2$. ■

Corollary 3. *For $m \geq 1$ and $n \geq 1$,*

$$P(K_m[K_n]) = \frac{mn(mn-1)}{2}.$$

4. The Complete Bipartite Graph with Other Graphs

Theorem 4. *Let $G = K_{m,n}[H]$, where $m \leq n$ and H is a graph of order l. Then*

$$P(G) = mnl^2 + \frac{ml(ml-1)}{2} + nP(H).$$

Proof. Assume the two partite sets of vertices in $K_{m,n}$ are $\{v_1, v_2, \dots, v_n\}$ and $\{v_{n+1}, v_{n+2}, \dots, v_{n+m}\}$. Also assume that g is a profile numbering of H. Consider a numbering f such that $f(v_{i,j}) = g(v_j) + (i-1)l$ for $1 \leq i \leq n+m$. Then

$$\begin{aligned} P_f(G) &= nP(H) + \sum_{i=nl+1}^{(m+n)l} (i-1) \\ &= mnl^2 + \frac{ml(ml-1)}{2} + nP(H). \end{aligned}$$

Now assume that h is a profile numbering of G. For $1 \leq i \leq n$ and $n+1 \leq j \leq n+m$, every vertex in $\bigcup_{i=1}^{n} R_i$ is adjacent to every vertex in $\bigcup_{j=n+1}^{n+m} R_j$ and for $m \leq n$, the vertex $v = h^{-1}(1)$ should be in $\bigcup_{i=1}^{n} R_i$. So

$$\sum_{i=n+1}^{m+n} \sum_{j=1}^{l} w_h(v_{i,j}) = mnl^2 + \frac{ml(ml-1)}{2}.$$

Since $\min \sum_{i=1}^{n} \sum_{j=1}^{l} w_h(v_{i,j}) = nP(H)$, we have $P_h(G) = mnl^2 + ml(ml-1)/2 + nP(H)$. ■

A direct application of Theorem 4 leads to Corollary 4.

Corollary 4.

$$P(G) = \begin{cases} mnl^2 + nl - n + \dfrac{ml(ml-1)}{2} & for\ G = K_{m,n}[P_l], \\ mnl^2 + 2nl - 3n + \dfrac{ml(ml-1)}{2} & for\ G = K_{m,n}[C_l], \\ mnl^2 + \dfrac{nl(l-1)}{2} + \dfrac{ml(ml-1)}{2} & for\ G = K_{m,n}[K_l]. \end{cases}$$

The profile of the composition of a star with any other graph also follows directly from Theorem 4.

Corollary 5. *For $n \geq 1$, let H be a graph with l vertices. Then*

$$P(K_{1,n}[H]) = nl^2 + \frac{l(l-1)}{2} + nP(H).$$

Corollary 6. *For $n \geq 1$ and $l \geq 1$,*

$$P(G) = \begin{cases} nl^2 + nl - n + \dfrac{l(l-1)}{2} & for\ G = K_{1,n}[P_l], \\ nl^2 + 2nl - 3n + \dfrac{l(l-1)}{2} & for\ G = K_{1,n}[C_l], \\ nl^2 + \dfrac{l(l-1)(n+1)}{2} & for\ G = K_{1,n}[K_l]. \end{cases}$$

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