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ON STABILITY OF THE EQUAIONS $Bu'(t) = Au(t)$

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Abstract. The stability of solutions of the equation $Bu'(t) = Au(t)$ is considered, where A and B are closed linear operators on a Banach space. Under the well-posedness condition it is proved that if the imaginary part of the spectrum of the pencil $(\lambda B - A)$ is countable, then a bounded uniformly continuous solution $u(t)$ of the equation is asymptotically almost periodic if and only if the functions $e^{\lambda t}u(t)$, $(\lambda \in i\mathbb{R})$, have uniformly convergent means. A condition of exponential stability also is given when the generalized eigenvectors and associated root vectors of the linear pencil $(\lambda B - A)$ form a Riesz basis.

0. Introduction

Many problems in physics or engineering lead to differential equations in Hilbert spaces, with constant operator coefficients, or to integro-differential operators where the kernel depends on the difference of arguments. Such problems are usually studied using the Laplace transform methods. In the classical, most well studied case of the differential equation $u'(t) = Au(t)$, $u(0) = x$, the Laplace transform of the solution yields the resolvent, $R_A(\lambda) = (\lambda I - A)^{-1}$, which satisfies some nice properties (e.g., the resolvent equation, among other things). Methods of complex analysis, notably the Cauchy integral, and the theory of strongly continuous semigroups (C_0-semigroups) enable us to construct a complete theory with many deep results.

Laplace transform is still the main method in the study of more general differential or integro-differential equations. In particular, it leads normally to the notion of "generalized" resolvent, which is, in fact, the inverse of some operator-valued functions, or an operator pencil (whose values are, in general, unbounded linear

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operators). The study of such an operator pencil is in general a difficult problem, which, in particular, explains the absence of an advanced theory of general integro-differential equations including the solvability and the asymptotic behavior questions.

In this paper, we make an attempt to carry on a study of the Laplace transform and its applications to asymptotic behavior of solutions to the initial value problem of the following form

$$(1) \qquad \begin{cases} Bu'(t) &= Au(t) \\ u(0) &= u_0 \end{cases}$$

where A and B are, in general, unbounded linear operators on a Banach space H. If A and B are both bounded and B is invertible (i.e., has a bounded inverse), then equation (1) is completely equivalent to the regular equation $u'(t) = B^{-1}Au(t)$. However, when A is not bounded, even if B^{-1} exists as a bounded linear operator, the operator $B^{-1}A$ is not, in general, a generator of a C_0-semigroup (it may not even be closed). Therefore, the existing theory of C_0-semigroups is not directly applicable.

The paper is organized as follows. In Section 1, we introduce main definitions, and in Section 2 we present some standard results on the Laplace transform for equation (1) (cf. [8]). Section 3 contains our main results on the stability of solutions to equation (1).

While our study of the Laplace transform and stability of equation (1) may have independent theoretical interest and applications (because some problems in physics and engineering lead to equations of this form, see e.g. [6]), one of our primary objectives is to give a prototype for a more general study of Laplace transforms of more general integro-differential equations and their asymptotic behavior where our research is still going on.

1. Preliminaries

Let A and B be closed linear operators on a Hilbert (or Banach) space H. Consider the Cauchy problem

$$(1) \qquad \begin{cases} Bu'(t) &= Au(t) \\ u(0) &= u_0. \end{cases}$$

Definition 1. A function $u(\cdot) : \mathbb{R}_+ \to H$ is called a (classical) solution to (1), if, for all $t \geq 0$, $u(t) \in D(A) \cap D(B)$, $u(t)$ and $Bu(\cdot)$ are (strongly) continuous on $[0, \infty)$, the (strong) derivative $u'(t)$ exists, $u'(t) \in D(B)$, $Bu'(t)$ is strongly continuous on $(0, \infty)$, $Bu'(t) = Au(t)$, and $u(0) = u_0$.

Definition 2. The Cauchy problem (1) is well posed if for all $u_0 \in D(A) \cap D(B)$ there exists a unique solution $u(t)$ of (1), and if for all $u_0^{(n)} \subset D(A) \cap D(B)$ such that $\lim_{n\to\infty} u_0^{(n)} = 0$, we have $\lim_{n\to\infty} u_n(t) = 0$ for all $t > 0$, where $u_n(t)$ is the solution with $u_n(0) = u_0^{(n)}$.

Assume that (1) is well posed. Define a family $U(t)$ of mappings from $D(A) \cap D(B)$ into itself by

$$U(t)u_0 := u(t),$$

where $u(t)$ is the solution of (1).

Proposition 1 ([8]). $U(t)$ *satisfy the following*:

(i) $U(t)$ *are linear*;

(ii) *If* $x_n \in D(A) \cap D(B)$, $\lim_{n\to\infty} x_n = x_0 \in D(A) \cap D(B)$, *then*

$$\lim_{n\to\infty} U(t)x_n = U(t)x_0, \quad \text{for all } t \geq 0.$$

(iii) $U(0) = I$; $U(t+s)x = U(t)U(s)x$, *for all* $x \in D(A) \cap D(B)$.

Proposition 2 ([8]). *Assume that* $D(A) \cap D(B)$ *is dense in* H *and* (1) *is well posed. Then there exists a unique family of operators* $V(t) \in L(H)$ *such that*

$$V(t)x = U(t)x \quad \text{for all } x \in D(A) \cap D(B); \quad V(0) = I; \; V(t+s) = V(t)V(s).$$

It follows from the definitions that $V(t)x = U(t)x$ are strongly continuous on $[0, \infty)$ for all $x \in D(A) \cap D(B)$. One can show, by a standard argument (see e.g. [7], p. 26), that, for every $\delta > 0$, there exists $M_\delta > 0$, such that $\|V(t)\| \leq M_\delta$ for all t, $\delta \leq t \leq \frac{1}{\delta}$. From this it follows that $V(t)x$ is continuous on $(0, \infty)$ for all $x \in H$.

Proposition 3. *Assume that* $D(A) \cap D(B)$ *is dense and* (1) *is well posed. If* u *is a solution of* (1), *then* $u'(t)$ *is continuous on* $(0, \infty)$.

Proof: We have

$$u'(t) = \frac{d}{dt}U(t)u_0 = \lim_{h\downarrow 0} U(t)\frac{u(h) - u(0)}{h} = \lim_{h\downarrow 0} V(t)\frac{u(h) - u(0)}{h} = V(t)u'_+(0);$$

and continuity of $u'(t)$ *on* $(0, \infty)$ *follows from the above remark.* ■

Proposition 4. *Assume* $D(A) \cap D(B)$ *is dense and* (1) *is well posed. Then*

$$-\infty \leq \lim_{t\to\infty} \frac{1}{t} \ln \|V(t)\| = \omega < +\infty. \tag{2}$$

The proof of Proposition 4 is standard (*see e.g.* [7], *p.* 28).

Proposition 5. *Assume* $D(A) \cap D(B)$ *is dense and* (1) *is well posed*. *Then for all* λ; *such that* $\operatorname{Re}\lambda > \omega$; *the operator* $(\lambda B - A)$ *is one-to-one.*

Proof: Let $z \in D(A) \cap D(B)$, $z \neq 0$; and $\lambda Bz - Az = 0$. Consider

$$u(t) = e^{\lambda t} z:$$

We have $Bu'(t) = Au(t)$, for all $t \geq 0$, $u(0) = z$: Hence

$$u(t) = V(t)z; \quad \text{and so } \|e^{\lambda t} z\| \leq \|V(t)\|\,\|z\|;$$

which implies

$$\operatorname{Re}\lambda \; \frac{1}{t} \lim \|V(t)\| \;\Rightarrow\; \operatorname{Re}\lambda \leq \omega: \qquad \blacksquare$$

2. The Laplace Transform

Proposition 6. *Assume that* $u(t)$ *is a solution of* (1) *such that* $\|u(t)\| \leq Me^{t\alpha}$; $t \geq 0$; *for some* M *and* α. *Then for all* λ; $\operatorname{Re}\lambda > \alpha$, *the following transforms exist*

$$\hat{u}(\lambda) = \int_0^\infty e^{-\lambda t} u(t)\, dt$$

$$\lim_{\varepsilon \downarrow 0} \lim_{n \to \infty} \int_\varepsilon^N e^{-\lambda t} u'(t)\, dt := \int_{0^+}^\infty e^{-\lambda t} u'(t)\, dt$$

and

$$\int_{0^+}^\infty e^{-\lambda t} u'(t)\, dt = \lambda \hat{u}(\lambda) - u(0): \tag{3}$$

Proof: The proof for the existence of $\hat{u}(t)$ is standard. We have

$$\int_\varepsilon^N e^{-\lambda t} u'(t)\, dt = e^{-\lambda N} u(N) - e^{-\lambda \varepsilon} u(\varepsilon) + \lambda \int_\varepsilon^N e^{-\lambda t} u(t)\, dt:$$

Since $\|u(t)\| \leq Me^{\alpha t}$, we have

$$\|e^{-\lambda N} u(N)\| \leq Me^{(-(\operatorname{Re}\lambda)+\alpha)N} \to 0 \quad \text{as } N \to \infty;$$

hence

$$\lim_{N \to \infty} \int_\varepsilon^N e^{-\lambda t} u'(t)\, dt = -e^{-\lambda \varepsilon} u(\varepsilon) + \lambda \int_\varepsilon^\infty e^{-\lambda t} u(t)\, dt:$$

Since $u(t)$ is continuous on $[0, \infty)$, we have

$$\lim_{\varepsilon \downarrow 0} \lim_{n \to \infty} \int_{\varepsilon}^{N} e^{-\lambda t} u'(t)dt := \int_{0^+}^{\infty} e^{-\lambda t} u'(t)\, dt = -u(0) + \lambda \hat{u}(\lambda). \qquad \blacksquare$$

Proposition 7. *Assume that there exists* $\mu \in \rho(B)$ *such that* $(\mu - B)^{-1}A$ *is closed. Then for every solution* $u(t)$ *such that* $\|u(t)\| \le Me^{\alpha t}$, $t \ge 0$, *for some* M, α, *we have* $\hat{u}(\lambda) \in D(A) \cap D(B)$ *and*

$$(\lambda B - A)\hat{u}(\lambda) = Bu(0), \quad \text{for all } \mathrm{Re}\,\lambda > \alpha. \tag{4}$$

Proof (cf. [3], p. 10): From $Bu'(t) = Au(t)$, the continuity of $u'(t)$ on $(0, \infty)$ (Proposition 3), and closedness of B, it follows

$$\begin{aligned} \int_{\varepsilon}^{N} e^{-\lambda t} Au(t)dt &= B \int_{\varepsilon}^{N} e^{-\lambda t} u'(t)\, dt \\ &= B\left(e^{-\lambda t} u(t)\Big|_{\varepsilon}^{N} + \lambda \int_{\varepsilon}^{N} e^{-\lambda t} u(t)\, dt \right) \\ &= e^{-\lambda N} Bu(N) - e^{-\lambda \varepsilon} Bu(\varepsilon) + \lambda \int_{\varepsilon}^{N} e^{-\lambda t} Bu(t)\, dt. \end{aligned} \tag{5}$$

From (5) we have

$$\begin{aligned} \int_{\varepsilon}^{N} e^{-\lambda t} Au(t)\, dt = {}& e^{-\lambda N}(B - \mu)u(N) - e^{-\lambda \varepsilon}(B - \mu)u(\varepsilon) \\ &+ \lambda \int_{\varepsilon}^{N} e^{-\lambda t}(B - \mu)u(t)\, dt + \mu e^{-\lambda N} u(N) - \mu e^{-\lambda \varepsilon} u(\varepsilon) \\ &+ \mu \lambda \int_{\varepsilon}^{N} e^{-\lambda t} u(t)\, dt; \end{aligned}$$

or

$$\begin{aligned} (B - \mu)^{-1} \int_{\varepsilon}^{N} e^{-\lambda t} Au(t)\, dt = {}& e^{-\lambda N} u(N) - e^{-\lambda \varepsilon} u(\varepsilon) \\ &+ \lambda \int_{\varepsilon}^{N} e^{-\lambda t} u(t)\, dt + \mu (B - \mu)^{-1} e^{-\lambda N} u(N) \\ &- \mu e^{-\lambda \varepsilon}(B - \mu)^{-1} u(\varepsilon) \\ &+ \mu \lambda (B - \mu)^{-1} \int_{\varepsilon}^{N} e^{-\lambda t} u(t)\, dt. \end{aligned}$$

Letting $N \to \infty$, and $\varepsilon \downarrow 0$, we see that the following integral converges and

$$\int_{0^+}^{\infty} e^{-\lambda t}(B-\mu)^{-1}Au(t)\,dt = -u(0)+\lambda\hat{u}(\lambda)-\mu(B-\mu)^{-1}u(0)+\lambda\mu(B-\mu)^{-1}\hat{u}(\lambda).$$

This implies that $\hat{u}(\lambda) \in D(B)$ and

$$\begin{aligned}(B-\mu)\textstyle\int_{0^+}^{\infty} e^{-\lambda t}(B-\mu)^{-1}Au(t)\,dt &= -(B-\mu)u(0)+(B-\mu)\lambda\hat{u}(\lambda)-\mu u(0)\\ &\quad +\lambda\mu\hat{u}(\lambda)\\ &= -Bu(0)+\lambda B\hat{u}(\lambda).\end{aligned}$$

Since $(B-\mu)^{-1}A$ is closed, it follows that $\hat{u}(\lambda) \in D(A)$ and

$$A\int_{0^+}^{\infty} e^{-\lambda t}u(t)\,dt = \lambda B\hat{u}(\lambda) - Bu(0),$$

or

$$A\hat{u}(\lambda) = \lambda B\hat{u}(\lambda) - Bu(0),$$
$$(\lambda B - A)\hat{u}(\lambda) = Bu(0). \qquad \blacksquare$$

If equation (1) is well posed and ω is defined by (2), then, since $(\lambda B - A)$ is one-to-one for all $\lambda > \omega$ (Proposition 5), formula (4) can be written as:

$$\int_0^{\infty} e^{-\lambda t}V(t)x\,dt = (\lambda B - A)^{-1}Bx \quad \text{for all } x \in D(A)\cap D(B). \tag{6}$$

Note that from (6) it follows that the operator $(\lambda B - A)^{-1}B$ is a bounded operator from $D(A)\cap D(B)$ to H for all $\mathrm{Re}\,\lambda > \omega$, but the operator $(\lambda B - A)^{-1}$ may not be bounded.

A complex number λ is called (A,B)-*regular*, if $(\lambda B - A)$ is one-to-one and the operator $(\lambda B - A)^{-1}B$ is bounded. The set of all (A,B)-regular points is denoted by $\rho(A,B)$ and called the (A,B)-*resolvent set*. Its complement in $\mathbb{C}$ is called the spectrum of (A,B) and denoted by $\sigma(A,B)$. The function $R(\lambda) := (\lambda B - A)^{-1}B$, $\lambda \in \rho(A,B)$ is called the (A,B)-*resolvent*. From (6) we have

Proposition 8. *Assume that* $D(A)\cap D(B)$ *is dense*, (1) *is well posed*, *and condition in Proposition 7 holds. Then* $\{\lambda : \mathrm{Re}\,\lambda > \omega\} \subset \rho(A,B)$.

Proposition 9. *The* (A,B)-*resolvent set* $\rho(A,B)$ *is open and* $R(\lambda)$ *is an analytic function on* $\rho(A,B)$. *Moreover*, *the following identity holds*

$$R(\lambda) - R(\mu) = (\mu - \lambda)R(\lambda)R(\mu). \tag{7}$$

Proof: Let $\lambda \in \rho(A,B)$, and $r := \|(\lambda B - A)^{-1}B\|^{-1}$. From

$$(\mu B - A) = (\lambda B - A)[I - (\lambda - \mu)(\lambda B - A)^{-1}B]$$

it follows that $(\mu B - A)$ is one-to-one and $(\mu B - A)^{-1}B$ is bounded whenever $|\mu - \lambda| < r$, i.e. $\rho(A,B)$ is an open set. Since

$$(\mu B - A)^{-1}B - (\lambda B - A)^{-1}B = [(I - (\lambda - \mu)(\lambda B - A)^{-1}B)^{-1} - I](\lambda B - A)^{-1}B$$

and

$$[I - (\lambda - \mu)(\lambda B - A)^{-1}B]^{-1} - I = \sum_{k=1}^{\infty} (\lambda - \mu)^k[(\lambda B - A)^{-1}B]^k \to 0 \text{ as } \mu \to \lambda,$$

it follows that $R(\lambda)$ is continuous. Furthermore,

$$\begin{aligned}(\lambda B - A)^{-1}B - (\mu B - A)^{-1}B &= (\lambda B - A)^{-1}[I - (\lambda B - A)(\mu B - A)^{-1}]B\\ &= (\lambda B - A)^{-1}[(\mu B - A) - (\lambda B - A)](\mu B - A)^{-1}B\\ &= (\lambda B - A)^{-1}(\mu - \lambda)B(\mu B - A)^{-1}B;\end{aligned}$$

which implies (7), as well as the fact that $R(\lambda)$ has derivative in $\rho(A,B)$, hence is an analytic function. ■

From (7) it also follows that the resolvents are commuting, i.e. $R(\lambda)R(\mu) = R(\mu)R(\lambda)$.

Since every solution $u(t)$ is continuosly differentiable on $[t_1, t_2]$ $(t_1 > 0)$ (Proposition 3), we have by the formula for the inverse Laplace transform

$$V(t)x = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{\lambda t}(\lambda B - A)^{-1}Bx\,d\lambda, \quad x \in D(A) \cap D(B). \tag{8}$$

3. Stability

Let $u : \mathbb{R}_+ \to H$ be a bounded uniformly continuous function. Then the function $\hat{u}(\lambda) := \int_0^\infty e^{-\lambda t}u(t)\,dt$ is defined and analytic in $\{\lambda \in \mathbb{C} : \mathrm{Re}\,\lambda > 0\}$. A point $\lambda_0 \in i\mathbb{R}$ is called a *regular point* of u, if $\hat{u}(\lambda)$ has an analytic continuation into a neighborhood of λ_0. The complement in $i\mathbb{R}$ of the set of regular points is called *spectrum* of u and denoted by $\sigma(u)$.

The following Lemma is a special case of Proposition 7.

Lemma 10. *Assume the condition in Proposition* 7 *hold and that* $u(t)$ *is a bounded solution of* (1). *Then, for every* λ; $\mathrm{Re}\,\lambda > 0$, $\hat{u}(\lambda) \in D(A) \cap D(B)$ *and* $(\lambda B - A)\hat{u}(\lambda) = Bu_0$. *In particular*; *if* $\lambda \in \rho(A;B)$; *then* $\hat{u}(\lambda) = R(\lambda)u_0$.

Lemma 11. *Assume that* $D(A) \cap D(B)$ *is dense*, (1) *is well posed*; *and the conditions in Proposition* 7 *hold. Then, for every bounded uniformly continuous solution* $u(t)$,

$$\sigma(u) \subset \sigma(A;B):$$

Proof: Since

$$\hat{u}(\lambda) = (\lambda B - A)^{-1}Bu_0; \quad \mathrm{Re}\,\lambda > 0;$$

it follows, by Lemma 10, that if $\lambda \in i\mathbb{R} \setminus \rho(A;B)$; then $\hat{u}$ has analytic continuation into λ. Therefore,

$$\sigma(u) \subset \sigma(A;B): \qquad \blacksquare$$

Theorem 12. *Assume that* $D(A) \cap D(B)$ *is dense*, (1) *is well posed*; *and* $\sigma(A;B) \cap i\mathbb{R}$ *is countable. Let* $u(t)$ *be a uniformly continuous and bounded solution to* (1). *Then* $\|u(t)\| \to 0\,(t \to \infty)$ *if and only if*; *for each* $\lambda \in \sigma(A;B) \cap i\mathbb{R}$;

$$\lim_{T \to \infty} \frac{1}{T} \int_h^{h+T} e^{-\lambda t} u(t)\, dt = 0$$

uniformly in h.

Proof: The statement follows from Lemma 11 and an individual stability theorem in [1], [2], or [4].

Corollary 13. *Assume* $\sigma(A;B) \cap i\mathbb{R} = \emptyset$. *Then every bounded uniformly continuous solution of* (1) *satisfies* $\lim_{t\to\infty} \|u(t)\| = 0$.

In practice, sometimes it is known that $\sigma(A;B)$ consists of generalized eigenvalues λ_1, λ_2, $\ldots$; λ_m; $\ldots$ and the corresponding eigenvectors y_k, $y_k \in D(A) \cap D(B)$,

$$(\lambda_k B - A)y_k = 0$$

form a Riesz basis. Recall that $\{y_k\}_{k=1}^{\infty}$ form a Riesz basis in a Hilbert space H if there exists an invertible operator Q and an orthonormal basis e_1, e_2, $\ldots$ such that $y_k = Qe_k$, $k = 1; 2; \ldots$.

Theorem 14. *Assume that equation* (1) *is well posed and* $\sigma(A;B)$ *consists of generalized eigenvalues* λ_k *such that* $\mathrm{Re}\,\lambda_k \leq -\varepsilon < 0$ *for some* $\varepsilon > 0$; *for all* k;

and the corresponding eigenvectors form a Riesz basis. Then for every solution $u(t)$ *of* (1), *we have*

$$\|u(t)\| \le M e^{-\varepsilon t}\|u(0)\|, \quad \text{for all } t \ge 0, \quad \text{where } M = \|Q\|\|Q^{-1}\|.$$

Proof: We have

$$\begin{aligned}(\lambda B - A)y_k &= (\lambda_k B - A)y_k + (\lambda - \lambda_k)By_k\\ &= (\lambda - \lambda_k)By_k;\end{aligned}$$

hence

$$(\lambda B - A)^{-1}By_k = (\lambda - \lambda_k)^{-1}y_k; \quad \text{for all } \operatorname{Re}\lambda > \omega.$$

Let $x \in D(A) \cap D(B)$, and $u(t) = V(t)x$ be the solution of (1) (with $u_0 = x$). We have $x = \sum_{n=1}^{\infty} C_n y_n$. By (8),

$$\begin{aligned}V(t)x &= \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{\lambda t}(\lambda B - A)^{-1}Bx\, d\lambda = \sum_{n=1}^{\infty} C_n V(t)y_n\\ &= \sum_{n=1}^{\infty} C_n \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{\lambda t}(\lambda - \lambda_k)^{-1}y_n\, d\lambda = \sum_{n=1}^{\infty} e^{\lambda_n t} C_n y_n.\end{aligned}$$

Hence

$$\begin{aligned}\|V(t)x\| &= \left\|\sum_{n=1}^{\infty} e^{\lambda_n t}C_n y_n\right\| = \left\|\sum_{n=1}^{\infty} e^{\lambda_n t}C_n Q e_n\right\|\\ &\le \|Q\|\left\|\sum_{n=1}^{\infty} C_n e^{\lambda_n t} e_n\right\| \le \|Q\|\left\{\sum_{n=1}^{\infty}\left|C_n e^{\lambda_n t}\right|^2\right\}^{1/2}\\ &\le \|Q\|e^{-\varepsilon t}\left(\sum_{n=1}^{\infty}|C_n|^2\right)^{1/2} = \|Q\|e^{-\varepsilon t}\left\|\sum_{n=1}^{\infty} C_n e_n\right\|\\ &= \|Q\|e^{-\varepsilon t}\left\|Q^{-1}\sum_{n=1}^{\infty} C_n y_n\right\| \le \|Q\|\,\|Q^{-1}\|e^{-\varepsilon t}\|x\| = M e^{-\varepsilon t}\|x\|. \quad \blacksquare\end{aligned}$$

Theorem 14 can be generalized to include the case when the Riesz basis consists of eigenvectors and associated root vectors.

Definition 3. Let λ_0 be a generalized eigenvalue of $\lambda B - A$ and $y_0 (\neq 0)$ be the corresponding eigenvector, i.e., $\lambda_0 By_0 - Ay_0 = 0$. Vectors $y_1, y_2, \ldots, y_{m-1}$ are called root vectors associated with y_0, if

$$\begin{aligned}(\lambda_0 B - A)y_1 &= By_0\\ (\lambda_0 B - A)y_2 &= By_1\\ &\vdots\\ (\lambda_0 B - A)y_{m-1} &= By_{m-2}.\end{aligned}$$

The number m is the length of the chain y_0, y_1, $y_2, \ldots, y_{m-1}$ of the generalized eigenvector y_0 and the associated root vectors $y_1, \ldots, y_{m-1}$. The maximal length of chain of root vectors associated with y_0 is called the multiplicity of y_0 and denoted by $m(y_0)$.

Assume that there is a Riesz basis consisting of eigenvectors and associated root vectors, such that each of the eigenvectors has a finite multiplicity $\le m$. We can arrange them as,

$$\begin{gathered}y_1^{(0)}, y_1^{(1)}, \ldots, y_1^{(m_1)}\\ y_2^{(0)}, y_2^{(1)}, \ldots, y_2^{(m_2)}\\ \vdots\\ y_n^{(0)}, y_n^{(1)}, \ldots, y_n^{(m_n)}\\ \vdots\end{gathered}$$

where

$$\begin{aligned}(\lambda_n B - A)y_n^{(0)} &= 0,\\ (\lambda_n B - A)y_n^{(k)} &= By_n^{(k-1)}, \quad 1 \le k < m_n.\end{aligned}$$

Moreover, assume that $m_n = 0$ for all but finitely many n.

Theorem 15. *Assume that* Re $\lambda_k \le -\varepsilon < 0$ *for some* $\varepsilon > 0$, *and all* k. *Then every solution* $u(t)$ *of* (1) *satisfies the following*: *for every* $\delta < \varepsilon$, *there exists* $M = M(\delta)$ *such that*

$$\|u(t)\| \le Me^{-\delta t}\|u(0)\|, \quad \text{for all } t \ge 0.$$

Proof: We have, for every n,

$$\begin{aligned}(\lambda B - A)^{-1}By_n^{(0)} &= (\lambda - \lambda_n)^{-1}y_n^{(0)}\\ (\lambda B - A)y_n^{(1)} &= (\lambda_n B - A)y_n^{(1)} + (\lambda - \lambda_n)By_n^{(1)},\\ &= By_n^{(0)} + (\lambda - \lambda_n)By_n^{(1)},\end{aligned}$$

hence

$$\begin{aligned} y_n^{(1)} &= (\lambda B - A)^{-1}By_n^{(0)} + (\lambda - \lambda_n)(\lambda B - A)^{-1}By_n^{(1)} \\ &= (\lambda - \lambda_n)^{-1}y_n^{(0)} + (\lambda - \lambda_n)(\lambda B - A)^{-1}By_n^{(1)} \end{aligned}$$

or,

$$(\lambda B - A)^{-1}By_n^{(1)} = (\lambda - \lambda_n)^{-1}y_n^{(1)} - (\lambda - \lambda_n)^{-2}y_n^{(0)}.$$

Analogously

$$\begin{aligned} (\lambda B - A)^{-1}By_n^{(2)} &= (\lambda - \lambda_n)^{-1}y_n^{(2)} - (\lambda - \lambda_n)^{-2}y_n^{(1)} + (\lambda - \lambda_n)^{-3}y_n^{(0)} \\ &\vdots \\ (\lambda B - A)^{-1}By_n^{(k)} &= (\lambda - \lambda_n)^{-1}y_n^{(k)} - (\lambda - \lambda_n)^{-2}y_n^{(k-1)} + \cdots \\ &\quad + (-1)^k(\lambda - \lambda_n)^{-k-1}y_n^{(0)} \quad 1 \le k \le m_n. \end{aligned}$$

Let $x \in D(A) \cap D(B)$, and $u(t) = V(t)x$ be the solution of (1) (with $u_0 = x$). We have

$$x = \sum_{n=1}^{\infty} \sum_{k=0}^{m_n} C_n^{(k)} y_n^{(k)}.$$

Hence

$$\begin{aligned} V(t)x &= \sum_{n=1}^{\infty} \sum_{k=0}^{m_n} C_n^{(k)} V(t) y_n^{(k)} \\ &= \sum_{n=1}^{\infty} \sum_{k=0}^{m_n} C_n^{(k)} \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{\lambda t} (\lambda B - A)^{-1} B y_n^{(k)} \, d\lambda \\ &= \sum_{n=1}^{\infty} \sum_{k=0}^{m_n} C_n^{(k)} \sum_{j=0}^{k} \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{\lambda t} (\lambda - \lambda_n)^{-j-1} y_n^{(k-j)} \, d\lambda \\ &= \sum_{n=1}^{\infty} \sum_{k=0}^{m_n} C_n^{(k)} \sum_{j=0}^{k} \frac{1}{j!} t^j e^{\lambda_n t} y_n^{(k-j)} \\ &= \sum_{n=1}^{\infty} \left(\sum_{k=0}^{m_n} \left(\sum_{j=0}^{m_n - k} \frac{1}{j!} t^j C_n^{(j+k)} \right) y_n^{(k)} \right) e^{\lambda_n t}. \end{aligned}$$

The statement now follows easily from (9). ■

In the case $B = I$, Theorem 15 remains through without the assumption that $m_n = 0$ for all but finitely many n, if we assume $m_n \le m < +\infty$ for all n.

Theorem 16. *Assume* $B = I$; *and* $(\lambda I - A)$ *has a Riesz basis consisting of eigenvectors and associated root vectors*; *such that* $\sup m(y_n^{(0)}) = m < \infty$. *If* $\operatorname{Re} \lambda_n \leq -\varepsilon < 0$ *for all* n; *then for every* $0 < \delta < \varepsilon$ *there exists* M *such that* $\|u(t)\| \leq M e^{-\delta t}\|u(0)\|$ *for all* $t \geq 0$.

Under the condition of Theorem 16 the operator A is a spectral operator in the sense of Dunford, i.e., A has the representation

$$A = T + N$$

where T is a spectral operator of scalar type (T is similar to a normal operator) and N is a nilpotent operator, such that $N^{m+1} = 0$. Therefore, the statement in Theorem 16 follows from the following general result.

Recall that a closed linear operator A is called *spectral operator* if A can be represented in the form $A = T + N$, where T is a spectral operator of scalar type and N is a quasinilpotent operator such that $TN = NT$ (see [5]).

Theorem 17. *If* A *is a spectral operator such that*

$$A = T + N$$

and $N^k = 0$ *for some* $k > 1$; *then* $\omega(A) = s(A)$; *where* $\omega(A)$ *is the growth type of* A *and* $s(A)$ *is the spectral abscissa.*

Proof: Since T and N commute, we have

$$e^{tA} = e^{tT} \cdot e^{tN}; \quad t \geq 0.$$

Therefore $\|e^{tA}\| \leq \|e^{tT}\|\|e^{tN}\|$. Since $N^k = 0$,

$$\|e^{tN}\| = \left\| I + tN + \cdots + t^{k-1}N^{k-1} \right\| \leq p(|t|)$$

for some polynomial p. Hence

$$\|e^{tA}\| \leq p(|t|)\,\|e^{tT}\|.$$

From this it follows

$$\omega(A) = \omega(T) = s(T) = s(A)$$

(it is well known that $\sigma(A) = \sigma(T)$, hence $s(T) = s(A)$). ■

Theorem 17 is not true if A is an arbitrary spectral operator.

Example: We construct an example of a spectral operator A such that $s(A) = -1$, but e^{tA} is not exponentially stable. The operator A will have the form

$$A = -I + N$$

where N is quasinilpotent operator. Let

$$H = C^1 \oplus C^2 \oplus \cdots \oplus C^n \oplus \cdots$$

and

$$A = A_1 \oplus A_2 \oplus \cdots \oplus A_n \oplus \cdots$$

where

$$A_1 = -1; \quad A_2 = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}; \quad A_3 = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}; \quad \cdots .$$

It can be seen easily that

$$e^{tA} = e^{tA_1} \oplus e^{tA_2} \oplus \cdots \oplus e^{tA_n} \oplus \cdots$$

where

$$e^{tA_1} = e^{-t}; \quad e^{tA_2} = \begin{pmatrix} e^{-t} & te^{-t} \\ 0 & e^{-t} \end{pmatrix}; \quad e^{tA_3} = \begin{pmatrix} e^{-t} & te^{-t} & \frac{t^2}{2!}e^{-t} \\ 0 & e^{-t} & te^{-t} \\ 0 & 0 & e^{-t} \end{pmatrix}; \quad \cdots .$$

For a corresponding vector

$$x_n = \Big(0; \ldots; 0; \underbrace{\tfrac{1}{\sqrt{n}}; \tfrac{1}{\sqrt{n}}; \ldots; \tfrac{1}{\sqrt{n}}}_{n \text{ terms}}; 0; 0; 0; \ldots\Big)$$

we have $\|x_n\| = 1$ and

$$e^{At}x_n = \Big(0; \ldots; 0; \tfrac{1}{\sqrt{n}}\Big(1 + t + \tfrac{t^2}{2!} + \cdots + \tfrac{t^{n-1}}{(n-1)!}\Big)e^{-t};$$
$$\tfrac{1}{\sqrt{n}}\Big(1 + t + \cdots + \tfrac{t^{n-2}}{(n-2)!}\Big)e^{-t}; \ldots; \tfrac{1}{\sqrt{n}}e^{-t}; 0; \ldots\Big) .$$

We have for large odd n

$$\begin{aligned}\|e^{At}x_n\| = \frac{1}{\sqrt{n}}&\left[\left(1+t+\frac{t^2}{2!}+\cdots+\frac{t^{n-1}}{(n-1)!}\right)^2\right.\\ &+\left(1+t+\frac{t^2}{2!}+\cdots+\frac{t^{n-2}}{(n-2)!}\right)^2\\ &\left.+\cdots+(1+t)^2+1^2\right]^{1/2}e^{-t}\\ \geq \frac{1}{\sqrt{n}}&\left[\left(1+t+\frac{t^2}{2!}+\cdots+\frac{t^{n-1}}{(n-1)!}\right)^2+\cdots\right.\\ &\left.+\left(1+t+\frac{t^2}{2!}+\cdots+\frac{t^{\frac{n-1}{2}}}{\left(\frac{n-1}{2}\right)!}\right)^2\right]^{1/2}e^{-t}\\ \geq \frac{1}{\sqrt{n}}&\left[\frac{n-1}{2}\cdot\left(1+t+\frac{t^2}{2}+\cdots+\frac{t^{\frac{n-1}{2}}}{\left(\frac{n-1}{2}\right)!}\right)^2\right]^{1/2}e^{-t}.\end{aligned}$$

Hence

$$\sup_{\|x\|\leq 1}\|e^{At}x\| \geq \frac{1}{\sqrt{2}}. \qquad \blacksquare$$

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References

1. W. Arendt and C. J. K. Batty, Asymptotically almost periodic solutions of inhomogeneous Cauchy problems on the half-line. *Bull. London Math. Soc.* **31** (1999), no. 3, 291-304.
2. C. J. K. Batty, J. van Neerven and F. Räbiger, Local spectra and individual stability of uniformly bounded C_0-semigroups. *Trans. Amer. Math. Soc.* **350** (1998), no. 5, 2071-2085.
3. R. deLaubenfels, Existence Families, Functional Calculi and Evolution Equations, *Lecture Notes in Math.*, vol. 1570, Springer, 1994.
4. R. deLaubenfels and Vu Quoc Phong, Stability and almost periodicity of solutions of ill-posed abstract Cauchy problems. *Proc. Amer. Math. Soc.* **125** (1997), no. 1, 235-241.

5. N. Dunford and J. T. Schwartz, Linear operators. Part III. Spectral operators. John Wiley & Sons, Inc., New York, 1988.

6. R. O. Griniv, S. Yu. Dobrokhotov, and A. A. Shkalikov, An operator model for the oscillation problem of liquids on an elastic bottom. *Mathematical Notes*, **68** (2000), N1, 57-70.

7. S. G. Krein, *Linear Differential Equations in Banach spaces*. American Mathematical Society, Providence, RI, 1972.

8. S. Zaidman, *Topics in abstract differential equations*. Pitman Research Notes in Mathematics Series, 304. Longman Scientific & Technical, Harlow; copublished in the United States with John Wiley & Sons, Inc., New York, 1994.

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