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FIXED POINTS AND APPROXIMATE FIXED POINTS IN PRODUCT SPACES

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Abstract. The paper deals with the general theme of what is known about the existence of fixed points and approximate fixed points for mappings which satisfy geometric conditions in product spaces. In particular it is shown that if X and Y are metric spaces each of which has the fixed point property for nonexpansive mappings, then the product space $(X \times Y)_\infty$ has the fixed point property for nonexpansive mappings satisfying various contractive conditions. It is also shown that the product space $H = (M \times K)_\infty$ has the approximate fixed point property for nonexpansive mappings whenever M is a metric space which has the approximate fixed point property for such mappings and K is a bounded convex subset of a Banach space.

1. Introduction

The study of fixed point theory for nonexpansive mappings in product spaces is an outgrowth of its analog for continuous mappings. A topological space is said to have the *fixed point property* if every continuous self-map of the space has a fixed point. It has been known for some time that if both X and Y have the fixed point property for continouous mappings, then it need *not* be the case that $X \times Y$ has the fixed point property for mappings $f : X \times Y \to X \times Y$ which are continuous relative to the product topology. Indeed, an example is given in [4] of a metric space X which has the fixed point property, yet the space $X \times X$ fails to have the fixed point property. See, for example, [6] (specifically, Theorem 4.9) for a more extensive discussion.

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In 1968, Nadler [17] initiated a study of fixed point properties of mappings $T : X \times Y \to X \times Y$, where X is a topological space with the fixed point property, Y is a metric space, and T is a continuous mapping which is also a local contraction in its second coordinate. Fora continues this approach in [7].

The above results lead naturally to the question of what happens if both X and Y are metric spaces with the contractive conditions placed directly on T. For this discussion we need to fix some terminology. A mapping f of a metric space (M, d) into a metric space (N, r) is said to be *nonexpansive* if $r(f(x), f(y)) \leq d(x, y)$ for all $x, y \in M$. If $r(f(x), f(y)) < d(x, y)$ for all $x, y \in M$ with $x \neq y$, then f is said to be *strictly contractive*. A mapping f is said to be a *generalized contraction* if for each $x \in M$ there exists $\alpha(x) \in (0, 1)$ such that for each $y \in M$, $r(f(x), f(y)) \leq \alpha(x)d(x, y)$. If α is a constant map, then of course f is a *contraction mapping* in the sense of Banach.

We shall use fix (f) to denote the set of fixed points of a mapping $f : M \to M$.

If (X, ρ) and (Y, d) are metric spaces, then the metric d_∞ on $X \times Y$ is defined in the usual way:

$$d_\infty((x, u), (y, v)) = \max\{\rho(x, y), d(u, v)\}$$

for $(x, u), (y, v) \in X \times Y$. We shall confine ourselves to the metric d_∞ in this paper, although all of the results, indeed in some instances even stronger ones, seem to hold for the metrics d_p, $p \in [1, \infty)$,

$$d_p((x, u), (y, v)) = [(\rho(x, y))^p + (d(u, v))^p]^{1/p}.$$

A basic question now becomes: If (X, ρ) and (Y, d) have the fixed point property for nonexpansive mappings and if $T : X \times Y \to X \times Y$ is nonexpansive relative to the metric d_∞, then does T necessarily have a fixed point? Although sharp results have been obtained, the full answer to this question remains open.

In the next section we summarize what is known about metric fixed point theory in product spaces. In Section 3 we prove some new results for mappings satisfying 'contractive' conditions. In Section 4 we prove a new result about the existence of 'approximate fixed points' for nonexpansive mappings in product spaces by applying a well-known result about asymptotic regularity of 'averaged' nonexpansive mappings.

2. Overview

We begin by summarizing the results of Nadler and Fora. Here and throughout we use P_1 (resp., P_2) to denote the natural coordinate projection of $X \times Y$ onto X

(resp., onto Y). Version (C) of this result is due to Nadler; version (C$'$) to Fora. Alternate proofs of these results are given in [11].

Theorem 2.1. *Suppose X is a topological space which has the fixed point property with respect to continuous mappings, suppose Y is a complete metric space, and suppose $T : X \times Y \to X \times Y$ is a continuous mapping which satisfies*

(C) *for each $x \in X$, there exists a number $\lambda(x) \in (0, 1)$ such that for all $u, v \in Y$,*

$$d(P_2 \circ T(x, u), P_2 \circ T(x, v)) \le \lambda(x)d(u, v).$$

Then T has a fixed point if either

(a) *T is uniformly continuous; or*

(b) *Y is locally compact.*

Assumptions (a) *and* (b) *can be dropped if condition* (C) *is strengthened to*

(C$'$) *for each $x \in X$, there exists a number $\lambda(x) \in (0, 1)$ and a neighborhood V_x such that for each $w \in V_x$ and all $u, v \in Y$,*

$$d(P_2 \circ T(w, u), P_2 \circ T(w, v)) \le \lambda(x)d(u, v).$$

We now turn to nonexpansive mappings in product spaces. In [15], it was shown that if a bounded closed convex subset H of a Banach space has the fixed point property for nonexpansive mappings, and if K is a bounded closed convex subset of either a uniformly convex or uniformly smooth Banach space, then every nonexpansive $T : H \times K \to H \times K$ has a fixed point. This result led to a sequence of generalizations, culminating in a remarkable result of T. Kuczumow [16].

In order to describe Kuczumow's result, we need some additional facts. It is known that, in general, a weakly compact convex subset of a Banach space need not have the fixed point property for nonexpansive mappings (Alspach [1]), but at the same time weak compactness (or reflexivity of the underlying space) in conjunction with a variety of other geometric conditions (e.g., see [9]) does in fact assure that *any* closed convex set has the fixed point property for nonexpansive mappings. In view of this, the following definition is quite natural.

Definition 2.1. *A closed convex subset K is said to have the generic fixed point property* (*for nonexpansive mappings*) *if for every nonexpansive $T : K \to K$ and every T-invariant nonempty closed convex $H \subseteq K$,* $\mathrm{fix}\,(T) \cap H \neq \emptyset$.

Kuczumow used a retraction approach based on a method of Bruck [3] to prove the following.

Theorem 2.2. *Let X be a Banach space. Suppose $K \subseteq X$ is weakly compact convex and has the generic fixed point property, and suppose (Y, d) is a metric* *space which has the fixed point property for nonexpansive mappings. Then every nonexpansive* $T : (K \times Y)_\infty \to (K \times Y)_\infty$ *has a fixed point.*

Kuczumow observed that if X is a conjugate space, then the weak topology in the above result can be replaced by the weak* topology.

Bruck's paper [3] is remarkably rich in ideas, and in fact a different approach found in the same paper can be modified to prove the following result. The details are found in [12].

Theorem 2.3. *Let E be a Banach space. Suppose $X \subseteq E$ is a separable closed convex subset of E which has the generic fixed point property, and suppose (Y, d) is a separable metric space which has the fixed point property for nonexpansive mappings. Then every nonexpansive $T : (X \times Y)_\infty \to (X \times Y)_\infty$ has a fixed point.*

While its method of proof is different, it is not clear to what extent, if any, Theorem 2.3 is actually qualitatively more general than Theorem 2.2. This is because there is no known example of a closed convex subset of a Banach space which has the generic fixed point property yet fails to be weakly compact.

There are perhaps two additional results which should be mentioned. While we are basically interested here in the case $p = \infty$, it is quite easy to prove the following for $1 \le p < \infty$.

Theorem 2.4. *Let E and F be Banach spaces. Suppose $X \subseteq E$ and $Y \subseteq F$ both have the fixed point property for nonexpansive mappings. Then every nonexpansive $T : (X \times Y)_p \to (X \times Y)_p$ has a fixed point for $1 \le p < \infty$.*

A proof of the above result is given in [14], based on an argument given for the following result in [11].

Theorem 2.5. *Let E and F be Banach spaces. Suppose $X \subseteq E$ and $Y \subseteq F$ both have the fixed point property for generalized contractions. Then every generalized contraction $T : (X \times Y)_p \to (X \times Y)_p$ has a fixed point for $1 \le p \le \infty$.*

This completes an overview of what appear to be the most important known results. We now turn to some new observations.

3. Contractive Mappings in Product Spaces

If the assumption of nonexpansiveness is strengthened, then it is possible to prove additional results in a fairly direct manner.

Theorem 3.1. *Let (X, ρ) and (Y, d) be metric spaces. Suppose Y has the fixed point property for nonexpansive mappings and suppose X has the fixed point property for strictly contractive mappings, and suppose $T : (X \times Y)_\infty \to (X \times Y)_\infty$ is a nonexpansive mapping which satisfies the additional condition*

$$\rho(P_1 \circ T(x,u), P_1 \circ T(y,v)) < d_\infty((x,u),(y,v))$$

for all $(x,u),(y,v) \in X \times Y$ satisfying $\rho(x,y) \neq d(u,v)$. Then T has a fixed point.

Theorem 3.2. *Let (X, ρ) and (Y, d) be metric spaces, each of which has the fixed point property for strictly contractive mappings. Then every strictly contractive mapping $T : (X \times Y)_\infty \to (X \times Y)_\infty$ has a fixed point.*

Proof of Theorem 3.1. Fix $u \in Y$ and define $T_u : X \to X$ by setting

$$T_u(x) = P_1 \circ T(x,u), \quad x \in X.$$

Then if $x \neq y$, it follows that $\rho(x,y) \neq d(u,u) = 0$, and we have

$$\begin{aligned}\rho(T_u(x), T_u(y)) &= \rho(P_1 \circ T(x,u)), \rho(P_1 \circ T(y,u)) < d_\infty((x,u),(y,u)) \\ &= \rho(x,y).\end{aligned}$$

Thus T_u is strictly contractive and by assumption has a unique fixed $g(u) \in X$.

Now define

$$\varphi(u) = P_2 \circ T(g(u), u).$$

We show that φ is nonexpansive. Note that since $T_u(g(u)) = g(u)$ and $T_v(g(v)) = g(v)$, we have

$$g(u) = P_1 \circ T(g(u), u); \; g(v) = P_1 \circ T(g(v), v),$$

and, moreover, if $\rho(g(u),\ g(v)) \neq d(u,v)$, then

$$\begin{aligned}\rho(g(u), g(v)) &= \rho(P_1 \circ T(g(u), u), P_1 \circ T(g(v), v)) \\ &< d_\infty((g(u), u), (g(v), v)) \\ &= \max\{\rho(g(u), g(v)), d(u,v)\} \\ &= d(u,v).\end{aligned}$$

Therefore, $\rho(g(u), g(v)) \leq d(u,v)$ for all $u, v \in Y$. It follows that

$$\begin{aligned}&d(\varphi(u), \varphi(v)) = d(P_2 \circ T(g(u), u), P_2 \circ T(g(v), v)) \\ &\leq \max\{\rho(P_1 \circ T(g(u), u), P_1 \circ T(g(v), v)), d(P_2 \circ T(g(u), u), P_2 \circ T(g(v), v))\} \\ &= d_\infty(T(g(u), u), T(g(v), v)) \leq d_\infty((g(u), u), (g(v), v)) \\ &= \max\{\rho(g(u), g(v)), d(u,v)\} = d(u,v).\end{aligned}$$

Therefore, $\varphi : Y \to Y$ is nonexpansive. Since Y has the fixed point property for nonexpansive mappings, there exists $u \in Y$ such that $\varphi(u) = u$; whence

$$u = \varphi(u) = P_2 \circ T(g(u), u).$$

Since by assumption $g(u) \in \operatorname{fix}(T_u)$, we have $T_u(g(u)) = P_1 \circ T(g(u), u)$. ■

Proof of Theorem 3.2. The argument follows the previous one, except in this case we must show that φ is strictly contractive. The fact that T is strictly contractive assures that

$$\begin{aligned}&\max\{\rho(P_1 \circ T(x,u), P_1 \circ T(y,v)), d(P_2 \circ T(x,u), P_2 \circ T(y,v))\}\\&< \max\{\rho(x,y), d(u,v)\}\end{aligned}$$

if $x \neq y$ or $u \neq v$. Following the previous argument step-by-step, we conclude that for $u \in Y$ and $x \neq y$, the mapping T_u is strictly contractive and has a unique fixed point $g(u)$. Also, if $u \neq v$ we have

$$\begin{aligned}d(\varphi(u), \varphi(v)) &= d(P_2 \circ T(g(u), u), P_2 \circ T(g(v), v))\\&\leq d_\infty(T(g(u), u), T(g(v), v))\\&< d_\infty((g(u), u), (g(v), v))\\&= d(u, v).\end{aligned}$$

The conclusion now follows as in Theorem 3.1. ■

The following is a variant of Theorem 3.1. The assumptions on the mapping T do not seem to be comparable.

Theorem 3.3. *Let (X, ρ) and (Y, d) be metric spaces, each of which has the fixed point property for nonexpansive mappings, and suppose $T : (X \times Y)_\infty \to (X \times Y)_\infty$ is a nonexpansive mapping which satisfies the additional condition*

$$\rho(P_1 \circ T(x, u, P_1 \circ T(y, v)) < d_\infty((x, u), (y, v))$$

for all $(x, u), (y, v) \in X \times Y$ satisfying $u \neq v$ and $x \neq y$. Then T has a fixed point.

Theorem 3.3 has the following immediate corollary.

Corollary 3.1. *Let (X, ρ) and (Y, d) be metric spaces, each of which has the fixed point property for nonexpansive mappings, and suppose $T : (X \times Y)_\infty \to (X \times Y)_\infty$ is a nonexpansive mapping which is quasi-contractive in the sense that*

$$d_\infty(T(x, u), T(y, v)) < d_\infty((x, u), (y, v))$$

for all $(x, u), (y, v) \in X \times Y$ *satisfying* $u \neq v$ *and* $x \neq y$. *Then* T *has a* $($*unique*$)$ *fixed point.*

Note that the condition of the corollary is weaker than the contractive condition of Theorem 3.2. In exchange, a little more is assumed about the spaces; specifically that X has the fixed point property for nonexpansive mappings.

Proof of Theorem 3.3. Fix $u \in Y$ and as before define $T_u : X \to X$ by setting

$$T_u(x) = P_1 \circ T(x, u), \quad x \in X.$$

Then

$$\begin{aligned}
&\rho(T_u(x), T_u(y)) \\
&\leq \max\{\rho(P_1 \circ T(x, u), P_1 \circ T(y, u)), d(P_2 \circ T(x, u), P_2 \circ T(y, u))\} \\
&= d_\infty(T(x, u), T(y, u)) \\
&\leq d_\infty((x, u), (y, u)) \\
&= \rho(x, y).
\end{aligned}$$

Thus T_u is nonexpansive and by assumption has a nonempty fixed point set $\mathrm{fix}\,(T_u) \subseteq X$. Let g be any selection of the mapping

$$u \mapsto \mathrm{fix}\,(T_u)$$

and define φ as in Theorem 3.1. We show that φ is nonexpansive. Since $g(u) \in \mathrm{fix}\,(T_u)$ and $g(v) \in \mathrm{fix}\,(T_v)$, we have

$$g(u) = P_1 \circ T(g(u), u); g(v) = P_1 \circ T(g(v), v).$$

Now let $u, v \in Y$. There are two cases.

1. If $g(u) = g(v)$, then obviously $\rho(g(u), g(v)) \leq d(u, v)$ and we have

$$\begin{aligned}
&d(\varphi(u), \varphi(v)) = d(P_2 \circ T(g(u), u), P_2 \circ T(g(v), v)) \\
&\leq \max\{\rho(P_1 \circ T(g(u), u), P_1 \circ T(g(v), v)), d(P_2 \circ T(g(u), u), P_2 \circ T(g(v), v))\} \\
&= d_\infty(T(g(u), u), T(g(v), v)) \leq d_\infty((g(u), u), (g(v), v)) \leq d(u, v).
\end{aligned}$$

2. On the other hand, if $g(u) \neq g(v)$, then it must also be the case that $u \neq v$. Therefore,

$$\begin{aligned}
\rho(g(u), g(v)) &= \rho(P_1 \circ T(g(u), u), P_1 \circ T(g(v), v)) \\
&< d_\infty((g(u), u), (g(v), v)) \\
&= \max\{\rho(g(u), g(v)), d(u, v)\} \\
&= d(u, v)
\end{aligned}$$

and it follows that

$$\begin{aligned}
&d(\varphi(u),\varphi(v)) = d(P_2\circ T(g(u),u), P_2\circ T(g(v),v))\\
&\le \max\{\rho(P_1\circ T(g(u),u), P_1\circ T(g(v),v)), d(P_2\circ T(g(u),u), P_2\circ T(g(v),v))\}\\
&= d_\infty(T(g(u),u), T(g(v),v))\\
&\le d_\infty((g(u),u),(g(v),v))\\
&= \max\{\rho(g(u),g(v)), d(u,v)\} = d(u,v).
\end{aligned}$$

Therefore, in either case, $d(\varphi(u),\varphi(v)) \le d(u,v)$, and $\varphi : Y \to Y$ is nonexpansive. The conclusion again follows as in Theorem 3.1. ■

In the preceding proof, the question might arise as to whether fix (T_u) is a singleton. Suppose otherwise, and let $g_1(u)$ and $g_2(u)$ be distinct choices for the selection $u \mapsto$ fix (T_u). Then according to case 2, for any $v \in Y$,

$$d(g_1(u), g_2(u)) \le d(g_1(u), v) + d(g_2(u), v) < 2d(u,v).$$

Obviously, this can happen only if u is an isolated point of Y.

To facilitate comparison, we summarize the foregoing results as follows:

Theorem 3.4. *Let (X,ρ) and (Y,d) be metric spaces, and suppose $T : (X\times Y)_\infty \to (X\times Y)_\infty$ is a nonexpansive mapping. Then T has a fixed point if any one of the following conditions holds.*

(a) *X and Y have the fixed point property for nonexpansive mappings and T satisfies*

$$d_\infty(T(x,u),T(y,v)) < d_\infty((x,u),(y,v))$$

for all $(x,u),(y,v)\in X\times Y$ satisfying $u\ne v$ and $x\ne y$.

(b) *Y has the fixed point property for nonexpansive mappings, X has the fixed point property for strictly contractive mappings, and $T : (X\times Y)_\infty \to (X\times Y)_\infty$ satisfies*

$$\rho(P_1\circ T(x,u), P_1\circ T(y,v)) < d_\infty((x,u),(y,v))$$

for all $(x,u),(y,v)\in X\times Y$ satisfying $\rho(x,y)\ne d(u,v)$.

(c) *X and Y have the fixed point property for strictly contractive mappings and T is strictly contractive.*

4. Approximate Fixed Points in Product Spaces

In this section we prove an approximate fixed point theorem for nonexpansive mappings in certain product spaces. A metric space (M, d) is said to have the *approximate fixed point property* if any nonexpansive mapping $T : M \to M$ has an approximate fixed point sequence, that is, a sequence $\{u_n\}$ in M for which $\lim_n d(u_n, T(u_n)) = 0$. This of course is equivalent to saying

$$\inf\{d(x, T(x)) : x \in M\} = 0.$$

Our theorem is based on the following result, which was proved for a single mapping (and for a more general convergence process) by Ishikawa [10]. Edelstein and O'Brien [5] showed that the convergence is uniform over K, and subsequently Goebel and Kirk [8] showed that in fact the convergence is uniform over x_0 in K and over the class of all nonexpansive mappings $T : K \to K$. Another proof of this fact is given in [13]. For a technical study of the rate of uniform convergence and a comprehensive review of the literature, see [2].

Theorem 4.1. *Let K be a bounded convex subset of a Banach space and let $\varepsilon > 0$. Then there exists $N \in \mathbb{N}$ such that if $n \geq N$, if $x_0 \in K$, and if $T : K \to K$ is nonexpansive, then*

$$\|f^n(x_0) - f^{n+1}(x_0)\| \leq \varepsilon,$$

where $f = (1/2)(I + T)$.

An interesting feature of the proof given below is the fact that the penultimate step of the proof requires the *uniformity* of the convergence of $\{f^n(x_0)\}$ in the above result over the class of all nonexpansive $T : K \to K$.

Theorem 4.2. *Suppose M is a metric space which has the approximate fixed point property for nonexpansive mappings and suppose K is a bounded closed convex subset of a Banach space X. Let*

$$H = (K \times M)_\infty.$$

Then H has the approximate fixed point property for nonexpansive mappings.

Proof. Let $T : H \to H$ be nonexpansive and let P_1 and P_2 denote the respective coordinate projections of H onto K and M. Fix $y \in M$ and define $T_y : K \to K$ by setting

$$T_y(x) = P_1 \circ T(x, y), \qquad x \in K.$$

Now fix $x_0 \in K$ and set $f_y = (I + T_y)/2$.

Now, if $u, v \in M$, then

$$\begin{aligned}\|f_u(x_0) - f_v(x_0)\| &= (1/2)\|T_u(x_0) - T_v(x_0)\| \\ &= (1/2)\|P_1 \circ T(u, x_0) - P_1 \circ T(v, x_0)\| \\ &\leq (1/2)d_\infty(T(u, x_0), T(v, x_0)) \\ &\leq (1/2)d_\infty((u, x_0), (v, x_0)) \\ &= (1/2)d(u, v).\end{aligned}$$

Suppose $\|f_u^n(x_0) - f_v^n(x_0)\| \leq d(u, v)$. Then

$$\begin{aligned}&\|f_u^{n+1}(x_0) - f_v^{n+1}(x_0)\| \\ &= \|f_u \circ f_u^n(x_0) - f_v \circ f_v^n(x_0)\| \\ &= (1/2)\|f_u^n(x_0) - T_u(f_u^n(x_0)) + f_v^n(x_0) - T_v(f_v^n(x_0))\| \\ &\leq (1/2)\|f_u^n(x_0) - f_v^n(x_0)\| \\ &+ (1/2)\|T_u(f_u^n(x_0)) - T_v(f_v^n(x_0))\| \\ &\leq (1/2)d(u, v) + (1/2)\|P_1 \circ T(u, f_u^n(x_0)) - P_1 \circ T(v, f_v^n(x_0))\| \\ &\leq (1/2)d(u, v) + (1/2)\|T(u, f_u^n(x_0)) - T(v, f_v^n(x_0))\| \\ &\leq (1/2)d(u, v) + (1/2)\|(u, f_u^n(x_0)) - (v, f_v^n(x_0))\| \\ &\leq d(u, v).\end{aligned}$$

By induction, we conclude that

$$\|f_u^n(x_0) - f_v^n(x_0)\| \leq d(u, v)$$

for all $u, v \in M$; $n \in \mathbb{N}$. Now for each n, define $\varphi_n : M \to M$ via the relation

$$\varphi_n(x) = P_2 \circ T(f_x^n(x_0), x).$$

Then

$$\begin{aligned}d(\varphi_n(u), \varphi_n(v)) &= d(P_2 \circ T(f_u^n(x_0), u), P_2 \circ T(f_v^n(x_0), v)) \\ &\leq d_\infty((f_u^n(x_0), u), (f_v^n(x_0), v)) \\ &\leq d(u, v).\end{aligned}$$

By assumption, there exists $y_n \in M$ such that $d(\varphi_n(y_n), y_n) \leq 1/n$, $n = 1, 2, \cdots$. Thus we have

$$\begin{aligned} d(P_2 \circ T(f^n_{y_n}(x_0), y_n), y_n) &= d(P_2 \circ T(f^n_{y_n}(x_0), y_n), P_2(f^n_{y_n}(x_0), y_n)) \\ &\leq 1/n. \end{aligned}$$

Moreover, by Theorem 4.1,

$$\begin{aligned} &\|P_1(f^n_{y_n}(x_0), y_n) - P_1 \circ T(f^n_{y_n}(x_0), y_n)\| \\ &= \|f^n_{y_n}(x_0) - P_1 \circ T(f^n_{y_n}(x_0), y_n)\| \\ &= \|f^n_{y_n}(x_0) - T_{y_n}(f^n_{y_n}(x_0))\| \\ &= (1/2)\|f^n_{y_n}(x_0) - f^{n+1}_{y_n}(x_0)\| \to 0. \end{aligned}$$

Therefore,

$$\begin{aligned} &d_\infty((f^n_{y_n}(x_0), y_n), T(f^n_{y_n}(x_0), y_n)) \\ &= \max\{(1/2)\|f^n_{y_n}(x_0) - f^{n+1}_{y_n}(x_0)\|, 1/n\} \to 0, \end{aligned}$$

and $\{(f^n_{y_n}(x_0), y_n)\}$ is an approximate fixed point sequence for T. ∎

The proof above is suggested by the approach of [15].

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