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THE BEST APPROXIMATION BY PROJECTIONS IN BANACH SPACES

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Dedicated to Professor Kôzô Yabuta on his sixtieth birthday

Abstract. We consider the best approximation by projections in Banach spaces under certain suitable conditions. Furthermore, applications are discussed for multiplier operators and convolution type operators associated with strongly continuous families of bounded linear operators as well as for homogeneous Banach spaces which include the classical function spaces, as particular cases.

1. Introduction

Let X be a Banach space with norm $\|\cdot\|$, and let $B[X]$ denote the Banach algebra of all bounded linear operators of X into itself with the usual operator norm, which will be denoted by the same symbol $\|\cdot\|$. Let $\mathbb{Z}$ denote the set of all integers, and let $\mathfrak{P} = \{P_j : j \in \mathbb{Z}\}$ be a sequence of projection operators in $B[X]$ satisfying the following conditions:

(P-1) $\mathfrak{P}$ is orthogonal, i.e., $P_j P_n = \delta_{j,n} P_n$ for all $j, n \in \mathbb{Z}$, where $\delta_{j,n}$ denotes Kronecker's symbol.

(P-2) $\mathfrak{P}$ is fundamental, i.e., the linear span of the set $\cup_{j\in\mathbb{Z}} P_j(X)$ is dense in X.

(P-3) $\mathfrak{P}$ is total, i.e., if $f \in X$ and $P_j(f) = 0$ for all $j \in \mathbb{Z}$, then $f = 0$.

Let $\mathbb{N}$ be the set of all nonnegative integers. For each $n \in \mathbb{N}$, M_n stands for the linear span of the set $\{P_j(X) : |j| \leq n\}$, which is a closed linear subspace of X. Let $\mathfrak{T}_n$ denote the set of all bounded linear operators T of X into M_n such that $T(f) = f$ for all $f \in M_n$. In other words, $\mathfrak{T}_n$ is the set of all bounded linear projections of X onto M_n.

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In this paper, we consider the best approximation by operators in $\mathfrak{T}_n$ under certain suitable conditions. Moreover, applications are discussed for multiplier operators (cf. [1, 4, 5, 11]) and convolution type operators associated with strongly continuous families of operators in $B[X]$ (cf. [4]) as well as for homogeneous Banach spaces (cf. [2, 4, 8, 12]), which include the Banach space $C_{2\pi}$ of all 2π-periodic, continuous functions f on the real line $\mathbb{R}$ with the norm

$$\|f\|_\infty = \max\{|f(t)| : |t| \le \pi\}$$

and the Banach space $L^p_{2\pi}$ of all 2π-periodic, pth power Lebesgue integrable functions f on $\mathbb{R}$ with the norm

$$\|f\|_p = \Big(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(t)|^p dt\Big)^{1/p} \qquad (1 \le p < \infty),$$

as special cases.

For the general theory of the best approximation in normed linear spaces, we refer to [10].

2. Best Approximation by Projections

Let (Ω, μ) be a probability measure space. Let $\mathfrak{T} = \{T_t : t \in \Omega\}$ and $\mathfrak{U} = \{U_t : t \in \Omega\}$ be uniformly bounded families of operators in $B[X]$ such that for all $f \in X$ and all $T \in B[X]$, the mapping $t \mapsto T_t T U_t(f)$ is strongly μ-measurable on Ω. For any $T \in B[X]$, we define

$$\Phi_T(f) = \Phi_T(\mathfrak{T}, \mathfrak{U}; f) = \int_\Omega T_t T U_t(f)\, d\mu(t) \qquad (f \in X),$$

which always exists as a Bochner integral in X. Then Φ_T belongs to $B[X]$ and the uniform boundedness of $\mathfrak{T}$ and $\mathfrak{U}$ yields

$$\|\Phi_T\| \le AB\|T\|,$$

where

$$A = \sup\{\|T_t\| : t \in \Omega\} < \infty \tag{1}$$

and

$$B = \sup\{\|U_t\| : t \in \Omega\} < \infty. \tag{2}$$

From now on we suppose that the following additional conditions

$$T_t P_j = P_j T_t \quad \text{for all } j \in \mathbb{Z},\ t \in \Omega, \tag{3}$$

(4) $$U_t P_j = P_j U_t \quad \text{for all } j \in \mathbb{Z},\ t \in \Omega,$$

and

(5) $$T_t U_t = I \quad \text{for all } t \in \Omega,$$

where I is the identity operator on X.

Lemma 2.1. *Let $T \in B[X]$. If $T_t T = T T_t$ or $U_t T = T U_t$ for all $t \in \Omega$, then $\Phi_T = T$.*

Proof. Let $f \in X$ and suppose that $T_t T = T T_t$ for every $t \in \Omega$. Then by (5), we have

$$\Phi_T(f) = \int_\Omega (T T_t) U_t(f)\, d\mu(t) = \int_\Omega T I(f)\, d\mu(t) = T(f).$$

The case of $U_t T = T U_t$ is similar. ■

For each $n \in \mathbb{N}$, we define

$$S_n = \sum_{j=-n}^{n} P_j,$$

which belongs to $\mathfrak{T}_n$. Then (3) and (4) imply

(6) $$S_n T_t = T_t S_n, \quad S_n U_t = U_t S_n \qquad (n \in \mathbb{N},\ t \in \Omega).$$

Lemma 2.2. *If $T \in \mathfrak{T}_n$, then $\Phi_T \in \mathfrak{T}_n$.*

Proof. Let $f \in X$. Then we have

$$T U_t(f) = S_n(T U_t(f)) \qquad (t \in \Omega),$$

and so (6) gives

$$\begin{aligned}\Phi_T(f) &= \int_\Omega T_t(S_n(T U_t(f)))\, d\mu(t) = \int_\Omega S_n(T_t T U_t(f))\, d\mu(t) \\ &= S_n\left(\int_\Omega T_t T U_t(f)\, d\mu(t)\right) = S_n(\Phi_T(f)).\end{aligned}$$

Therefore, Φ_T maps X into M_n. Also, if $f \in M_n$, then (6) gives

$$U_t(f) = U_t(S_n(f)) = S_n(U_t(f)),$$

and so $T(U_t(f)) = T(S_nU_t(f)) = S_nU_t(f) = U_t(f)$. Thus by (5), we have

$$\begin{aligned}\Phi_T(f) &= \int_\Omega T_t(TU_t(f))\,d\mu(t)\\ &= \int_\Omega (T_tU_t(f))\,d\mu(t) = \int_\Omega I(f)\,d\mu(t) = f \quad (f \in M_n).\end{aligned}$$ ∎

For each $n \in \mathbb{N}$, we define

$$\mathfrak{T}_n^* = \{T \in \mathfrak{T}_n : \Phi_T P_j = 0 \text{ for all } j \in \mathbb{Z}, |j| > n\}.$$

By (P-1), (6) and Lemma 2.1, S_n belongs to $\mathfrak{T}_n^*$.

Lemma 2.3. *If $T \in \mathfrak{T}_n^*$, then $\Phi_T = S_n$.*

Proof. Let $T \in \mathfrak{T}_n$ and suppose that

$$\Phi_T P_j = 0 \quad \text{whenever } j \in \mathbb{Z},\ |j| > n. \tag{7}$$

Since Φ_T and S_n are continuous linear operators on X, it will suffice to show that $\Phi_T(P_j(f)) = S_n(P_j(f))$ for all $f \in X$ and all $j \in \mathbb{Z}$ because of Condition (P-2). If $|j| \le n$, then $S_n(P_j(f)) = P_j(f)$ and by Lemma 2.2, we have $\Phi_T(P_j(f)) = P_j(f)$. If $|j| > n$, then Condition (P-1) and (7) give

$$S_n(P_j(f)) = \sum_{k=-n}^{n} P_k(P_j(f)) = \sum_{k=-n}^{n} \delta_{k,j}P_j(f) = 0 = \Phi_T(P_j(f)).$$ ∎

We are now in a position to establish the following main result.

Theorem 2.4. *Let S be an operator in $B[X]$ such that $SU_t = U_tS$ or $ST_t = T_tS$ for all $t \in \Omega$. Then we have*

$$\|S - S_n\| \le AB \inf\{\|S - T\| : T \in \mathfrak{T}_n^*\}. \tag{8}$$

In particular, if $AB \le 1$, then

$$\|S - S_n\| = \min\{\|S - T\| : T \in \mathfrak{T}_n^*\},$$

which implies that S_n is an operator of best approximation to S from $\mathfrak{T}_n^*$.

Proof. Suppose that $SU_t = U_tS$ for all $t \in \Omega$. Let $f \in X$ and $T \in \mathfrak{T}_n^*$. Then by Lemma 2.3 and (5), we have

$$\begin{aligned}(S - S_n)(f) &= (S - \Phi_T)(f) = \int_\Omega (S - T_tTU_t)(f)\,d\mu(t)\\ &= \int_\Omega (T_tU_tS - T_tTU_t)(f)\,d\mu(t) = \int_\Omega (T_tSU_t - T_tTU_t)(f)\,d\mu(t)\\ &= \int_\Omega (T_t(S - T)U_t)(f)\,d\mu(t) = \Phi_{S-T}(f).\end{aligned}$$

Therefore, we obtain

$$\|S - S_n\| = \|\Phi_{S-T}\| \leq AB\|S - T\|,$$

which yields the desired inequality (8). The case of $ST_t = T_tS$ is similar. ■

Corollary 2.5. *Let α be a scalar. Then*

$$\|\alpha I - S_n\| \leq AB \inf\{\|\alpha I - T\| : T \in \mathfrak{T}_n^*\}.$$

In particular, if $AB \leq 1$, then

$$\|\alpha I - S_n\| = \min\{\|\alpha I - T\| : T \in \mathfrak{T}_n^*\}.$$

Let $\mathfrak{V} = \{V_t : t \in \Omega\}$ be a uniformly bounded family of operators in $B[X]$ such that for each $f \in X$, the mapping $t \mapsto V_t(f)$ is strongly μ-measurable on Ω and let χ be a μ-integrable function on Ω. Then we define the convolution type operator $W_{\mathfrak{V},\chi}$ associated with $\mathfrak{V}$ and χ by

$$W_{\mathfrak{V},\chi}(f) = \int_\Omega \chi(t)V_t(f)\, d\mu(t) \qquad (f \in X), \tag{9}$$

which exists as a Bochner integral in X (cf. [4]). Clearly, $W_{\mathfrak{V},\chi}$ belongs to $B[X]$ and

$$\|W_{\mathfrak{V},\chi}\| \leq C \int_\Omega |\chi(t)|\, d\mu(t),$$

where

$$C = \sup\{\|V_t\| : t \in \Omega\} < \infty. \tag{10}$$

Corollary 2.6. *Suppose that $V_uU_t = U_tV_u$ or $V_uT_t = T_tV_u$ for all $t, u \in \Omega$. Then the claim of Theorem 2.4 holds for $S = W_{\mathfrak{V},\chi}$.*

3. Applications

For any $f \in X$, we associate its (formal) Fourier series expansion

$$f \quad \sim \quad \sum_{j=-\infty}^{\infty} P_j(f). \tag{11}$$

An operator $T \in B[X]$ is called a multiplier operator on X if there exists a sequence $\{\tau_j : j \in \mathbb{Z}\}$ of scalars such that for every $f \in X$,

$$T(f) \quad \sim \quad \sum_{j=-\infty}^{\infty} \tau_j P_j(f),$$

and the following notation is used:

$$T \sim \sum_{j=-\infty}^{\infty} \tau_j P_j$$

(cf. [1, 4, 5, 11]). Let $M[X]$ denote the set of all multiplier operators on X, which is a commutative closed subalgebra of $B[X]$ containing I and S_n, which is the nth partial sum operator associated with the Fourier series (11).

From now on, let Ω be a separable topological space and μ a probability measure on Ω.

Let $\mathfrak{T} = \{T_t : t \in \Omega\}$ and $\mathfrak{U} = \{U_t : t \in \Omega\}$ be families of operators in $M[X]$ satisfying (1) and (2) and having the expansions

$$T_t \sim \sum_{j=-\infty}^{\infty} e_j(t) P_j \qquad (t \in \Omega) \tag{12}$$

and

$$U_t \sim \sum_{j=-\infty}^{\infty} f_j(t) P_j \qquad (t \in \Omega), \tag{13}$$

where $\{e_j : j \in \mathbb{Z}\}$ and $\{f_j : j \in \mathbb{Z}\}$ are sequences of scalar-valued continuous functions on Ω such that

$$e_j(t) f_j(t) = 1 \quad \text{for all } j \in \mathbb{Z},\ t \in \Omega. \tag{14}$$

By (12), we have

$$\lim_{t \to u} \|T_t(g) - T_u(g)\| = \lim_{t \to u} |e_j(t) - e_j(u)|\, \|g\| = 0 \quad (u \in \Omega)$$

for every $g \in P_j(X), j \in \mathbb{Z}$. Therefore, the mapping $t \mapsto T_t(f)$ is strongly continuous on Ω for each $f \in X$, since $\mathfrak{P}$ is fundamental and $\mathfrak{T}$ is uniformly bounded. Similarly, the mapping $t \mapsto U_t(f)$ is strongly continuous on Ω for each $f \in X$. Therefore, the mapping $t \mapsto T_t T U_t(f)$ is strongly continuous on Ω. Also, Conditions (3), (4) and (5) hold because of (12), (13), (14) and Condition (P-3). Consequently, all the results obtained in the preceding section hold under the above setting.

Now, we suppose that

$$\int_{\Omega} e_j(t) f_k(t)\, d\mu(t) = 0 \quad \text{whenever } j \neq k. \tag{15}$$

Lemma 3.1. $\mathfrak{T}_n^* = \mathfrak{T}_n$.

Proof. It will suffice to show that every $T \in \mathfrak{T}_n$ satisfies (7). Let $j \in \mathbb{Z}, |j| > n$ and $f \in X$. Then by (12), (13) and (15), we have

$$\begin{aligned}
\Phi_T(P_j(f)) &= \int_\Omega T_t T U_t(P_j(f))\, d\mu(t) \\
&= \int_\Omega (T_t T)(P_j U_t(f))\, d\mu(t) = \int_\Omega (T_t T)(f_j(t) P_j(f))\, d\mu(t) \\
&= \int_\Omega f_j(t) T_t(T P_j(f))\, d\mu(t) = \int_\Omega f_j(t) T_t(S_n(T P_j(f)))\, d\mu(t) \\
&= \int_\Omega f_j(t) S_n(T_t T P_j(f))\, d\mu(t) = \sum_{k=-n}^{n} \int_\Omega f_j(t) P_k(T_t T P_j(f))\, d\mu(t) \\
&= \sum_{k=-n}^{n} \int_\Omega f_j(t) e_k(t) P_k(T P_j(f))\, d\mu(t) \\
&= \sum_{k=-n}^{n} \Big\{ \int_\Omega f_j(t) e_k(t)\, d\mu(t) \Big\} P_k(T P_j(f)) = 0 \quad (|j| > n),
\end{aligned}$$

which implies (7). ■

Theorem 3.2. *Let $S \in M[X]$. Then*

$$\|S - S_n\| \leq AB \inf\{\|S - T\| : T \in \mathfrak{T}_n\}.$$

In particular, if $AB \leq 1$, then S_n is an operator of best approximation to S from $\mathfrak{T}_n$.

Proof. Since S commutes with U_t and T_t for every $t \in \Omega$, this follows from Lemma 3.1 and Theorem 2.4. ■

Let $\mathfrak{V} = \{V_t : t \in \Omega\}$ be a family of operators in $M[X]$ satisfying (10) and having the expansions

$$V_t \sim \sum_{j=-\infty}^{\infty} v_j(t) P_j \qquad (t \in \Omega), \tag{16}$$

where $\{v_j : j \in \mathbb{Z}\}$ is a sequence of scalar-valued continuous functions on Ω. Then the convolution type operator $W_{\mathfrak{V},\chi}$ given by (9) belongs to $M[X]$ and

$$W_{\mathfrak{V},\chi} \sim \sum_{j=-\infty}^{\infty} c_j(\mathfrak{V}, \chi) P_j, \tag{17}$$

where

$$c_j(\mathfrak{V}, \chi) = \int_{\Omega} \chi(t) v_j(t)\, d\mu(t) \qquad (j \in \mathbb{Z}).$$

Thus we have the following corollary.

Corollary 3.3. *The claim of Theorem* 3.2 *holds for* $S = W_{\mathfrak{V},\chi}$.

Theorem 3.4. *Let* α *be a scalar. Then we have*

$$\|\alpha I - S_n\| \le AB \inf\{\|\alpha I - T\| : T \in \mathfrak{T}_n\}.$$

In particular, *if* $AB \le 1$, *then*

$$\|\alpha I - S_n\| = \min\{\|\alpha I - T\| : T \in \mathfrak{T}_n\}.$$

Proof. This follows from Lemma 3.1 and Corollary 2.5. ∎

Remark 1. Suppose that

$$A = \sup\{\|T_t\| : t \in \mathbb{R}\} < \infty$$

and

$$T_t \quad \sim \quad \sum_{j=-\infty}^{\infty} e^{\lambda_j t} P_j \qquad (t \in \mathbb{R}), \tag{18}$$

where $\{\lambda_j : j \in \mathbb{Z}\}$ is a sequence of scalars. Then $\mathfrak{T} = \{T_t : t \in \mathbb{R}\}$ becomes a strongly continuous group of operators in $B[X]$ and

$$G(f) \quad \sim \quad \sum_{j=-\infty}^{\infty} \lambda_j P_j(f) \qquad (f \in D(G)),$$

where G is the infinitesimal generator of $\mathfrak{T}$ with domain $D(G)$ [4, Proposition 2]. Let $\Omega = [a, b] \subseteq \mathbb{R}$. Then in view of (14) and (18), (13) reduces to

$$U_t \quad \sim \quad \sum_{j=-\infty}^{\infty} e^{-\lambda_j t} P_j \qquad (t \in [a, b]).$$

Also, typical examples of the sequences $\{e_j\}$ and $\{f_j\}$ satisfying (14) and (15) are given by

$$e_j(t) = e^{-im_j \varphi(t)}, \quad f_j(t) = e^{im_j \varphi(t)} \quad (t \in [a, b],\ j \in \mathbb{Z}),$$

where

$$\varphi(t) = \frac{2\pi}{b-a}\left(t - \frac{1}{2}(b-a)\right) \quad (t \in [a,b])$$

and $\{m_j : j \in \mathbb{Z}\}$ is a sequence of integers such that $m_j \neq m_k$ whenever $j \neq k$.

Next, we consider a fundamental, total, biorthogonal system $\mathfrak{G} = \{g_j, g_j^*\}_{j\in\mathbb{Z}}$, where $\{g_j : j \in \mathbb{Z}\}$ and $\{g_j^* : j \in \mathbb{Z}\}$ are sequences of elements in X and X^* (the dual space of X), respectively (cf. [3, 9]). That is, $\mathfrak{G}$ satisfies the following conditions:

(G-1) $\mathfrak{G}$ is fundamental, i.e., the linear span of $\{g_j : j \in \mathbb{Z}\}$ is dense in X.

(G-2) $\mathfrak{G}$ is total, i.e., if $f \in X$ and $g_j^*(f) = 0$ for all $j \in \mathbb{Z}$, then $f = 0$.

(G-3) $\mathfrak{G}$ is biorthogonal, i.e., $g_j^*(g_n) = \delta_{j,n}$ for all $j, n \in \mathbb{Z}$.

Then we define

$$P_j(f) = g_j^*(f)g_j \qquad (j \in \mathbb{Z},\ f \in X),$$

which satisfies Conditions (P-1), (P-2) and (P-3). Therefore, Theorems 3.2 and 3.4 and Corollary 3.3 are applied in this setting.

Now, we restrict ourselves to the case where X is a homogeneous Banach space (cf. [2, 4, 8, 12]). That is, X is a space which satisfies the following conditions:

(H-1) X is a linear subspace of $L^1_{2\pi}$ and it is a Banach space with norm $\|\cdot\|_X$.

(H-2) X is continuously embedded in $L^1_{2\pi}$, i.e., there exists a constant $K > 0$ such that

$$\|f\|_1 \le K\|f\|_X \quad \text{for all } f \in X.$$

(H-3) The right translation operator T_t defined by

$$T_t(f)(\cdot) = f(\cdot - t) \qquad (f \in X)$$

is isometric on X for each $t \in \mathbb{R}$.

(H-4) For each $f \in X$, the mapping $t \mapsto T_t(f)$ is strongly continuous on $\mathbb{R}$.

Typical examples of homogeneous Banach spaces are $C_{2\pi}$ and $L^p_{2\pi}, 1 \le p < \infty$. For other examples, see [4] (cf. [2, 8, 12]).

Now take

$$(\Omega,\mu) = \left([-\pi,\pi], \frac{1}{2\pi}\,dt\right), \quad e_j(t) = e^{-ijt}, \quad f_j(t) = g_j(t) = e^{ijt},$$
$$g_j^*(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-ijt}\,dt,$$

which is the jth Fourier coefficient of f (cf. Remark 1). Then $\mathfrak{T}_n$ is the set of all bounded linear projections of X onto the closed linear subspace of X consisting of all trigonometric polynomials of degree at most n. Also, we have $U_t = T_{-t}$ for all $t \in [-\pi, \pi]$ and $A = B = 1$. Let $\mathfrak{V} = \mathfrak{T}$ and $\chi \in L^1_{2\pi}$. Then we have

$$W_{\mathfrak{V},\chi}(f)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \chi(t) f(x-t)\,dt = (\chi * I)(f)(x) \quad (f \in X).$$

Consequently, by Corollary 3.3 and Theorem 3.4 we have the following:

Theorem 3.5. *Let* $\chi \in L^1_{2\pi}$ *and let* α *be a scalar. Then we have*:

(a) $$\|\chi * I - S_n\| = \min\{\|\chi * I - T\| : T \in \mathfrak{T}_n\},$$

(b) $$\|\alpha I - S_n\| = \min\{\|\alpha I - T\| : T \in \mathfrak{T}_n\}.$$

Here, we mention several concrete examples of χ in Theorem 3.5 (a), which induce the classical important approximation processes of convolution operators (cf. [1, 4, 5, 6, 7, 11]).

1° (Fejér). Let $\alpha > 0, m \in \mathbb{N}$ and

$$\chi(t) = F_{m,\alpha}(t) = \sum_{j=-m}^{m} \frac{A^{(\alpha)}_{m-|j|}}{A^{(\alpha)}_m} e^{ijt},$$

where

$$A^{(\beta)}_m = \binom{m+\beta}{m} = \frac{(\beta+1)(\beta+2)\cdots(\beta+m)}{m!}, \quad \beta > -1.$$

2° (Riesz). Let $m \in \mathbb{N}, \kappa, \lambda > 0$ and

$$\chi(t) = r_{m,\kappa,\lambda}(t) = \sum_{j=-m}^{m} \Big(1 - \Big|\frac{j}{m+1}\Big|^{\kappa}\Big)^{\lambda} e^{ijt}.$$

3° (de la Vallée-Poussin). Let $m \in \mathbb{N}$ and

$$\chi(t) = v_m(t) = \frac{(m!)^2}{(2m)!}\Big(2\cos\frac{1}{2}t\Big)^{2m}.$$

4° (Jackson). Let $m \in \mathbb{N} \setminus \{0\}, r \in \mathbb{N} \setminus \{0, 1\}$ and

$$\chi(t) = j_{m,r}(t) = c_{m,r}\Big\{\frac{\sin \frac{1}{2}mt}{\sin \frac{1}{2}t}\Big\}^{2r},$$

where the normalizing constant $c_{m,r} > 0$ is taken in such a way that

$$\widehat{j_{m,r}}(0) = \frac{1}{\pi}\int_0^{\pi} j_{m,r}(t)\,dt = 1.$$

5° (Fejér-Korovkin). Let $m \in \mathbb{N}$ and

$$\chi(t) = K_m(t) = \Lambda_m \Big| \sum_{j=0}^{m} \lambda_m(j) e^{ijt} \Big|^2,$$

where

$$\lambda_m(j) = \sin\Big(\frac{j+1}{m+2}\Big)\pi \quad (j = 0, 1, 2, \dots, m), \quad \Lambda_m = \Big(\sum_{j=0}^{m} \lambda_m^2(j)\Big)^{-1}.$$

6° (Gauss-Weierstrass). Let $\lambda > 0$ and

$$\chi(t) = w_\lambda(t) = \sqrt{\frac{\pi}{\lambda}} \sum_{j=-\infty}^{\infty} \exp\Big\{-\frac{(t-2\pi j)^2}{4\lambda}\Big\} = \sum_{j=-\infty}^{\infty} e^{-\lambda j^2} e^{ijt}.$$

7° (Poisson). Let $0 \le r < 1$ and

$$\chi(t) = p_r(t) = 1 + 2\sum_{j=1}^{\infty} r^j \cos jt = \frac{1-r^2}{1-2r\cos t + r^2}.$$

Finally, it should be noticed that other applications can be devoted to certain negative problems of estimetes for the degree of the best approximation, and we omit the details (cf. [6, 7]).

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