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ON THE EXTENSION OF k-NUC NORMS

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Dedicated to Ky Fan on the Occasion of His Eighty-Fifth Birthday

It is well-known that if Y is a closed subspace of a separable Banach space X, and if Y admits a locally uniformly rotund (LUR) norm, then this LUR norm on Y can be extended to a LUR norm on X. (See [4,5] and II.8 of [1].) In [2], Fabian used a new technique to extend the result to other forms of rotundity without requiring X to be separable. However in [2], the subpace Y has to be reflexive. The second-named author later showed that the condition that Y is reflexive could be removed. (See [8] and [9].) It is worth noting that the techniques used in the above three instances are completely different and each has its own strength. Fabian's extension preserves moduli of convexity of power type. The extension given in [8] is simple and turns out to be very useful in many situations.

In [2] and [9], the authors expressed their opinions that the rotund extension holds for all forms of rotundity. In this note, by modifying the proof in [9], we verify that the rotund extension holds for a more complicated form of rotundity, namely, k-nearly uniform convexity (k-NUC).

Let $(X, \|\cdot\|)$ be a Banach space. Given $k \in \mathbb{N}$, the norm $\|\cdot\|$ is said to be k-NUC if for every $\varepsilon > 0$, there exists δ, $0 < \delta < 1$, such that for every ε-separated sequence (x_n) in $B_X(\|\cdot\|)$, i.e., $\mathrm{sep}(x_n) = \inf\{\|x_n - x_{n'}\| : n \neq n'\} > \varepsilon$, there are indices $(n_i)_{i=1}^k$ such that

$$\frac{1}{k}\left\|\sum_{i=1}^{k} x_{n_i}\right\| \leq 1 - \delta.$$

Equivalently, the norm $\|\cdot\|$ is k-NUC if whenever $\left\{(x_n^{(m)})_{n=1}^{\infty} : m \in \mathbb{N}\right\}$ is a collection of sequences in $B_X(\|\cdot\|)$ such that $\lambda_m \to 1$, where

$$\lambda_m = \inf\left\{\frac{1}{k}\left\|\sum_{i=1}^{k} x_{n_i}^{(m)}\right\| : n_1, n_2, ..., n_k \in \mathbb{N}\right\},$$

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we have $\operatorname{sep}\Big((x_n^{(m)})_n\Big) = \inf\Big\{\|x_n^{(m)} - x_{n'}^{(m)}\| : n \neq n'\Big\} \to 0$ as $m \to \infty$.

The definition of k-NUC is due to Kutzarova [6], which is a generalization of nearly uniform convexity (NUC). (See, e.g., [3] and [7].) We refer the reader to [6] for some interesting results on k-NUC spaces.

We need the following simple lemma.

Lemma 1. *Let p be a positive convex function on a bounded convex set $C \subset X$, and $\left(x_1^{(m)}\right)$, $\left(x_2^{(m)}\right)$, ..., $\left(x_k^{(m)}\right)$ be k sequences in C. Then the following are equivalent*:

(1) $\lim\limits_{m\to\infty} \left|p^2\left(\frac{1}{k}\sum_{i=1}^k x_i^{(m)}\right) - \frac{1}{k}\sum_{i=1}^k p^2\left(x_i^{(m)}\right)\right| = 0,$

(2) $\lim\limits_{m\to\infty} \left|p\left(x_i^{(m)}\right) - p\left(x_i^{(m)}\right)\right| = 0$ *for any* $1 \leq i < j \leq k$.

Theorem 1. *Let Y be a closed subspace of a Banach space X. Suppose X and Y both admit k-NUC norms, say, $\|\cdot\|$ and $|\cdot|_Y$, respectively, then $|\cdot|_Y$ can be extended to a k-NUC norm on X, in other words, $|\cdot|_Y$ is a restriction of a k-NUC norm of X to Y.*

Proof. We prove as in [9]. Let $|\cdot|$ be any extension of $|\cdot|_Y$ onto X such that $\|\cdot\| \leq (1/\sqrt{2})|\cdot|$. Define a function $p(\cdot)$ on $B_X(|\cdot|)$ by

$$p^2(x) = |x|^2 + q(x) + \operatorname{dist}(x, Y)^2,$$

where $q(x) = \operatorname{dist}(x, Y)^2 \exp(\|x\|^2)$. It can be shown that p is a convex symmetric function and the set $B = \{x \in X : p(x) \leq 1\}$ is an equivalent ball on X. Let $|||\cdot|||$ denote the corresponding norm defined by B.

To see that $|||\cdot|||$ is k-NUC, let $\left\{(x_n^{(m)})_{n=1}^\infty : m \in \mathbb{N}\right\}$ be a collection of sequences in $B_X(|||\cdot|||)$ such that $\lim\limits_{m\to\infty} \lambda_m = 1$, where

$$\lambda_m = \inf\left\{\frac{1}{k}\left|\left|\left|\sum_{i=1}^k x_{n_i}^{(m)}\right|\right|\right| : n_1, n_2, ..., n_k \in \mathbb{N}\right\}.$$

Suppose $\operatorname{sep}\Big((x_n^{(m)})_{n=1}^\infty\Big)$ does not converge to zero as $m \to \infty$. Then there exists a subset of $\left\{(x_n^{(m)})_{n=1}^\infty : m \in \mathbb{N}\right\}$ which we label again as $\left\{(x_n^{(m)})_{n=1}^\infty : m \in \mathbb{N}\right\}$ such that $\operatorname{sep}\Big((x_n^{(m)})_{n=1}^\infty\Big)$ is bounded from zero, i.e, there exists an $\varepsilon > 0$ such that

(1) $\quad \operatorname{sep}\left((x_n^{(m)})_{n=1}^\infty\right) = \inf\left\{|||x_n^{(m)} - x_{n'}^{(m)}||| : n \neq n'\right\} > \varepsilon,$ for all $m \in \mathbb{N}$.

Since $\lim_{m\to\infty} \lambda_m = 1$, we have

$$\lim_m \frac{1}{k} \left|\left|\left| \sum_{i=1}^{k} x_{n_i}^{(m)} \right|\right|\right| = 1 \text{ for any } n_1, n_2, ..., n_k \in \mathbb{N}.$$

Therefore we have

$$\lim_{m\to\infty} \left| p^2 \left(\frac{1}{k} \sum_{i=1}^{k} x_{n_i}^{(m)} \right) - \frac{1}{k} \sum_{i=1}^{k} p^2 \left(x_{n_i}^{(m)} \right) \right| = 0 \text{ for any } n_1, n_2, ..., n_k \in \mathbb{N}.$$

Then by convexity, we have for any $n_1, n_2, ..., n_k \in \mathbb{N}$,

(2) $$\lim_{m\to\infty} \left| \left| \frac{1}{k} \sum_{i=1}^{k} x_{n_i}^{(m)} \right|^2 - \frac{1}{k} \sum_{i=1}^{k} \left| x_{n_i}^{(m)} \right|^2 \right| = 0,$$

(3) $$\lim_{m\to\infty} \left| q \left(\frac{1}{k} \sum_{i=1}^{k} x_{n_i}^{(m)} \right) - \frac{1}{k} \sum_{i=1}^{k} q \left(x_{n_i}^{(m)} \right) \right| = 0$$

and

(4) $$\lim_{m\to\infty} \left| \text{dist}^2 \left(\frac{1}{k} \sum_{i=1}^{k} x_{n_i}^{(m)}, Y \right) - \frac{1}{k} \sum_{i=1}^{k} \text{dist}^2 \left(x_{n_i}^{(m)}, Y \right) \right| = 0.$$

Applying Lemma 1 to (4), we have

(5) $$\lim_{m\to\infty} (\text{dist}(x_{n_i}^{(m)}, Y) - \text{dist}(x_{n_j}^{(m)}, Y)) = 0 \text{ for } 1 \le i < j \le k,$$

and

(6) $$\lim_{m\to\infty} \left(\text{dist}(x_{n_1}^{(m)}, Y) - \text{dist} \left(\frac{1}{k} \sum_{i=1}^{k} x_{n_i}^{(m)}, Y \right) \right) = 0.$$

Now we consider two cases:

Case I: Suppose $\lim_{m\to\infty} \text{dist}(x_1^{(m)}, Y) = 0$. Then by (5), we get $\lim_{m\to\infty} \text{dist}(x_{n_{m_1}}^{(m)}, Y) = 0$. For each m, let $\left\{ \tilde{x}_n^{(m)} \right\}_{n=1}^{\infty} \subseteq Y$ with $|\tilde{x}_n^{(m)}|_Y \le 1$ such that for each n, $|x_n^{(m)} - \tilde{x}_n^{(m)}| \to 0$ as $m \to \infty$. Since the sequence $\left\{ x_n^{(m)} \right\}$ is bounded, by (2),

$$\lim_{m\to\infty} \left| \left| \frac{1}{k} \sum_{i=1}^{k} \tilde{x}_{n_i}^{(m)} \right|_Y^2 - \frac{1}{k} \sum_{i=1}^{k} \left| \tilde{x}_{n_i}^{(m)} \right|_Y^2 \right| = 0 \text{ for any } n_1, n_2, ..., n_k \in \mathbb{N}.$$

By using Lemma 1 and passing to a subsequence if necessary, we get $\lim_{m\to\infty}\left|\frac{1}{k}\sum_{i=1}^{k} \tilde{x}_{n_i}^{(m)}\right|_Y = \lim_{m\to\infty} |\tilde{x}_1^{(m)}|_Y$ for any $n_1, n_2, ..., n_k \in \mathbb{N}$. By the k-nearly uniform convexity of $|\cdot|_Y$,

$$\text{sep}_Y\left(\left(\tilde{x}_n^{(m)}\right)_{n=1}^{\infty}\right) = \inf\left\{|\tilde{x}_n^{(m)} - \tilde{x}_{n'}^{(m)}|_Y : n \neq n'\right\} \to 0 \text{ as } m \to \infty.$$

Thus

$$\text{sep}\left((x_n^{(m)})_{n=1}^{\infty}\right) = \inf\left\{|||x_n^{(m)} - x_{n'}^{(m)}||| : n \neq n'\right\} \to 0 \text{ as } m \to \infty,$$

which contradicts Equation (1).

Case II: Suppose $\overline{\lim_m}\ \text{dist}(x_1^{(m)}, Y) = d > 0$. By passing to a subsequence of $\left(\text{dist}(x_1^{(m)}, Y)\right)_{m=1}^{\infty}$ if necessary, we may assume that $\lim_{m\to\infty} \text{dist}(x_1^{(m)}, Y) = d$. Then by (3), for any $n_1, n_2, ..., n_k \in \mathbb{N}$, we have

$$\lim_{m\to\infty}\left|\text{dist}\left(\frac{1}{k}\sum_{i=1}^{k} x_{n_i}^{(m)}, Y\right) e^{\|\frac{1}{k}\sum_{i=1}^{k} x_{n_i}^{(m)}\|^2} - \frac{1}{k}\sum_{i=1}^{k}\text{dist}\left(x_{n_i}^{(m)}, Y\right) e^{\|x_{n_i}^{(m)}\|^2}\right| = 0,$$

which implies

$$\lim_{m\to\infty}\left|e^{\|\frac{1}{k}\sum_{i=1}^{k} x_{n_i}^{(m)}\|^2} - \frac{1}{k}\sum_{i=1}^{k} e^{\|x_{n_i}^{(m)}\|^2}\right| = 0.$$

Hence

$$\sum_{p=1}^{\infty}\frac{1}{p!}\left(\left\|\frac{1}{k}\sum_{i=1}^{k} x_{n_i}^{(m)}\right\|^{2p} - \frac{1}{k}\sum_{i=1}^{k}\left\|x_{n_i}^{(m)}\right\|^{2p}\right) \to 0 \text{ as } m \to \infty.$$

Consequently, we have

$$\lim_{m\to\infty}\left(\left\|\frac{1}{k}\sum_{i=1}^{k} x_{n_i}^{(m)}\right\|^{2} - \frac{1}{k}\sum_{i=1}^{k}\left\|x_{n_i}^{(m)}\right\|^{2}\right) = 0, \quad \text{for any } n_1,\ n_2, ..., n_k \in \mathbb{N}.$$

By Lemma 1 and passing to a subsequence if necessary, we get $\lim_{m\to\infty}\left\|\frac{1}{k}\sum_{i=1}^{k} x_{n_i}^{(m)}\right\|$ = $\lim_{m\to\infty}\|x_1^{(m)}\|$. Then by the k-nearly uniform convexity of $\|\cdot\|$,

$$\inf\left\{\|x_n^{(m)} - x_{n'}^{(m)}\| : n \neq n'\right\} \to 0$$

as $m \to \infty$, contradicting Equation (1). Hence our assertion is proved. ■

Remark 1. In the same manner, we can extend k-β and k-UR norms on a subspace to the whole space while preserving their respective rotundity.

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