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COUPLED SINE-GORDON EQUATIONS AS NONLINEAR SECOND ORDER EVOLUTION EQUATIONS

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Abstract. The existence, uniqueness and continuous dependence of global weak solutions of coupled sine-Gordon equations are established in the framework of variational method due to Dautray and Lions. As an application of weak solutions, we solve the quadratic optimal control problems for the control systems described by coupled sine-Gordon equations.

1. Introduction

In [6] and [3], we proved the existence and uniqueness of weak global solutions of a single damped sine-Gordon equation

$$\frac{\partial^2 y}{\partial t^2} + \alpha \frac{\partial y}{\partial t} - \beta \Delta y + \gamma \sin y = f \tag{1.1}$$

and studied the numerical analysis based on the finite element method. The equation (1.1) describes the dynamics of a Josephson junction driven by a current source by taking account of damping effect. It is numerically verified in Bishop *et al.* [1] that this equation shows the most interesting physical phenomena. That is, the numerical solutions of this equation with periodic boundary conditions lead to the nontrivial dynamics which is called the chaotic behaviour. However, there are no proofs of existence, uniqueness and chaotic behaviour of solutions in [1]. After [1], Levi [7] studied a system of coupled sine-Gordon equations of the form

$$\begin{cases} \dfrac{\partial^2 y_1}{\partial t^2} + \dfrac{\partial y_1}{\partial t} - \Delta y_1 + \sin y_1 + k(y_1 - y_2) = f_1, \\ \dfrac{\partial^2 y_2}{\partial t^2} + \dfrac{\partial y_2}{\partial t} - \Delta y_2 + \sin y_2 + k(y_2 - y_1) = f_2, \end{cases} \tag{1.2}$$

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and observed the chaotic behaviour of numerical solutions under similar periodic boundary conditions as in [1]. Also in Temam [12], the non-gradient coupled sine-Gordon equations of the form

$$
\left\{
\begin{aligned}
&\frac{\partial^2 y_1}{\partial t^2} + \frac{\partial y_1}{\partial t} - \Delta y_1 + \sin(y_1 + y_2) = f_1 \\
&\frac{\partial^2 y_2}{\partial t^2} + \frac{\partial y_2}{\partial t} - \Delta y_2 + \sin(y_1 - y_2) = f_2
\end{aligned}
\right.
\tag{1.3}
$$

is studied. The existence and uniqueness of the strong solutions of the Cauchy problem for (1.1), (1.2) and (1.3) with Dirichlet and Neumann boundary conditions has been studied by Lions [8] and Temam [12] in the evolution equation setting.

In this paper, we study the system of coupled sine-Gordon equations described by

$$
\left\{
\begin{aligned}
&\frac{\partial^2 y_1}{\partial t^2} + \alpha_{11}\frac{\partial y_1}{\partial t} + \alpha_{12}\frac{\partial y_2}{\partial t} - \beta_1 \Delta y_1 + \gamma_1 \sin(\delta_{11} y_1 + \delta_{12} y_2) \\
&\qquad + k_{11} y_1 + k_{12} y_2 = f_1, \\
&\frac{\partial^2 y_2}{\partial t^2} + \alpha_{21}\frac{\partial y_1}{\partial t} + \alpha_{22}\frac{\partial y_2}{\partial t} - \beta_2 \Delta y_2 + \gamma_2 \sin(\delta_{21} y_1 + \delta_{22} y_2) \\
&\qquad + k_{21} y_1 + k_{22} y_2 = f_2,
\end{aligned}
\right.
\tag{1.4}
$$

where $\alpha_{ij} \in \mathbb{R}$, $\beta_i > 0$, $\gamma_i, \delta_{ij}, k_{ij} \in \mathbb{R}$ are physical constants and f_i are forcing functions, $i, j = 1, 2$. This system is proposed to describe the dynamics of coupled Josephson junctions driven by current sources, in which the constants α_{ij}, δ_{ij}, k_{ij} in (1.4) are chosen suitably to represent the effects of coupling and damping. This system covers (1.2) and (1.3). The numerical analysis of (1.4) based on the finite element method is studied in Elgamal and Nakagiri [4].

The chaotic behaviour suggests that the problem of controlling the solutions of equations for (1.4) by forcing and initial functions is very delicate and important. For this, we should take a weak solution approach of the equation (1.4) to obtain the solutions under less regularities of data. Thus we utilize the variational formulation of weak solutions due to Dautray and Lions [2] and formulate the weak solution setting for the nonlinear system (1.4). Under the setting, we state and prove the results of existence, uniqueness and continuous dependence of weak solutions. We note that the existence proof by Temam in [12] is a sketch for more general equations and the detailed proof and the proof of continuous dependence are not given in there.

As an application of the results, we solve quadratic optimal control problems for the control system described by (1.4).

2. Existence, Uniqueness and Continuous Dependence

Let Ω be an open bounded set of $\mathbb{R}^n$ with a piecewise smooth boundary $\Gamma = \partial\Omega$. Let $Q = (0,T) \times \Omega$ and $\Sigma = (0,T) \times \Gamma$. We consider the coupled and damped sine-Gordon equations described by

$$\left\{\begin{array}{l} \dfrac{\partial^2 y_1}{\partial t^2} + \alpha_{11}\dfrac{\partial y_1}{\partial t} + \alpha_{12}\dfrac{\partial y_2}{\partial t} - \beta_1 \Delta y_1 + \gamma_1 \sin(\delta_{11} y_1 + \delta_{12} y_2) \\ \qquad + k_{11} y_1 + k_{12} y_2 = f_1 \text{ in } Q, \\ \dfrac{\partial^2 y_2}{\partial t^2} + \alpha_{21}\dfrac{\partial y_1}{\partial t} + \alpha_{22}\dfrac{\partial y_2}{\partial t} - \beta_2 \Delta y_2 + \gamma_2 \sin(\delta_{21} y_1 + \delta_{22} y_2) \\ \qquad + k_{21} y_1 + k_{22} y_2 = f_2 \text{ in } Q, \end{array}\right. \tag{2.1}$$

where $\alpha_{ij} \in \mathbb{R}, \beta_i > 0, \gamma_i, \delta_{ij}, k_{ij} \in \mathbb{R}, i,j = 1,2$, and Δ is a Laplacian and $f_i, i = 1,2$, are given functions. The boundary condition is the Dirichlet condition

$$y_i = 0 \quad \text{on} \quad \Sigma, \quad i = 1,2, \tag{2.2}$$

and the initial values are given by

$$y_i(0,x) = y_0^i(x) \quad \text{in} \quad \Omega \quad \text{and} \quad \frac{\partial y_i}{\partial t}(0,x) = y_1^i(x) \quad \text{in} \quad \Omega, \quad i = 1,2. \tag{2.3}$$

We define two Hilbert spaces H and V by $H = L^2(\Omega)$ and $V = H_0^1(\Omega)$, respectively. We endow these spaces with the usual inner products and norms

$$(\psi,\phi) = \int_\Omega \psi(x)\phi(x)dx, \quad |\psi| = (\psi,\psi)^{1/2}, \quad \text{for all } \phi,\psi \in L^2(\Omega), \tag{2.4}$$

$$(\!(\psi,\phi)\!) = \sum_{i=1}^n \int_\Omega \frac{\partial}{\partial x_i}\psi(x)\frac{\partial}{\partial x_i}\phi(x)dx, \quad \|\psi\| = (\!(\psi,\psi)\!)^{1/2}, \quad \text{for all } \phi,\psi \in H_0^1(\Omega). \tag{2.5}$$

Then the pair (V,H) is a Gelfand triple space with a notation, $V \hookrightarrow H \equiv H' \hookrightarrow V'$ and $V' = H^{-1}(\Omega)$, which means that embeddings $(V \subset H)$ and $H \subset V'$ are continuous, dense and compact. To use a variational formulation, let us introduce the bilinear form

$$a(\phi,\varphi) = \int_\Omega \nabla\phi \cdot \nabla\varphi dx = (\!(\phi,\varphi)\!), \quad \forall \phi,\varphi \in V = H_0^1(\Omega). \tag{2.6}$$

The form (2.6) is symmetric, bounded on $V \times V = H_0^1(\Omega)^2$ and coercive

$$a(\phi,\phi) \geq \|\phi\|^2, \quad \forall \phi \in V. \tag{2.7}$$

Then we can define the bounded operator $A = -\Delta \in \mathcal{L}(V, V')$ and the problem (2.1)-(2.3) is reduced to the following system of Cauchy problems in H:

$$
\text{(2.8)} \qquad \begin{cases} \dfrac{d^2y_1}{dt^2} + \alpha_{11}\dfrac{dy_1}{dt} + \alpha_{12}\dfrac{dy_2}{dt} + \beta_1 A y_1 \\ \qquad +\gamma_1 \sin(\delta_{11}y_1 + \delta_{12}y_2) + k_{11}y_1 + k_{12}y_2 = f_1(t) \ \text{ in } \ (0,T), \\ \dfrac{d^2y_2}{dt^2} + \alpha_{21}\dfrac{dy_1}{dt} + \alpha_{22}\dfrac{dy_2}{dt} + \beta_2 A y_2 \\ \qquad +\gamma_2 \sin(\delta_{21}y_1 + \delta_{22}y_2) + k_{21}y_1 + k_{22}y_2 = f_2(t) \ \text{ in } \ (0,T), \\ y_i(0) = y_0^i \in V, \ \ \dfrac{dy_i}{dt}(0) = y_1^i \in H, \qquad i = 1, 2. \end{cases}
$$

The operator A in (2.8) is an isomorphism from V onto V' and it is also considered as a self-adjoint operator in H with dense domain $\mathcal{D}(A)$ in V and in H,

$$\mathcal{D}(A) = \{\phi \in V |\ A\phi \in H\}.$$

In this case, A is an unbounded selfadjoint operator in H.

We introduce the solution space and the space of distributions. The space $W(0,T)$ is defined by

$$W(0,T) = \{g | g \in L^2(0,T;V), g' \in L^2(0,T;H), g'' \in L^2(0,T;V')\}.$$

The notation $\mathcal{D}'(0,T)$ denotes the space of distributions on $(0,T)$.

Now we give a vectorial representation of (2.8). For the sake of simplicity, we shall write the coupled and damped sine-Gordon equations (2.8) as the following vectorial form

$$
\text{(2.9)} \qquad \begin{cases} \mathbf{y}'' + \boldsymbol{\alpha}\mathbf{y}' + \boldsymbol{\beta}\mathbf{A}\mathbf{y} + \boldsymbol{\gamma}\sin\boldsymbol{\delta}\mathbf{y} + \mathbf{k}\mathbf{y} = \mathbf{f} \ \text{ in } \ (0,T), \\ \mathbf{y}(0) = \mathbf{y}_0, \ \ \mathbf{y}'(0) = \mathbf{y}_1, \end{cases}
$$

where

$$
\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \ \mathbf{f} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \ \mathbf{y}' = \frac{d\mathbf{y}}{dt} = \begin{bmatrix} \frac{dy_1}{dt} \\ \frac{dy_2}{dt} \end{bmatrix}, \ \mathbf{y}'' = \frac{d^2\mathbf{y}}{dt^2} = \begin{bmatrix} \frac{d^2y_1}{dt^2} \\ \frac{d^2y_2}{dt^2} \end{bmatrix},
$$

$$
\mathbf{A} = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}, \ \boldsymbol{\delta} = \begin{bmatrix} \delta_{11} & \delta_{12} \\ \delta_{21} & \delta_{22} \end{bmatrix}, \ \sin\mathbf{y} = \begin{bmatrix} \sin y_1 \\ \sin y_2 \end{bmatrix},
$$

$$
\boldsymbol{\alpha} = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}, \ \boldsymbol{\beta} = \begin{bmatrix} \beta_1 & 0 \\ 0 & \beta_2 \end{bmatrix}, \ \boldsymbol{\gamma} = \begin{bmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{bmatrix}, \ \mathbf{k} = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix},
$$

$$
\mathbf{y}_0 = \begin{bmatrix} y_0^1 \\ y_0^2 \end{bmatrix}, \ \mathbf{y}_1 = \begin{bmatrix} y_1^1 \\ y_1^2 \end{bmatrix}.
$$

The norm $|\boldsymbol{\alpha}|$ of the 2×2 matrix $\boldsymbol{\alpha}$ is defined by $\sum_{i,j=1,2}|\alpha_{ij}|$. For the treatment of (2.9),we introduce the following two product spaces:

$$\mathcal{V} = V \times V \quad \text{and} \quad \mathcal{H} = H \times H$$

with the inner products defined respectively by

$$(\!(\boldsymbol{\phi}, \boldsymbol{\psi})\!) = (\!(\phi_1, \psi_1)\!) + (\!(\phi_2, \psi_2)\!), \quad \boldsymbol{\phi} = [\phi_1, \phi_2]^t, \ \boldsymbol{\psi} = [\psi_1, \psi_2]^t \in \mathcal{V},$$

$$(\boldsymbol{\phi}, \boldsymbol{\psi}) = (\phi_1, \psi_1) + (\phi_2, \psi_2), \quad \boldsymbol{\phi} = [\phi_1, \phi_2]^t, \ \boldsymbol{\psi} = [\psi_1, \psi_2]^t \in \mathcal{H},$$

where $[\cdot,\cdot]^t$ denotes the transpose of $[\cdot,\cdot]$. Then the dual space $\mathcal{V}' = V' \times V'$ and the dual pairing between $\mathcal{V}'$ and $\mathcal{V}$ are denoted by

$$\langle \boldsymbol{\phi}, \boldsymbol{\psi} \rangle = \langle \phi_1, \psi_1 \rangle + \langle \phi_2, \psi_2 \rangle, \quad \forall \boldsymbol{\phi} = [\phi_1, \phi_2]^t \in \mathcal{V}', \ \boldsymbol{\psi} = [\psi_1, \psi_2]^t \in \mathcal{V}.$$

By the embeddings $V \hookrightarrow H \hookrightarrow V'$, it is easily verified that the pair $(\mathcal{V}, \mathcal{H})$ is a Gelfand triple space with the notation $\mathcal{V} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{V}'$. The norms of $\mathcal{V}$ and $\mathcal{H}$ are denoted simply by $\|\boldsymbol{\psi}\|$ and $|\boldsymbol{\psi}|$, respectively.

Here we give a definition of weak solutions for (2.9).

Definition 2.1. A function $\mathbf{y}$ is said to be a weak solution of (2.9) if $\mathbf{y} \in \mathbf{W}(0,T) = W(0,T) \times W(0,T)$ and $\mathbf{y}$ satisfies

$$\begin{aligned} &\langle \mathbf{y}''(\cdot), \boldsymbol{\phi} \rangle + (\boldsymbol{\alpha}\mathbf{y}'(\cdot), \boldsymbol{\phi}) + (\!(\boldsymbol{\beta}\mathbf{y}(\cdot), \boldsymbol{\phi})\!) + (\boldsymbol{\gamma} \sin \boldsymbol{\delta}\mathbf{y}(\cdot), \boldsymbol{\phi}) + (\mathbf{k}\mathbf{y}(\cdot), \boldsymbol{\phi}) \\ &= (\mathbf{f}(\cdot), \boldsymbol{\phi}) \qquad\qquad \text{for all } \ \boldsymbol{\phi} \in \mathcal{V} \ \text{ in the sense of } \ \mathcal{D}'(0,T), \end{aligned} \tag{2.10}$$

$$\mathbf{y}(0) = \mathbf{y}_0, \quad \mathbf{y}'(0) = \mathbf{y}_1. \tag{2.11}$$

For the existence and uniqueness of weak solutions for (2.9), we can state the following theorem.

Theorem 2.1. *Let* $\alpha_{ij} \in \mathbb{R},\ \beta_i > 0, \gamma_i,\ \delta_{ij},\ k_{ij} \in \mathbb{R},\ i,j = 1,2,$ *and* $\mathbf{f},\ \mathbf{y}_0,\ \mathbf{y}_1$ *be given satisfying*

$$\mathbf{f} \in L^2(0,T;\mathcal{H}), \quad \mathbf{y}_0 \in \mathcal{V}, \quad \mathbf{y}_1 \in \mathcal{H}. \tag{2.12}$$

Then the problem (2.9) *has a unique weak solution* $\mathbf{y}$ *in* $\mathbf{W}(0,T)$*. The solution* $\mathbf{y}$ *has the regularity*

$$\mathbf{y} \in C([0,T];\mathcal{V}), \quad \mathbf{y}' \in C([0,T];\mathcal{H}). \tag{2.13}$$

The existence and uniqueness of *strong* solutions of (2.8) is also proved in Temam [12] under the stronger assumption that $f_i \in C^1([0,T];H)$, $y_0^i \in \mathcal{D}(A)$,

$y_1^i \in H_0^1(\Omega)$. Since the proof is a sketch and the detailed proof is not given in there, we give a complete proof of Theorem 2.1 in the next section.

For the continuous dependence of weak solutions for (2.9), we have the following theorem.

Theorem 2.2. *Assume that the assumption in Theorem* 2.1 *holds. Let* $\mathbf{y}_A = [y_{A1}, y_{A2}]^t$ *(resp.,* $\mathbf{y}_B = [y_{B1}, y_{B2}]^t$*) be a weak solution of* (2.9) *with initial values* $(\mathbf{y}_{A0}, \mathbf{y}_{A1}) \in \mathcal{V} \times \mathcal{H}$ *(resp.,* $(\mathbf{y}_{B0}, \mathbf{y}_{B1}) \in \mathcal{V} \times \mathcal{H}$*) and* $\mathbf{f}_A \in L^2(0, T; \mathcal{H})$ *(resp.,* $\mathbf{f}_B \in L^2(0, T; \mathcal{H})$*). Then there exists a constant* $C > 0$ *depending only on* $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta}$ *and* T *such that, for each* $t \in [0, T]$,

$$
\begin{aligned}
&\|\mathbf{y}_A(t) - \mathbf{y}_B(t)\|^2 + |\mathbf{y}'_A(t) - \mathbf{y}'_B(t)|^2 \\
&\leq C \left(\|\mathbf{y}_{A0} - \mathbf{y}_{B0}\|^2 + |\mathbf{y}_{A1} - \mathbf{y}_{B1}|^2 + \int_0^t |\mathbf{f}_A(\sigma) - \mathbf{f}_B(\sigma)|^2 d\sigma \right).
\end{aligned}
\tag{2.14}
$$

Lastly in this section we give the meaning of weak solutions for (2.9). We suppose that

$$
f_i \in L^2(Q) \quad \text{and} \quad y_0^i \in H_0^1(\Omega), \ \ y_1^i \in L^2(\Omega), \quad i = 1, 2.
$$

Then by standard manipulations (cf. Lions and Magenes [10]), we can verify that the weak solutions $(y_1(t, x), y_2(t, x))$ satisfy

$$
\begin{cases}
\dfrac{\partial^2 y_1}{\partial t^2} + \alpha_{11} \dfrac{\partial y_1}{\partial t} + \alpha_{12} \dfrac{\partial y_2}{\partial t} - \beta_1 \Delta y_1 + \gamma_1 \sin(\delta_{11} y_1 + \delta_{12} y_2) \\
\qquad + k_{11} y_1 + k_{12} y_2 = f_1 \ \text{ in } \ Q, \\
\dfrac{\partial^2 y_2}{\partial t^2} + \alpha_{21} \dfrac{\partial y_1}{\partial t} + \alpha_{22} \dfrac{\partial y_2}{\partial t} - \beta_2 \Delta y_2 + \gamma_2 \sin(\delta_{21} y_1 + \delta_{22} y_2) \\
\qquad + k_{21} y_1 + k_{22} y_2 = f_2 \ \text{ in } \ Q, \\
y_i = 0 \ \text{ on } \ \Sigma, \\
y_i(0, x) = y_0^i(x) \ \text{ in } \ \Omega \ \text{ and } \ \dfrac{\partial y_i}{\partial t}(0, x) = y_1^i(x) \ \text{ in } \ \Omega, \quad i = 1, 2,
\end{cases}
\tag{2.15}
$$

in the sense of distribution $\mathcal{D}'(Q)$, and

$$
y_i, \ \frac{\partial y_i}{\partial t}, \ \frac{\partial y_i}{\partial x_j} \in L^2(Q), \quad i = 1, 2, \ j = 1, \cdots, n.
$$

3. Proofs of Theorems

Since the embedding of V into H is compact and A is selfadjoint, there exists an orthonormal basis of H, $\{w_j\}_{j=1}^{\infty}$, consisting of eigenfunctions of A such that

$$(3.1) \qquad \begin{cases} Aw_j = \lambda_j w_j, \quad \forall j, \\ 0 < \lambda_1 \le \lambda_2 \le \cdots, \quad \lambda_j \to \infty \quad \text{as} \quad j \to \infty. \end{cases}$$

We denote by P_m the orthogonal projection on H(or V) onto the space spanned by $\{w_1, \cdots, w_m\}$. We define $\mathbf{P}_m = (P_m, P_m)$. Then $\mathbf{P}_m$ is an orthogonal projection on $\mathcal{H}$ (or $\mathcal{V}$).

Proof of Theorem 2.1. First we consider the existence part of Theorem 2.1. The proof of existence is divided into 3 steps.

Step 1. Approximate solutions

We use the Faedo-Galerkin method. As a basis $\{w_m\}_{m=1}^{\infty}$ we use the set of eigenfunctions w_j of the operator A which is orthonormal in H.

For each $m \in N$, we define an approximate solution of the problem (2.9) by

$$(3.2) \qquad \mathbf{y}_m(t) = [y_m^1(t), y_m^2(t)]^t = \left[\sum_{j=1}^{m} g_{jm}^1(t) w_j, \sum_{j=1}^{m} g_{jm}^2(t) w_j \right]^t,$$

where $\mathbf{y}_m(t)$ satisfies

$$(3.3) \qquad \begin{cases} \dfrac{d^2}{dt^2}(\mathbf{y}_m(t), [w_j, 0]^t) + \dfrac{d}{dt}(\boldsymbol{\alpha}\mathbf{y}_m(t), [w_j, 0]^t) + (\!(\boldsymbol{\beta}\mathbf{y}_m(t), [w_j, 0]^t)\!) \\ \qquad + (\boldsymbol{\gamma} \sin \boldsymbol{\delta}\mathbf{y}_m(t), [w_j, 0]^t) + (\mathbf{k}\mathbf{y}_m(t), [w_j, 0]^t) \\ \qquad = (\mathbf{f}(t), [w_j, 0]^t), \qquad t \in [0, T],\ 1 \le j \le m, \\ \dfrac{d^2}{dt^2}(\mathbf{y}_m(t), [0, w_j]^t) + \dfrac{d}{dt}(\boldsymbol{\alpha}\mathbf{y}_m(t), [0, w_j]^t) + (\!(\boldsymbol{\beta}\mathbf{y}_m(t), [0, w_j]^t)\!) \\ \qquad + (\boldsymbol{\gamma} \sin \boldsymbol{\delta}\mathbf{y}_m(t), [0, w_j]^t) + (\mathbf{k}\mathbf{y}_m(t), [0, w_j]^t) \\ \qquad = (\mathbf{f}(t), [0, w_j]^t), \qquad t \in [0, T],\ 1 \le j \le m, \\ \mathbf{y}_m(0) = \mathbf{P}_m \mathbf{y}_0 = [P_m y_0^1, P_m y_0^2]^t, \\ \dfrac{d}{dt}\mathbf{y}_m(0) = \mathbf{P}_m \mathbf{y}_1 = [P_m y_1^1, P_m y_1^2]^t. \end{cases}$$

We set $\mathbf{y}_{0m} = [y_{0m}^1, y_{0m}^2]^t = \mathbf{P}_m \mathbf{y}_0$ and $\mathbf{y}_{1m} = [y_{1m}^1, y_{1m}^2]^t = \mathbf{P}_m \mathbf{y}_1$. Then

$$(3.4) \qquad \mathbf{y}_{0m} \to \mathbf{y}_0 \ \text{ in } \ \mathcal{V}, \quad \mathbf{y}_{1m} \to \mathbf{y}_1 \ \text{ in } \mathcal{H} \quad \text{as} \ \ m \to \infty.$$

Let

$$\vec{g}_m = [\vec{g}_m^1, \vec{g}_m^2]^t = [(g_{1m}^1, \cdots, g_{mm}^1), (g_{1m}^2, \cdots, g_{mm}^2)]^t.$$

Then the equation (3.3) can be written as the system of two m vector differential equations

$$\frac{d^2}{dt^2}\vec{g}_m + \boldsymbol{\alpha}\frac{d}{dt}\vec{g}_m + \boldsymbol{\beta}[\Lambda, \Lambda]^t\vec{g}_m = \vec{F}(t) - \vec{N}(t, \vec{g}_m) \tag{3.5}$$

with initial values

$$\vec{g}_m(0) = [(y_{0m}^1, w_1), \cdots, (y_{0m}^1, w_m), (y_{0m}^2, w_1), \cdots, (y_{0m}^2, w_m)]^t,$$

$$\frac{d}{dt}\vec{g}_m(0) = [(y_{1m}^1, w_1), \cdots, (y_{1m}^1, w_m), (y_{1m}^2, w_1), \cdots, (y_{1m}^2, w_m)]^t.$$

Here in (3.5), $\Lambda = \text{diag}\ (\lambda_i : i = 1, \cdots, m)$,

$$\vec{F}(t) = [(f_1(t), w_1), \cdots, (f_1(t), w_m), (f_2(t), w_1), \cdots, (f_2(t), w_m)]^t$$

and

$$\vec{N}(t, \vec{g}_m) = [\vec{N}_1(t, \vec{g}_m^1, \vec{g}_m^2), \vec{N}_2(t, \vec{g}_m^1, \vec{g}_m^2)]^t$$

with

$$\begin{aligned}\vec{N}_1(t, \vec{g}_m^1, \vec{g}_m^2) = \Bigg[&\gamma_1\Bigg(\sin\Bigg(\delta_{11}\sum_{j=1}^m g_{jm}^1 w_j + \delta_{12}\sum_{j=1}^m g_{jm}^2 w_j\Bigg), w_1\Bigg) \\ &+\Bigg(k_{11}\sum_{j=1}^m g_{jm}^1 w_j + k_{12}\sum_{j=1}^m g_{jm}^2 w_j, w_1\Bigg), \cdots, \\ &\gamma_1\Bigg(\sin\Bigg(\delta_{11}\sum_{j=1}^m g_{jm}^1 w_j + \delta_{12}\sum_{j=1}^m g_{jm}^2 w_j\Bigg), w_m\Bigg) \\ &+\Bigg(k_{11}\sum_{j=1}^m g_{jm}^1 w_j + k_{12}\sum_{j=1}^m g_{jm}^2 w_j, w_m\Bigg)\Bigg]^t,\end{aligned}$$

and

$$\begin{aligned}\vec{N}_2(t, \vec{g}_m^1, \vec{g}_m^2) = \Bigg[&\gamma_2\Bigg(\sin\Bigg(\delta_{21}\sum_{j=1}^m g_{jm}^1 w_j + \delta_{22}\sum_{j=1}^m g_{jm}^2 w_j\Bigg), w_1\Bigg) \\ &+\Bigg(k_{21}\sum_{j=1}^m g_{jm}^1 w_j + k_{22}\sum_{j=1}^m g_{jm}^2 w_j, w_1\Bigg), \cdots, \\ &\gamma_2\Bigg(\sin\Bigg(\delta_{21}\sum_{j=1}^m g_{jm}^1 w_j + \delta_{22}\sum_{j=1}^m g_{jm}^2 w_j\Bigg), w_m\Bigg) \\ &+\Bigg(k_{21}\sum_{j=1}^m g_{jm}^1 w_j + k_{22}\sum_{j=1}^m g_{jm}^2 w_j, w_m\Bigg)\Bigg]^t,\end{aligned}$$

The nonlinear forcing function vector $\vec{N}$ is Lipschitz continuous. Indeed, for $\vec{g}_m = [\sum_{j=1}^m g_{jm}^1 w_j, \sum_{j=1}^m g_{jm}^2 w_j]^t$ and $\vec{h}_m = [\sum_{j=1}^m h_{jm}^1 w_j, \sum_{j=1}^m h_{jm}^2 w_j]^t$, it follows by

$$\int_\Omega |\sin\psi(x) - \sin\phi(x)|^2 dx \le \int_\Omega |\psi(x) - \phi(x)|^2 dx, \ \forall \psi, \phi \in H, \tag{3.6}$$

and Schwarz inequality that

$$\begin{aligned}
&|\vec{N}_1(t, \vec{g}_m^1, \vec{g}_m^2) - \vec{N}_1(t, \vec{h}_m^1, \vec{h}_m^2)| \\
&\le |\gamma_1| \biggl(\sum_{i=1}^m \biggl| \Bigl(\sin\bigl(\delta_{11} \sum_{j=1}^m g_{jm}^1 w_j + \delta_{12} \sum_{j=1}^m g_{jm}^2 w_j\bigr) \\
&\qquad\qquad - \sin\bigl(\delta_{11} \sum_{j=1}^m h_{jm}^1 w_j + \delta_{12} \sum_{j=1}^m h_{jm}^2 w_j\bigr), w_i \Bigr) \biggr|^2 \biggr)^{\frac{1}{2}} \\
&\quad + \biggl(\sum_{i=1}^m \biggl| \Bigl(\bigl(k_{11} \sum_{j=1}^m g_{jm}^1 w_j + k_{12} \sum_{j=1}^m g_{jm}^2 w_j\bigr) \\
&\qquad\qquad - \bigl(k_{11} \sum_{j=1}^m h_{jm}^1 w_j + k_{12} \sum_{j=1}^m h_{jm}^2 w_j\bigr), w_i \Bigr) \biggr|^2 \biggr)^{\frac{1}{2}} \\
&\le |\gamma_1| m^{\frac{1}{2}} \biggl| \sin(\delta_{11} \sum_{j=1}^m g_{jm}^1 w_j + \delta_{12} \sum_{j=1}^m g_{jm}^2 w_j) \\
&\quad - \sin(\delta_{11} \sum_{j=1}^m h_{jm}^1 w_j + \delta_{12} \sum_{j=1}^m h_{jm}^2 w_j) \biggr| \\
&\quad + m^{\frac{1}{2}} \biggl| (k_{11} \sum_{j=1}^m g_{jm}^1 w_j + k_{12} \sum_{j=1}^m g_{jm}^2 w_j) - (k_{11} \sum_{j=1}^m h_{jm}^1 w_j + k_{12} \sum_{j=1}^m h_{jm}^2 w_j) \biggr| \\
&\le |\gamma_1| m^{\frac{1}{2}} \biggl| (\delta_{11} \sum_{j=1}^m g_{jm}^1 w_j + \delta_{12} \sum_{j=1}^m g_{jm}^2 w_j) - (\delta_{11} \sum_{j=1}^m h_{jm}^1 w_j + \delta_{12} \sum_{j=1}^m h_{jm}^2 w_j) \biggr| \\
&\quad + m^{\frac{1}{2}} \biggl| k_{11} \sum_{j=1}^m (g_{jm}^1 - h_{jm}^1) w_j \biggr| + m^{\frac{1}{2}} \biggl| k_{12} \sum_{j=1}^m (g_{jm}^2 - h_{jm}^2) w_j \biggr| \\
&\le |\gamma_1| m^{\frac{1}{2}} \biggl\{ |\delta_{11}| \biggl| \sum_{j=1}^m (g_{jm}^1 - h_{jm}^1) w_j \biggr| + |\delta_{12}| \biggl| \sum_{j=1}^m (g_{jm}^2 - h_{jm}^2) w_j \biggr| \biggr\} \\
&\quad + m^{\frac{1}{2}} \biggl\{ |k_{11}| \biggl| \sum_{j=1}^m (g_{jm}^1 - h_{jm}^1) w_j \biggr| + |k_{12}| \biggl| \sum_{j=1}^m (g_{jm}^2 - h_{jm}^2) w_j \biggr| \biggr\} \\
&\le (1 + |\gamma_1|) m (|\delta_{11}| + |\delta_{12}| + |k_{11}| + |k_{12}|) \\
&\quad \times \biggl\{ \Bigl(\sum_{j=1}^m |g_{jm}^1 - h_{jm}^1|^2 \Bigr)^{\frac{1}{2}} + \Bigl(\sum_{j=1}^m |g_{jm}^2 - h_{jm}^2|^2 \Bigr)^{\frac{1}{2}} \biggr\}
\end{aligned}$$

$$
\begin{aligned}
&\leq \sqrt{2}(1+|\gamma_1|)(|\delta_{11}|+|\delta_{12}|+|k_{11}|+|k_{12}|)m \\
&\quad\times\left[\sum_{j=1}^{m}|g_{jm}^1-h_{jm}^1|^2+|g_{jm}^2-h_{jm}^2|^2)\right]^{\frac{1}{2}} \\
&= \sqrt{2}(1+|\gamma_1|)(|\delta_{11}|+|\delta_{12}|+|k_{11}|+|k_{12}|)m|\vec{g}_m-\vec{h}_m|.
\end{aligned}
$$

By similar calculations, we can verify

$$
\begin{aligned}
&|\vec{N}_2(t,\vec{g}_m^1,\vec{g}_m^2)-\vec{N}_2(t,\vec{h}_m^1,\vec{h}_m^2)| \\
&\leq \sqrt{2}(1+|\gamma_2|)(|\delta_{21}|+|\delta_{22}|+|k_{21}|+|k_{22}|)m|\vec{g}_m-\vec{h}_m|.
\end{aligned}
$$

So that

$$
\begin{aligned}
|\vec{N}(t,\vec{g}_m)-\vec{N}(t,\vec{h}_m)| &\leq |\vec{N}_1(t,\vec{g}_m^1,\vec{g}_m^2)-\vec{N}_1(t,\vec{h}_m^1,\vec{h}_m^2)| \\
&\quad +|\vec{N}_2(t,\vec{g}_m^1,\vec{g}_m^2)-\vec{N}_2(t,\vec{h}_m^1,\vec{h}_m^2)| \\
&\leq \sqrt{2}(1+|\boldsymbol{\gamma}|)(|\boldsymbol{\delta}|+|\mathbf{k}|)m|\vec{g}_m-\vec{h}_m|.
\end{aligned}
$$

This proves the Lipschitz continuity of the nonlinear term $\vec{N}$. Therefore, this system of second order vector differential equation admits a unique solution $\vec{g}_m=[\vec{g}_m^1,\vec{g}_m^2]^t$ on $[0,T]$, by reducing this to a first order system and applying Carathéodory-type existence theorem. Hence we can construct the approximate solutions $\mathbf{y}_m(t)=[y_m^1(t),y_m^2(t)]^t$ of (2.9).

Step 2. A priori estimates

In this step, we shall derive a priori estimates of approximate solutions $\mathbf{y}_m(t)=[y_m^1(t),y_m^2(t)]^t$. We multiply both sides of the first and second equations of (3.3) by $g_{jm}^{1'}(t)$ and $g_{jm}^{2'}(t)$, respectively, sum over j, and add these equations to have

$$
\begin{aligned}
&(\mathbf{y}_m''(t),\mathbf{y}_m'(t))+(\boldsymbol{\alpha}\mathbf{y}_m'(t),\mathbf{y}_m'(t))+(\!(\boldsymbol{\beta}\mathbf{y}_m(t),\mathbf{y}_m'(t))\!) \\
&\quad+(\boldsymbol{\gamma}\sin\boldsymbol{\delta}\mathbf{y}_m(t),\mathbf{y}_m'(t))+(\mathbf{k}\mathbf{y}_m(t),\mathbf{y}_m'(t))=(\mathbf{f}(t),\mathbf{y}_m'(t)).
\end{aligned}
\tag{3.7}
$$

Since

$$
(\!(\boldsymbol{\beta}\mathbf{y}_m(t),\mathbf{y}_m'(t))\!)=\frac{1}{2}\frac{d}{dt}\|\sqrt{\boldsymbol{\beta}}\mathbf{y}_m(t)\|^2,\quad (\mathbf{y}_m''(t),\mathbf{y}_m'(t))=\frac{1}{2}\frac{d}{dt}|\mathbf{y}_m'(t)|^2,
\tag{3.8}
$$

by substituting (3.8) to (3.7), we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left[|\mathbf{y}_m'|^2+\|\sqrt{\boldsymbol{\beta}}\mathbf{y}_m\|^2\right]+(\boldsymbol{\alpha}\mathbf{y}_m',\mathbf{y}_m')+(\mathbf{k}\mathbf{y}_m,\mathbf{y}_m') \\
&\quad+(\boldsymbol{\gamma}\sin\boldsymbol{\delta}\mathbf{y}_m,\mathbf{y}_m')=(\mathbf{f},\mathbf{y}_m').
\end{aligned}
\tag{3.9}
$$

It is easily verified that

$$|(\boldsymbol{\alpha}\mathbf{y}'_m, \mathbf{y}'_m)| \le |\boldsymbol{\alpha}||\mathbf{y}'_m|^2, \tag{3.10}$$

$$2|(\mathbf{f}, \mathbf{y}'_m)| \le (|\mathbf{f}|^2 + |\mathbf{y}'_m|^2). \tag{3.11}$$

Let c_1 be the imbedding constant such that $|\phi| \le c_1\|\phi\|$ for all $\phi \in V$. Then by (3.6) the remaining terms appearing in (3.9) are estimated as follows:

$$2|(\mathbf{k}\mathbf{y}_m, \mathbf{y}'_m)| \le 2|\mathbf{k}||\mathbf{y}_m| \cdot |\mathbf{y}'_m| \le |\mathbf{k}|c_1(\|\mathbf{y}_m\|^2 + |\mathbf{y}'_m|^2), \tag{3.12}$$

$$2|(\boldsymbol{\gamma}\sin\boldsymbol{\delta}\mathbf{y}_m, \mathbf{y}'_m)| \le 2|\boldsymbol{\gamma}||\sin\boldsymbol{\delta}\mathbf{y}_m| \cdot |\mathbf{y}'_m| \le |\boldsymbol{\gamma}||\boldsymbol{\delta}|c_1(\|\mathbf{y}_m\|^2 + |\mathbf{y}'_m|^2). \tag{3.13}$$

Integrating (3.9) on $[0, t]$ and using the estimates (3.10)-(3.12), we obtain the following inequality

(3.14)
$$\begin{aligned}|\mathbf{y}'_m(t)|^2 + \|\sqrt{\boldsymbol{\beta}}\mathbf{y}_m(t)\|^2 \le& |\mathbf{y}_{1m}|^2 + \|\sqrt{\boldsymbol{\beta}}\mathbf{y}_{0m}\|^2 + \int_0^t |\mathbf{f}(\sigma)|^2 d\sigma \\ &+(|\mathbf{k}| + |\boldsymbol{\gamma}||\boldsymbol{\delta}|)c_1\int_0^t \|\mathbf{y}_m(\sigma)\|^2 d\sigma \\ &+(1 + 2|\boldsymbol{\alpha}| + |\mathbf{k}|c_1 + |\boldsymbol{\gamma}||\boldsymbol{\delta}|c_1)\int_0^t |\mathbf{y}'_m(\sigma)|^2 d\sigma.\end{aligned}$$

Since $\|\mathbf{y}_{0m}\| \le \|\mathbf{y}_0\|$ and $|\mathbf{y}_{1m}| \le |\mathbf{y}_1|$, it follows from (3.14) that

$$\begin{aligned}|\mathbf{y}'_m(t)|^2 + \|\sqrt{\boldsymbol{\beta}}\mathbf{y}_m(t)\|^2 \le& |\mathbf{y}_1|^2 + \|\sqrt{\boldsymbol{\beta}}\mathbf{y}_0\|^2 + \|\mathbf{f}\|^2_{L^2(0,T;\mathcal{H})} \\ &+C_0\int_0^t (|\mathbf{y}'_m(\sigma)|^2 + \|\mathbf{y}_m(\sigma)\|^2)d\sigma,\end{aligned} \tag{3.15}$$

where $C_0 = 1 + 2|\boldsymbol{\alpha}| + |\mathbf{k}|c_1 + |\boldsymbol{\gamma}||\boldsymbol{\delta}|c_1$. By the inequality $\min\{\beta_1, \beta_2\}\|\mathbf{y}_m(t)\|^2 \le \|\sqrt{\boldsymbol{\beta}}\mathbf{y}_m(t)\|^2$, we divide (3.15) by $\beta = \min\{\beta_1, \beta_2, 1\} > 0$ to have

$$\|\mathbf{y}_m(t)\|^2 + |\mathbf{y}'_m(t)|^2 \le C_1 + C_2\int_0^t (\|\mathbf{y}_m(\sigma)\|^2 + |\mathbf{y}'_m(\sigma)|^2)d\sigma, \tag{3.16}$$

where

$$C_1 = \frac{1}{\beta}[|\mathbf{y}_1|^2 + \|\sqrt{\boldsymbol{\beta}}\mathbf{y}_0\|^2 + \|\mathbf{f}\|_{L^2(0,T;\mathcal{H})}], \quad C_2 = \frac{C_0}{\beta}.$$

Therefore, by Bellman-Gronwall's inequality we have the uniform boundedness

$$\|\mathbf{y}_m(t)\|^2 + |\mathbf{y}'_m(t)|^2 \le C_1\exp(C_2 t) \le C_1\exp(C_2 T), \quad \forall t \in [0, T]. \tag{3.17}$$

Step 3. Passage to the limit

The estimate (3.17) implies that

$$\{\mathbf{y}_m\} \text{ is bounded in } L^\infty(0,T;\mathcal{V}),$$
$$\{\mathbf{y}'_m\} \text{ is bounded in } L^\infty(0,T;\mathcal{H}).$$

Hence, by the extraction theorem of Rellich's, we can find a subsequence $\{\mathbf{y}_{m_k}\}$ of $\{\mathbf{y}_m\}$ and find

$$\mathbf{z} \in L^\infty(0,T;\mathcal{V}) \subset L^2(0,T;\mathcal{V}), \quad \bar{\mathbf{z}} \in L^\infty(0,T;\mathcal{H}) \subset L^2(0,T;\mathcal{H})$$

such that

(3.18) $$\mathbf{y}_{m_k} \to \mathbf{z} \text{ weak star in } L^\infty(0,T;\mathcal{V}) \text{ and weakly in } L^2(0,T;\mathcal{V}),$$

(3.19) $$\mathbf{y}'_{m_k} \to \bar{\mathbf{z}} \text{ weak star in } L^\infty(0,T;\mathcal{H}) \text{ and weakly in } L^2(0,T;\mathcal{H}).$$

By the classical compactness theorem (cf. Temam [11, Thm. 2.3, Chap. III]), these convergences imply

(3.20) $$\mathbf{y}_{m_k} \to \mathbf{z} \text{ strongly in } L^2(0,T;\mathcal{H}).$$

Hence, by (3.20),

(3.21) $$\sin\boldsymbol{\delta}\mathbf{y}_{m_k} \to \sin\boldsymbol{\delta}\mathbf{z} \text{ strongly in } L^2(0,T;\mathcal{H}).$$

We shall show that $\bar{\mathbf{z}} = \mathbf{z}'$ and $\mathbf{z}(0) = \mathbf{y}_0$. For $t \in [0,T)$,

(3.22) $$\mathbf{y}_{m_k}(t) = \mathbf{y}_{m_k}(0) + \int_0^t \mathbf{y}'_{m_k}(\sigma)d\sigma$$

in the $\mathcal{V}$ (and hence $\mathcal{H}$) sense. Moreover, $\mathbf{y}_{m_k}(0) = \mathbf{y}_{0m_k} \to \mathbf{y}_0$ in the $\mathcal{V}$ and hence $\mathcal{H}$ sense, whereas for each t, $\int_0^t \mathbf{y}'_{m_k}(\sigma)d\sigma \to \int_0^t \bar{\mathbf{z}}(\sigma)d\sigma$ weakly in $\mathcal{H}$ by (3.19). Hence, taking the limit in the weak $\mathcal{H}$ sense in (3.22) we obtain

(3.23) $$\mathbf{z}(t) = \mathbf{y}_0 + \int_0^t \bar{\mathbf{z}}(\sigma)d\sigma \text{ for } t \in [0,T).$$

This shows that $\mathbf{z}'(t)$ exists a.e. in the $\mathcal{H}$ sense and $\bar{\mathbf{z}} = \mathbf{z}' \in L^2(0,T;\mathcal{H}), \mathbf{z}(0) = \mathbf{y}_0$ (cf. [2, p. 564]).

Let j be fixed. Multiply the first equations of (3.3) by the scalar function $\zeta(t)$ with

(3.24) $$\zeta \in C^1([0,T]), \quad \zeta(T) = 0,$$

and put $\boldsymbol{\phi}_j = \zeta(t)[w_j, 0]^t$. Integrating these over $[0,T]$ for $m_k > j$ and using integration by parts, we have

$$\begin{aligned}\int_0^T &[-(\mathbf{y}'_{m_k}(t), \boldsymbol{\phi}'_j(t)) + (\boldsymbol{\alpha}\mathbf{y}'_{m_k}(t), \boldsymbol{\phi}_j(t)) + (\!(\boldsymbol{\beta}\mathbf{y}_{m_k}(t), \boldsymbol{\phi}_j(t))\!) \\ &+(\boldsymbol{\gamma}\sin\boldsymbol{\delta}\mathbf{y}_{m_k}(t), \boldsymbol{\phi}_j(t)) + (\mathbf{k}\mathbf{y}_{m_k}(t), \boldsymbol{\phi}_j(t))]dt \\ &= \int_0^T (\mathbf{f}(t), \boldsymbol{\phi}_j(t))dt - (\mathbf{y}_{1m_k}, \boldsymbol{\phi}_j(0)).\end{aligned} \tag{3.25}$$

If we take $k \to \infty$ in (3.25) and use (3.4), (3.18)-(3.21), then we have

$$\begin{aligned}\int_0^T &[-(\mathbf{z}'(t), \boldsymbol{\phi}'_j(t)) + (\boldsymbol{\alpha}\mathbf{z}'(t), \boldsymbol{\phi}_j(t)) + (\!(\boldsymbol{\beta}\mathbf{z}(t), \boldsymbol{\phi}_j(t))\!) \\ &+(\boldsymbol{\gamma}\sin\boldsymbol{\delta}\mathbf{z}(t), \boldsymbol{\phi}_j(t)) + (\mathbf{k}\mathbf{z}(t), \boldsymbol{\phi}_j(t))]dt \\ &= \int_0^T (\mathbf{f}(t), \boldsymbol{\phi}_j(t))dt - (\mathbf{y}_1, \boldsymbol{\phi}_j(0)),\end{aligned} \tag{3.26}$$

so that

(3.27)

$$\begin{aligned}&\int_0^T \zeta'(t)(-\mathbf{z}'(t), [w_j, 0]^t)dt \\ &+ \int_0^T \zeta(t)\{(\boldsymbol{\alpha}\mathbf{z}'(t), [w_j,0]^t) + (\!(\boldsymbol{\beta}\mathbf{z}(t), [w_j,0]^t)\!) + (\boldsymbol{\gamma}\sin\boldsymbol{\delta}\mathbf{z}(t), [w_j,0]^t) \\ &+(\mathbf{k}\mathbf{z}(t), [w_j,0]^t) - (\mathbf{f}(t), [w_j,0]^t)\}dt = -\zeta(0)(\mathbf{y}_1, [w_j,0]^t).\end{aligned}$$

If we take $\zeta \in \mathcal{D}(0,T)$ in (3.27), then

$$\begin{aligned}&\frac{d}{dt}(\mathbf{z}'(\cdot), [w_j,0]^t) + (\boldsymbol{\alpha}\mathbf{z}'(\cdot), [w_j,0]^t) + (\!(\boldsymbol{\beta}\mathbf{z}(\cdot), [w_j,0]^t)\!) \\ &\quad+(\boldsymbol{\gamma}\sin\boldsymbol{\delta}\mathbf{z}(\cdot), [w_j,0]^t) + (\mathbf{k}\mathbf{z}(\cdot), [w_j,0]^t) = (\mathbf{f}(\cdot), [w_j,0]^t)\end{aligned} \tag{3.28}$$

in the sense of distribution $\mathcal{D}'(0,T)$. Similarly, we have

$$\begin{aligned}&\frac{d}{dt}(\mathbf{z}'(\cdot), [0,w_j]^t) + (\boldsymbol{\alpha}\mathbf{z}'(\cdot), [0,w_j]^t) + (\!(\boldsymbol{\beta}\mathbf{z}(\cdot), [0,w_j]^t)\!) \\ &\quad+(\boldsymbol{\gamma}\sin\boldsymbol{\delta}\mathbf{z}(\cdot), [0,w_j]^t) + (\mathbf{k}\mathbf{z}(\cdot), [0,w_j]^t) = (\mathbf{f}(\cdot), [0,w_j]^t)\end{aligned} \tag{3.29}$$

in the sense of distribution $\mathcal{D}'(0,T)$. Since $\{\sum_{j=1}^m \xi_j[w_j,0]^t + \sum_{j=1}^m \eta_j[0,w_j]^t \mid \xi_j, \eta_j \in \mathbb{R}, m \in \mathbb{N}\}$ is dense in $\mathcal{V}$, we conclude by (3.28) and (3.29) that for all $\boldsymbol{\phi} \in \mathcal{V}$,

$$\begin{aligned}&\langle \mathbf{z}''(\cdot), \boldsymbol{\phi}\rangle_{\mathcal{V}',\mathcal{V}} + (\boldsymbol{\alpha}\mathbf{z}'(\cdot), \boldsymbol{\phi}) + (\!(\boldsymbol{\beta}\mathbf{z}(\cdot), \boldsymbol{\phi})\!) + (\boldsymbol{\gamma}\sin\boldsymbol{\delta}\mathbf{z}(\cdot), \boldsymbol{\phi}) \\ &\quad+(\mathbf{k}\mathbf{z}(\cdot), \boldsymbol{\phi}) = (\mathbf{f}(\cdot), \boldsymbol{\phi})\end{aligned} \tag{3.30}$$

in the sense of $\mathcal{D}'(0,T)$, and that $\mathbf{z}'' = -\boldsymbol{\beta}\mathbf{A}\mathbf{z} - \boldsymbol{\alpha}\mathbf{z}' - \boldsymbol{\gamma}\sin\boldsymbol{\delta}\mathbf{z} - \mathbf{k}\mathbf{z} + \mathbf{f} \in L^2(0,T;\mathcal{V}')$, and hence $\mathbf{z} \in \mathbf{W}(0,T)$. Multiplying both sides of (3.28) and (3.29) by ζ in (3.24) and using integration by parts, we have from (3.26) that

$$(\mathbf{z}'(0), [w_j, 0]^t)\zeta(0) = (\mathbf{y}_1, [w_j, 0]^t)\zeta(0),$$
$$(\mathbf{z}'(0), [0, w_j]^t)\zeta(0) = (\mathbf{y}_1, [0, w_j]^t)\zeta(0)$$

and that $(\mathbf{z}'(0), [w_j, 0]^t) = (\mathbf{y}_1, [w_j, 0]^t)$, $(\mathbf{z}'(0), [0, w_j]^t) = (\mathbf{y}_1, [0, w_j]^t)$. Since the span of $\{[w_j, 0]^t, [0, w_j]^t\}_{j=1}^{\infty}$ is dense in $\mathcal{H}$, we obtain $\mathbf{z}'(0) = \mathbf{y}_1$. This proves that $\mathbf{z}$ is a weak solution of the problem (2.9). This completes the proof of the existence part of Theorem 2.1.

The uniqueness part of Theorem 2.1 follows immediately from Theorem 2.2. ■

Proof of Theorem 2.2.

For the proof of Theorem 2.2, we need the following proposition on energy equality.

Proposition 3.1. *Assume that the assumption in Theorem 2.1 holds. Let* $\mathbf{y} = [y_1, y_2]^t$ *be a weak solution of* (2.9). *Then, for each* $t \in [0,T]$, *we have the following equality*

$$\begin{aligned} &|\mathbf{y}'(t)|^2 + \|\sqrt{\boldsymbol{\beta}}\mathbf{y}(t)\|^2 + 2\int_0^t (\boldsymbol{\alpha}\mathbf{y}'(\sigma), \mathbf{y}'(\sigma))d\sigma \\ &+2\int_0^t (\boldsymbol{\gamma}\sin\boldsymbol{\delta}\mathbf{y}(\sigma), \mathbf{y}'(\sigma))d\sigma + 2\int_0^t (\mathbf{k}\mathbf{y}(\sigma), \mathbf{y}'(\sigma))d\sigma \\ &= |\mathbf{y}_1|^2 + \|\sqrt{\boldsymbol{\beta}}\mathbf{y}_0\|^2 + 2\int_0^t (\mathbf{f}(\sigma), \mathbf{y}'(\sigma))d\sigma. \end{aligned} \tag{3.31}$$

Since $\sin\boldsymbol{\delta}\mathbf{y}(t) \in L^2(0,T;\mathcal{H})$, by considering this nonlinear term as a forcing function term, the equality (3.31) can be proved by the regularization method for linear equations as proved in Lions and Magenus [10, pp. 276-279].

Now we give a proof of Theorem 2.2. Let $\mathbf{z} = \mathbf{y}_A - \mathbf{y}_B$. Since $\mathbf{z}$ is a weak solution of (2.9) with $\boldsymbol{\gamma} = O$ and $\mathbf{f}(t) = \boldsymbol{\gamma}(\sin\boldsymbol{\delta}\mathbf{y}_A(t) - \sin\boldsymbol{\delta}\mathbf{y}_B(t)) + (\mathbf{f}_A(t) - \mathbf{f}_B(t))$ with initial values $\mathbf{z}_0 = \mathbf{y}_{A0} - \mathbf{y}_{B0}$, $\mathbf{z}_1 = \mathbf{y}_{A1} - \mathbf{y}_{B1}$, by Proposition 3.1, we have

(3.32)
$$\begin{aligned} &|\mathbf{z}'(t)|^2 + \|\sqrt{\boldsymbol{\beta}}\mathbf{z}(t)\|^2 + 2\int_0^t (\boldsymbol{\alpha}\mathbf{z}'(\sigma), \mathbf{z}'(\sigma))d\sigma \\ &+2\int_0^t (\boldsymbol{\gamma}(\sin\boldsymbol{\delta}\mathbf{y}_A(\sigma) - \sin\boldsymbol{\delta}\mathbf{y}_B(\sigma)), \mathbf{z}'(\sigma))d\sigma + 2\int_0^t (\mathbf{k}\mathbf{z}(\sigma), \mathbf{z}'(\sigma))d\sigma \\ &= |\mathbf{z}_1|^2 + \|\sqrt{\boldsymbol{\beta}}\mathbf{z}_0\|^2 + 2\int_0^t (\mathbf{f}_A(\sigma) - \mathbf{f}_B(\sigma), \mathbf{z}'(\sigma))d\sigma. \end{aligned}$$

Since

$$2|(\gamma(\sin\boldsymbol{\delta}\mathbf{y}_A(t)-\sin\boldsymbol{\delta}\mathbf{y}_B(t)),\mathbf{z}'(t))|\le|\gamma||\boldsymbol{\delta}|c_1(\|\mathbf{z}(t)\|^2+|\mathbf{z}'(t)|^2),$$
$$2|(\mathbf{f}_A(t)-\mathbf{f}_B(t),\mathbf{z}'(t))|\le|\mathbf{f}_A(t)-\mathbf{f}_B(t)|^2+|\mathbf{z}'(t)|^2,$$

by substituting these into (3.32), as in the proof of Step 2 of Theorem 2.1, we have

$$\tag{3.33} \begin{aligned}|\mathbf{z}'(t)|^2+\|\sqrt{\boldsymbol{\beta}}\mathbf{z}(t)\|^2\le|\mathbf{z}_1|^2+\|\sqrt{\boldsymbol{\beta}}\mathbf{z}_0\|^2+\|\mathbf{f}_A-\mathbf{f}_B\|^2_{L^2(0,T;\mathcal{H})}\\ +C_0\textstyle\int_0^t(|\mathbf{z}'(\sigma)|^2+\|\mathbf{z}(\sigma)\|^2)d\sigma\end{aligned}$$

for some $C_0>0$. So that by $\min\{\beta_1,\beta_2\}\|\mathbf{z}\|^2\le\|\sqrt{\boldsymbol{\beta}}\mathbf{z}\|^2\le\max\{\beta_1,\beta_2\}\|\mathbf{z}\|^2$,

$$\tag{3.34} \begin{aligned}\|\mathbf{z}(t)\|^2+|\mathbf{z}'(t)|^2\le C_1(\|\mathbf{z}_0\|^2+|\mathbf{z}_1|^2+\|\mathbf{f}_A-\mathbf{f}_B\|^2_{L^2(0,T;\mathcal{H})})\\ +C_2\int_0^t(\|\mathbf{z}(\sigma)\|^2+|\mathbf{z}'(\sigma)|^2)d\sigma\end{aligned}$$

for some $C_1,C_2>0$. Then by applying Bellman-Gronwall's lemma to (3.34), we obtain

$$\tag{3.35} \begin{aligned}\|\mathbf{z}(t)\|^2+|\mathbf{z}'(t)|^2\le C_1(\|\mathbf{z}_0\|^2+|\mathbf{z}_1|^2+\|\mathbf{f}_A-\mathbf{f}_B\|^2_{L^2(0,T;\mathcal{H})})\exp C_2T,\\ t\in[0,T],\end{aligned}$$

which proves the inequality (2.14). This completes the proof of Theorem 2.2. ■

4. Optimal Control Problems

In this section, we give an application to optimal control problems for the coupled sine-Gordon equations. Consider the following control system described by the nonlinear sine-Gordon equations **(CS)**:

$$\tag{4.1} \begin{cases}\dfrac{\partial^2y_1}{\partial t^2}+\alpha_{11}\dfrac{\partial y_1}{\partial t}+\alpha_{12}\dfrac{\partial y_1}{\partial t}-\beta_1\Delta y_1+\gamma_1\sin(\delta_{11}y_1+\delta_{12}y_2)\\ \qquad +k_{11}y_1+k_{12}y_2=B_1v_1(t,x)\ \text{ in }\ Q,\\ \dfrac{\partial^2y_2}{\partial t^2}+\alpha_{21}\dfrac{\partial y_1}{\partial t}+\alpha_{22}\dfrac{\partial y_2}{\partial t}-\beta_2\Delta y_2+\gamma_2\sin(\delta_{21}y_1+\delta_{22}y_2)\\ \qquad +k_{21}y_1+k_{22}y_2=B_2v_2(t,x)\ \text{ in }\ Q,\\ y_i=0\ \text{ on }\ \Sigma,\\ y_i(0,x)=E_0^iw_0^i(x),\quad \dfrac{\partial y_i}{\partial t}(0,x)=E_1^iw_1^i(x)\ \text{ in }\ \Omega,\ \ i=1,2.\end{cases}$$

Here v_i and w_0^i,w_1^i are forcing and initial functions control variables, and B_i and E_j^i are bounded operators (controllers) from the Hilbert spaces V_i and W_j^i into the spaces which admit the unique existence of weak solutions of (4.1) ($i = 1, 2,\ j = 0, 1$). More precisely, we assume that

$$v_i \in V_i \equiv \text{Hilbert space}, \quad i = 1, 2, \tag{4.2}$$

$$w_0^i \in W_0^i \equiv \text{Hilbert space}, \quad i = 1, 2, \tag{4.3}$$

$$w_1^i \in W_1^i \equiv \text{Hilbert space}, \quad i = 1, 2, \tag{4.4}$$

and that

$$\begin{cases} B_i \in \mathcal{L}(V_i, L^2(0, T; L^2(\Omega))), \quad i = 1, 2, \\ E_0^i \in \mathcal{L}(W_0^i, H_0^1(\Omega)), \quad i = 1, 2, \\ E_1^i \in \mathcal{L}(W_1^i, L^2(\Omega)), \quad i = 1, 2. \end{cases} \tag{4.5}$$

We set $\mathbf{v} = [v_1, v_2]^t$, $\mathbf{w}_0 = [w_0^1, w_0^2]^t$, and $\mathbf{w}_1 = [w_1^1, w_1^2]^t$ and define the following product Hilbert spaces of control variables:

$$\mathbf{V} = V_1 \times V_2, \quad \mathbf{W}_0 = W_0^1 \times W_0^2, \quad \mathbf{W}_1 = W_1^1 \times W_1^2. \tag{4.6}$$

The inner products of $\mathbf{V},\ \mathbf{W}_0,\ \mathbf{W}_1$ are denoted by $(\cdot,\cdot)_{\mathbf{V}}$, $(\cdot,\cdot)_{\mathbf{W}_0}$, $(\cdot,\cdot)_{\mathbf{W}_1}$, respectively. Also we define the vectors of bounded operators $\mathbf{B}$, $\mathbf{E}_0$ and $\mathbf{E}_1$ by

$$\mathbf{B} = \begin{bmatrix} B_1 & O \\ O & B_2 \end{bmatrix} \in \mathcal{L}(\mathbf{V}, L^2(0, T; \mathcal{H})), \tag{4.7}$$

$$\mathbf{E}_0 = \begin{bmatrix} E_0^1 & O \\ O & E_0^2 \end{bmatrix} \in \mathcal{L}(\mathbf{W}_0, \mathcal{V}), \quad \mathbf{E}_1 = \begin{bmatrix} E_1^1 & O \\ O & E_1^2 \end{bmatrix} \in \mathcal{L}(\mathbf{W}_1, \mathcal{H}). \tag{4.8}$$

Then the system (4.1) can be written as the following system of vector form

$$\begin{cases} \dfrac{\partial^2 \mathbf{y}}{\partial t^2} + \boldsymbol{\alpha} \dfrac{\partial \mathbf{y}}{\partial t} - \boldsymbol{\beta} \Delta \mathbf{y} + \boldsymbol{\gamma} \sin \boldsymbol{\delta} \mathbf{y} + \mathbf{k} \mathbf{y} = \mathbf{B} \mathbf{v} \ \text{ in } \ Q, \\ \mathbf{y} = \mathbf{0} \ \text{ on } \ \Sigma, \\ \mathbf{y}(0, x) = \mathbf{E}_0 \mathbf{w}_0(x), \quad \dfrac{\partial \mathbf{y}}{\partial t}(0, x) = \mathbf{E}_1 \mathbf{w}_1(x), \quad x \in \Omega, \end{cases} \tag{4.9}$$

or of the evolution equation form

$$\begin{cases} \mathbf{y}'' + \boldsymbol{\alpha} \mathbf{y}' + \boldsymbol{\beta} \mathbf{A} \mathbf{y} + \boldsymbol{\gamma} \sin \boldsymbol{\delta} \mathbf{y} + \mathbf{k} \mathbf{y} = \mathbf{B} \mathbf{v} \ \text{ in } \ (0, T), \\ \mathbf{y}(0) = \mathbf{E}_0 \mathbf{w}_0, \quad \mathbf{y}'(0) = \mathbf{E}_1 \mathbf{w}_1. \end{cases} \tag{4.10}$$

Now we define the product Hilbert space of total control variables

$$\mathcal{U} = \mathbf{V} \times \mathbf{W}_0 \times \mathbf{W}_1. \tag{4.11}$$

The inner product and the norm of $\mathcal{U}$ are denoted by $(\cdot,\cdot)_{\mathcal{U}}$ and $\|\cdot\|_{\mathcal{U}}$, respectively. An element $\mathbf{u} = (\mathbf{v}, \mathbf{w}_0, \mathbf{w}_1) \in \mathcal{U}$ is called a control of **(CS)**.

By virtue of Theorems 2.1 and 2.2, for each $\mathbf{u} = (\mathbf{u}, \mathbf{w}_0, \mathbf{w}_1) \in \mathcal{U}$ there exists a unique weak solution $\mathbf{y}(\mathbf{u}) = \mathbf{y}(\mathbf{u}; t) \in \mathbf{W}(0, T)$ of (4.1), which is continuous in the control variables $\mathbf{u} = (\mathbf{v}, \mathbf{w}_0, \mathbf{w}_1) \in \mathcal{U}$. Then we can define uniquely the continuous solution map $\mathbf{u} \to \mathbf{y}(\mathbf{u})$ of $\mathcal{U}$ into $\mathbf{W}(0, T)$.

The observation of the state is assumed to be given by

$$\mathbf{z}(\mathbf{u}) = \mathbf{C}\mathbf{y}(\mathbf{u}) = \mathbf{C}[y_1(\mathbf{u}), y_2(\mathbf{u})]^t \in \mathcal{M}, \tag{4.12}$$

where $\mathbf{C}$ is an operator called the observer, and $\mathcal{M}$ is a Hilbert space of observation variables.

In this paper, we restrict ourselves the observation to the cases of distributive observations and terminal value observations. The cost $J(\mathbf{u})$ attached with **(CS)** is given by the following quadratic cost:

$$\begin{aligned} J(\mathbf{u}) = {} & \kappa_1 \int_Q |\mathbf{y}(\mathbf{u}; t, x) - \mathbf{z}_Q(t, x)|^2 dx dt \\ & + \kappa_2 \int_\Omega |\mathbf{y}(\mathbf{u}; T, x) - \mathbf{z}_\Omega(x)|^2 dx \\ & + (\mathbf{R}\mathbf{v}, \mathbf{v})_{\mathbf{V}} + (\mathbf{M}_0\mathbf{w}_0, \mathbf{w}_0)_{\mathbf{W}_0} + (\mathbf{M}_1\mathbf{w}_1, \mathbf{w}_1)_{\mathbf{W}_1}, \\ & \qquad\qquad \forall \mathbf{u} = (\mathbf{v}, \mathbf{w}_0, \mathbf{w}_1) \in \mathcal{U}, \end{aligned} \tag{4.13}$$

where κ_1, $\kappa_2 \geq 0$, $\mathbf{R}, \mathbf{M}_0, \mathbf{M}_1$ are nonnegative operators on $\mathbf{V}, \mathbf{W}_0, \mathbf{W}_1$, respectively, and $\mathbf{z}_Q = (z_Q^1, z_Q^2) \in L^2(Q)^2$, $\mathbf{z}_\Omega = (z_\Omega^1, z_\Omega^2) \in L^2(\Omega)^2$ are desired values in $L^2(Q)^2$ and $L^2(\Omega)^2$, respectively.

Our main concern in this section is to solve the optimal control problem for the nonlinear control system (4.1) with the cost (4.13). Let $\mathcal{U}_{ad}$ be a closed and convex subset of $\mathcal{U}$, which is called the admissible set. The quadratic cost optimal control problem is usually devide into two problems:

(i) Find an element $\mathbf{u}^* \in \mathcal{U}_{ad}$ such that $\inf_{\mathbf{u} \in \mathcal{U}_{ad}} J(\mathbf{u}) = J(\mathbf{u}^*)$.

(ii) Give a characterization of $\mathbf{u}^*$.

Since the control system (4.1) includes a nonlinear term, it is not easy to solve the problem (i). By using the compactness imbedding theorem in Temam [11], we can prove the following theorem.

Theorem 4.1. *Assume that the conditions of Theorem* 2.1 *hold*, *the cost* $J(\mathbf{u})$ *is given by* (4.13). *If all* $\mathbf{R}$, $\mathbf{M}_0$ *and* $\mathbf{M}_1$ *are positive or* $\mathcal{U}_{ad}$ *is bounded, then there exists at least one optimal control* $\mathbf{u}^*$ *for the cost* $J(\mathbf{u})$ *in* (4.13) *subject to the control system* $(\mathbf{CS})$.

For (ii) we solve the problem by giving necessary optimality conditions. We can prove the following criterion on necessary optimality conditions associated with the cost (4.13) along the line of Ha and Nakagiri [5] based on the work by Lions [9].

Theorem 4.2. *The optimal cost* $\mathbf{u}^* = (\mathbf{v}^*, \mathbf{w}_0^*, \mathbf{w}_1^*)$ *such that* $\min\{J(\mathbf{u}) : \mathbf{u} \in \mathcal{U}_{ad}\} = J(\mathbf{u}^*)$ *for the cost* (4.13) *is characterized by the following system of equations and inequality*:

$$\begin{cases} \dfrac{\partial^2 \mathbf{y}}{\partial t^2} + \boldsymbol{\alpha}\dfrac{\partial \mathbf{y}}{\partial t} - \boldsymbol{\beta}\Delta\mathbf{y} + \boldsymbol{\gamma}\sin\boldsymbol{\delta}\mathbf{y} + \mathbf{k}\mathbf{y} = \mathbf{B}\mathbf{v}^* \quad \text{in} \quad Q, \\ \mathbf{y} = \mathbf{0} \quad \text{on} \quad \Sigma, \\ \mathbf{y}(\mathbf{u}^*; 0, x) = \mathbf{E}_0\mathbf{w}_0^*(x), \quad \dfrac{\partial \mathbf{y}}{\partial t}(\mathbf{u}^*; 0, x) = \mathbf{E}_1\mathbf{w}_1^*(x), \quad x \in \Omega, \\ \mathbf{y} \in \mathbf{W}(0, T), \end{cases}$$

$$\begin{cases} \dfrac{\partial^2 \mathbf{p}}{\partial t^2} - \boldsymbol{\alpha}^t\dfrac{\partial \mathbf{p}}{\partial t} - \boldsymbol{\beta}\Delta\mathbf{p} + \boldsymbol{\gamma}\boldsymbol{\delta}^t\cos\boldsymbol{\delta}\mathbf{y}(\mathbf{u}^*; t)\mathbf{p} + \mathbf{k}^t\mathbf{p} \\ \qquad\qquad = \kappa_1(\mathbf{y}(\mathbf{u}^*) - \mathbf{z}_Q) \quad \text{in} \quad Q, \\ \mathbf{p} = \mathbf{0} \quad \text{on} \quad \Sigma, \\ \mathbf{p}(T, x) = \mathbf{0}, \quad \dfrac{\partial \mathbf{p}}{\partial t}(T, x) = \kappa_2(\mathbf{y}(\mathbf{u}^*; T) - \mathbf{z}_\Omega), \quad x \in \Omega, \\ \mathbf{p} \in \mathbf{W}(0, T), \end{cases}$$

$$\begin{aligned} &\int_Q (\mathbf{p}(\mathbf{u}^*; t, x))\mathbf{B}(\mathbf{v} - \mathbf{v}^*)(t, x)\,dxdt \\ &\quad + \int_\Omega (\boldsymbol{\alpha}^t\mathbf{p}(\mathbf{u}^*; 0, x) - \frac{\partial \mathbf{p}}{\partial t}(\mathbf{u}^*; 0, x))\mathbf{E}_0(\mathbf{w}_0 - \mathbf{w}_0^*)(x)\,dx \\ &\quad + \int_\Omega (\mathbf{p}(\mathbf{u}^*; 0, x))\mathbf{E}_1(\mathbf{w}_1 - \mathbf{w}_1^*)(x)\,dx \\ &\quad + (\mathbf{R}\mathbf{v}^*, \mathbf{v} - \mathbf{v}^*)_{\mathbf{V}} + (\mathbf{M}_0\mathbf{w}_0^*, \mathbf{w}_0 - \mathbf{w}_0^*)_{\mathbf{W}_0} + (\mathbf{M}_1\mathbf{w}_1^*, \mathbf{w}_1 - \mathbf{w}_1^*)_{\mathbf{W}_1} \geq 0, \\ &\qquad\qquad\qquad\qquad\qquad\qquad \forall \mathbf{u} = (\mathbf{v}, \mathbf{w}_0, \mathbf{w}_1) \in \mathcal{U}_{ad}. \end{aligned}$$

Since it requires too long calculations to prove this theorem, we omit the proof. The further study of optimal control problems for $(\mathbf{CS})$ will appear elsewhere.

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