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# ON THE STABILITY OF STEADY SURFACE-TENSION DRIVEN FLOWS

Ching-An Wang* and Tu-Cheng Wu

**Abstract.** In this paper, we study the stability of similarity solutions for the problem $f''' + Q(Aff'' - (f')^2) = \beta$, with $Q > 0$, $A \geq 1$ and $\beta \in \mathbb{R}$. The given problem was derived from the symmetric reduction of similarity transformations from the Navier-Stokes equation for the planar flows. By imposing additional eigenvalue problems, our numerical studies show that the resultant steady flows are unstable as $Q$ becomes large for various $A < 2$. Furthermore, our analytical result gives that the steady flows are stable for small $Q$, when $1 \leq A < 2$, or for any $Q > 0$ when $A \geq 2$. Moreover, the existence of asymmetric flows for various $A < 2$ is also found numerically.

## 1. Introduction

The surface-tension driven flows of low Prandtl number fluids are important in processes such as the production of silicon crystals and cylindrical rods of silicon. Surface-tension phenomena mainly occur along a floating rectangular slot or a floating disk due to the difference of temperatures of fluid from some imposed conditions. By means of floating, it was assumed that two opposite surfaces of the rectangular slot or the disk are free. For the physical consideration, the floating zones are assumed to be in a micro-gravity environment. Usually, the lateral solid surfaces (wall) of the slot or the disk and the free surfaces confine the flow in the low Prandtl number fluid. On the free surfaces, a temperature radiation, due to the difference of temperatures at the mid-line for the slot and at the lateral solid wall, is assumed, and the radiation

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drives the surface-tension driven flows. Assume a slot is described by the coordinate system such that $-l \leq x \leq l$ and $0 \leq y \leq 2d$, where $y = d$ represents half of the depth for the slot and $y = 0$, $2d$ denote the position at the free surfaces. The Navier-Stokes system was applied to describe the surface-tension flows for the distributions of velocity in a low Prandtl number fluid confined in the domain of a floating rectangular slot and let $u$ and $v$ denote the tangential and normal velocity of the planar flow. Therefore, the formulation of boundary layer approximation of the 2-dimensional Navier-Stokes equations for the surface-tension driven flows is given as follows:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{1}$$

$$\frac{\partial u}{\partial T} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \tag{2}$$

$$\frac{\partial v}{\partial T} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right), \tag{3}$$

where $P$, $\rho$, $\nu$, and $T$ represent the pressure, density, kinematic viscosity and time, respectively.

For the steady state, we recall that, as in [7, 9], $\eta = y/\delta(x)$ is an independent variable and the similarity function $f(\eta)$ corresponds to the stream function $\Psi(x, y) = u_R(x)\delta(x)f(\eta)$ with $u = \Psi_y$ and $v = -\Psi_x$, where $f(\eta) = \int_0^\eta h(s)ds$ and $h(\eta) = u(x, y)/u_R(x)$. Then, the $x$-momentum equation (2) for the steady flow leads to the following equation

$$f''' + Q(Aff'' - (f')^2) = \beta, \quad ('= d/d\eta), \tag{4}$$

for $0 \leq \eta \leq 2$, where $\delta(x)$, $u_R(x)$ are properly chosen and $Q$, $\beta$, $A$ are real with $A > 1/2$. The flows were confined by the free surfaces and, hence, the condition $v = 0$ was imposed. Moreover, the geometry of the free surface could be described by the condition $-\mu u_y = d\sigma/dx$, where $\mu$ and $\sigma$ represent the dynamical viscosity and the surface tension, respectively. Thus,

$$f(0) = f(2) = 0 \tag{5}$$

and the normalized conditions

$$f''(0) + 1 = f''(2) - 1 = 0 \tag{6}$$

are derived.

By the symmetry assumption for the planar flows crossing the center line $y = d$ in the slot, the equation (4) subject to

The bifurcation diagram in the parameter $(Q, \beta)$ space for the problem (1), (2) when $A = 1$, $3/2$, $5/4$, and 2.

$$f(0) = f''(0) + 1 = f(1) = f''(1) = 0 \tag{7}$$

was studied numerically in [7, 8]. Two types of nonnegative solutions and one type of oscillatory solutions on $[0, 1]$ were found. Here, the type $I$ denotes the positive and concave, while the type $II$ represents the positive and nonconcave solution on $(0, 1)$. Moreover, the type $III$ is labeled that the solution of (4), (7) has exactly one zero in $(0, 1)$. The corresponding bifurcation diagrams and profiles of type $I$,

FIG. 2. (a) The profile of symmstric solution $f$ corresponding to type $I$. (b) The streamline of a symmetric flow for the type $I$ solution.

FIG. 3. (a) The profile of symmstric solution $f$ corresponding to type $II$. (b) The streamline of a symmetric flow for the type $II$ solution.

$II$ and $III$ solutions were shown in Figures 1-4. In fact, Chen *et al.* [2, 10] verified the following properties:

(P1) The problem (4), (7) can only possess the type $I$, $II$ and $III$ solutions for $1 \leq A < 2$ and the type $I$ solutions if $A \geq 2$.

FIG. 4. (a) The profile of symmstric solution $f$ corresponding to type $III$. (b) The streamline of a symmetric flow for the type $III$ solution.

FIG. 5. (a) The profile of an symmstric solution $f$. (b) The streamline of an symmetric flow.

(P2) Suppose $A \geq 1$. The problem (4), (7) has at least one type $I$ or $II$ solution for $Q \geq 0$. Moreover, there is at least one $Q < 0$ such that (4), (7) has a type $I$ solution for $\beta > 1$.

(P3) Let $1 \leq A \leq 3/2$. The problem (4), (7) has at least a type $III$ solution for sufficiently large $Q > 0$.

It is our purpose here to study the stability of the solutions for the problem (4), (7) by imposing a corresponding eigenvalue problem on (4)-(6). The formulation of the eigenvalue problem and the numerical studies of the stability for various $A$ are given in Section 3. The asymptotic behavior of the principal eigenvalues of type $I$ solutions for some $Q$ is analyzed in Sections 4 and 5. Our study shows that the steady state solutions are unstable for large $Q$ when $1 \leq A < 2$. Furthermore, the numerical computation indicates the existence of some families of asymmetric solutions for various $A$, $1 \leq A < 2$.

## 2. Formulation of the Eigenvalue Problem

For the stability analysis, we write the stream function as

$$\Psi(x, y, T) = u_R(x)\delta(x)F(\eta, T),$$

where $F(\eta,\ T) = \int_0^\eta h(s,\ T)ds$ and $h(\eta,\ T) = u(x,\ y,\ T)/u_R(x)$. Let $t = (du_R(x)/dx)T$. Then we obtain that

$$QF_{\eta\eta t} = F_{\eta\eta\eta\eta} + Q(AFF_{\eta\eta\eta} + (A-2)F_\eta F_{\eta\eta}) \tag{8}$$

and

$$F(0, t) = F''(0, t) + 1 = F(2, t) = F''(2, t) - 1 = 0. \tag{9}$$

Treat the temporal stability of the steady state flows by expressing

$$F(\eta, t, Q) = f(\eta, Q) + g(\eta, t, Q), \tag{10}$$

linearizing (8) for small $g$, and taking normal modes with $g(\eta, t, Q) = e^{st}G(\eta, Q)$. This leads to the eigenvalue functional

$$G'''' = QsG'' - Q(AfG''' + (A-2)f'G'' + (A-2)f''G' + Af'''G) \tag{11}$$

for the determination of the eigenvalues $s$ and the corresponding eigenfunctions $G$. To be consistent with (9), $G$ must satisfy

$$G(0) = G''(0) = G(2) = G''(2) = 0. \tag{12}$$

For convenience, we may study the stability of the symmetric flow on the upper half, $0 \leq \eta \leq 1$. Therefore, the eigenfunction $G$ is imposed by the boundary conditions as

$$G(0) = G''(0) = G'(1) = G'''(1) = 0 \tag{13}$$

at the odd mode, and

$$G(0) = G''(0) = G(1) = G''(1) = 0 \tag{14}$$

at the even mode. Note that we predict instability of a specified basic flow $f$ with $Q > 0$ if $\mathrm{Re}(s) > 0$, or $\mathrm{Re}(s) < 0$ with $Q < 0$, for at least one eigenvalue. It indicates that the steady flows are unstable when there exist some positive eigenvalues.

## 3. Local Bifurcation and Numerical Simulation

By means of a small perturbation, we may perturb a steady solution $f_0$ of (4)-(6) at given $Q_0$ by $f(\eta, Q)$ with small $\epsilon = Q - Q_0$. In fact, it is known that a power series in $\epsilon$ is appropriate at a regular point, but in $\epsilon^{1/2}$ at a turning point or a pitchfork bifurcation. Assume the expansion

$$f(\eta, Q) = f_0(\eta) + \epsilon^{1/2} f_{1/2}(\eta) + \epsilon f_1(\eta) + \epsilon^{3/2} f_{3/2}(\eta) + \cdots \tag{15}$$

for small $\epsilon$. Differentiating (4), we get

$$f^{''''} + Q(Aff^{'''} + (A-2)f^{'}f^{''}) = 0. \tag{16}$$

By substituting (15) into the problem (16), (5), (6) and equating coefficients of terms in $\epsilon^0$, we obtain that

$$f_0^{''''} + Q_0(Af_0f_0^{'''} + (A-2)f_0^{'}f_0^{''}) = 0$$

subject to

$$f_0(0) = f_0^{''} + 1 = f_0(2) = f_0^{''}(2) - 1 = 0.$$

Next we equate coefficients of $\epsilon^{1/2}$ and obtain that

$$f_{1/2}^{''''} + Q_0(Af_0f_{1/2}^{'''} + (A-2)f_0^{'}f_{1/2}^{''} + (A-2)f_0^{''}f_{1/2}^{'} + Af_0^{'''}f_{1/2}) = 0 \tag{17}$$

subject to

$$f_{1/2}(0) = f_{1/2}^{''}(0) = f_{1/2}(2) = f_{1/2}^{''} = 0. \tag{18}$$

Suppose all the eigenvalues of (11), (12) are nonzero for a given $Q_0$. Then, it is clear that $f_{1/2} = 0$ is the only possible solution of (17), (18). Indeed, it shows that at such a point the perturbed solution is expressible as a series in integer powers of $\epsilon$, and is unique. Then no bifurcation may occur. This indicates that a turning point or a pitchfork bifurcation may be expected when obtaining a zero eigenvalue at a certain $Q_0$.

It is clear that, as in Section 4, the eigenvalues are discrete. Denote the eigenvalues at $i$th odd and even mode by $\sigma_i^o$ and $\sigma_i^e$, respectively. For numerical simulation, we compute the first three $\sigma_i^o$ and $\sigma_i^e$ by a multiple shooting code BVPSOL [3-6] to integrate the basic flow $f$ and eigenfunction $G$. Also, the eigenfunction is normalized by imposing $G^{'} = 1$ at $\eta = 0$.

The correlation diagrams of eigenvalues and $Q$ for the symmetric solutions of the types $I$, $II$ and $III$ when $A = 1$, where solid and dotted curves represent $\sigma_n^0$ and $\sigma_n^e$, respectively.

By applying the continuation scheme, we begin the computation at $Q = 0.1$ for various $A$. The initial data for the first three eigenvalues $\sigma_i^a$, $\sigma_i^s$, $i = 1$, 2, 3, have been chosen by the asymptotic values obtained in Section 4. The continuation scheme is applied along the type $I$, $II$ and $III$ branches, as in Figure 1, by increasing $Q$. The plots of the corresponding $(Q, \sigma_i^o)$ or $(Q, \sigma_i^e)$ curves, $i = 1$, 2, 3, along the type $I$ branch are given in Figures 6-9. By the corresponding $(Q, s)$ diagrams, it could be observed that the type $I$ symmetric solution is temporally stable and becomes unstable when $Q$ increases beyond a particular level $Q = Q^*$ at which $\sigma_1^o(Q^*) = 0$ for various $A < 2$, while it is temporally stable for $Q > 0$ when $A \geq 2$. Here $Q^* \approx 1607$, 2113, 2977 when $A = 1$, 5/4, 3/2, respectively, and, as shown in

The correlation diagrams of eigenvalues and $Q$ for the symmetric solutions of the types $I$, $II$ and $III$ when $A = 1.25$, where solid and dotted curves represent $\sigma_n^0$ and $\sigma_n^e$, respectively.

Figure 10, our computation indicates that $Q^*$ tends to infinity as $A$ approaches to 2. In fact, in Section 5, we shall verify that $\sigma_n^o$ and $\sigma_n^e$ for the type $I$ solution with $A \geq 1$ tend to zero as $Q$ increases and the solutions of types $II$ and $III$ are unstable.

Furthermore, it is known that the asymmetric solutions may bifurcate from some symmetric solutions with a zero eigenvalue. By imposing the data obtained from the eigenfunction with $\sigma_1^o(Q^*) = 0$, a continuous family of asymmetric solutions is also found that the corresponding branch bifurcates at $Q = Q^*$. It is clear that the mirror image of an asymmetric solution at the center line $\eta = 1$ is also a solution of (4) - (6) and this indicates that the pitchfork bifurcation occurs at $Q^*$. The

The correlation diagrams of eigenvalues and $Q$ for the symmetric solutions of the types $I$ and $III$ when $A = 1.5$, where solid and dotted curves represent $\sigma_n^0$ and $\sigma_n^e$, respectively.

FIG. 9. The correlation diagrams of eigenvalues and $Q$ for the symmetric solutions of the type $I$ when $A = 2$, where solid and dotted curves represent $\sigma_n^0$ and $\sigma_n^e$, respectively.

FIG. 10. The correlation diagrams of $\sigma_1^0$ and $Q$, when $\sigma_1^0 \geq 0$ for various $1 \leq A \leq 3/2$.

detailed bifurcation diagrams for the asymmetric flows when $Q$ is close to $Q^*$ are given in Figures 11-13 when $A = 1,\ 5/4,\ 3/2$. The profile of a selected asymmetric solution and its corresponding stream line are shown in Figure 5.

FIG. 11. The diagrams in the parameter $(Q, f'(0))$ and $(Q, \beta)$ spaces for the symmetric $(I)$ and asymmetric $(I_1, I_1')$ solutions when $A = 1$.

FIG. 12. The diagrams in the parameter $(Q, f'(0))$ and $(Q, \beta)$ spaces for the symmetric $(I)$ and asymmetric $(I_1, I_1')$ solutions when $A = 1.25$

## 4. Asymptotic Expansion of the Eigensolutions for Small $Q$

In this section, we begin with the analytical study of the steady flows $f$, the eigenfunctions $G$, and the eigenvalues $s$ for small $Q$. It is clear that the problem (4), (7) has a unique symmetric solution

$$f_0(\eta) = \frac{1}{3}\eta - \frac{1}{2}\eta^2 + \frac{1}{6}\eta^3$$

when $Q = 0$ for any $A$. We expand

The diagrams in the parameter $(Q, f'(0))$ and $(Q, \beta)$ spaces for the symmetric $(I)$ and asymmetric $(I_1, I_1')$ solutions when $A = 1.5$

$$f(\eta, Q) = f_0(\eta) + \sum_{n=1}^{\infty} f_n(\eta) Q^n \tag{19}$$

for small $Q$. Since $f$ satisfies the boundary conditions (7), the $f_n$'s in the expansion must satisfy the boundary conditions

$$f_0(0) = f_0''(0) + 1 = f_0(1) = f_0''(1) = 0$$

and

$$f_n(0) = f_n''(0) = f_n(1) = f_n''(1) = 0$$

for $n = 1, 2, \cdots$. Substituting (19) into (16) and equating coefficients of terms in $Q$, we get

$$f_1'''' + A f_0 f_0''' + (A-2) f_0 f_0'' = 0 \tag{20}$$

subject to the boundary conditions

$$f_1(0) = f_1''(0) = f_1(1) = f_1''(1) = 0. \tag{21}$$

Then, $f_1(\eta)$ is expressed as

$$\begin{aligned} f_1(\eta) \quad = \quad & -\frac{A+2}{945}\eta - \frac{A-4}{270}\eta^3 + \frac{A-2}{72}\eta^4 - \frac{5A-8}{360}\eta^5 \\ & + \frac{2A-3}{360}\eta^6 - \frac{2A-3}{2520}\eta^7 \end{aligned}$$

by solving (20), (21), and the other $f_n$'s can also be obtained by equating coefficients of terms in $Q^n$ for $n = 2, 3, \cdots$ . We proceed similarly with the eigensolution by writing

$$G(\eta, Q) = G_0(\eta) + \sum_{n=1}^{\infty} G_n(\eta) Q^n \tag{22}$$

and

$$Qs = -\sigma_0 - \sum_{n=1}^{\infty} \sigma_n Q^n \tag{23}$$

as $Q$ tends to zero. Substituting (19), (22) and (23) into (11), and equating coefficients of terms in $Q^n$ for $n = 0, 1, 2, \cdots$, we find that

$$\begin{aligned} G_0'''' + \sigma_0 G_0'' &= 0, \\ G_1'''' + \sigma_0 G_1'' &= -\sigma_1 G_0'' - (A f_0 G_0''' + (A-2) f_0' G_0'' \\ &\quad + (A-2) f_0'' G_0' + A f_0''' G_0), \\ &\vdots \end{aligned}$$

To be consistent with the boundary conditions of $G$, $G_n$ must satisfy

$$G_n(0) = G_n''(0) = G_n'(1) = G_n'''(1) = 0$$

for the odd modes and

$$G_n(0) = G_n''(0) = G_n(1) = G_n''(1) = 0$$

for the even modes, when $n = 0, 1, 2, \cdots$. Note that the equation for $G_n$ involves $G_0$, $G_1$,$\cdots$, $G_{n-1}$, $\sigma_0$, $\sigma_1$, $\cdots$, $\sigma_n$. As in most of our numerical results, we normalize the eigenfunction by imposing the condition that $G'(0) = 1$, and so take $G_0'(0) = 1$ and $G_n'(0) = 0$ for $n = 1, 2, \cdots$. For the convenience, we denote by $\sigma_{nm}^o$ the value of $\sigma_n$ in (23) corresponding to the $m$th odd mode, and, similarly, denote by $\sigma_{nm}^e$ the corresponding quantity for the even modes. Also, $G_{nm}^o$ and $G_{nm}^e$ are defined analogously for each type of modes.

For the odd modes, we obtain that

$$
\begin{aligned}
\sigma_{0m}^a &= \frac{(2m-1)^2\pi^2}{4}, \\
\sigma_{1m}^a &= \frac{4+5A}{2(2m-1)^2\pi^2}, \\
G_{0m}^a &= \frac{2}{(2m-1)\pi}\sin\frac{2m-1}{2}\pi\eta, \\
G_{1m}^a &= \frac{32+32A}{(2m-1)^4\pi^4} + \left[-\frac{A}{24(2m-1)\pi}\eta^4 + \frac{A}{6(2m-1)\pi}\eta^3\right. \\
&\quad + \left(-\frac{A}{6(2m-1)\pi} + \frac{12+11A}{2(2m-1)^3\pi^3}\right)\eta^2 - \frac{12+11A}{(2m-1)^3\pi^3}\eta \\
&\quad \left. - \left(\frac{4\sigma_{1m}^a}{(2m-1)^3\pi^3} - \frac{8+6A}{3(2m-1)^3\pi^3} + \frac{56+54A}{(2m-1)^5\pi^5}\right)\right]\sin\frac{2m-1}{2}\pi\eta \\
&\quad + \left[-\frac{4+3A}{6(2m-1)^2\pi^2}\eta^3 + \frac{4+3A}{4(2m-1)^2\pi^2}\eta^2 + \left(\frac{2\sigma_{1m}^a}{(2m-1)^2\pi^2}\right.\right. \\
&\quad \left.\left. - \frac{4+3A}{3(2m-1)^2\pi^2} + \frac{28+27A}{(2m-1)^4\pi^4}\right)\eta - \frac{32+32A}{(2m-1)^4\pi^4}\right]\cos\frac{2m-1}{2}\pi\eta.
\end{aligned}
$$

For the even modes, we find that

$$
\begin{aligned}
\sigma_{0m}^s &= m^2\pi^2, \\
\sigma_{1m}^s &= \frac{4+5A}{8m^2\pi^2}, \\
G_{0m}^s &= \frac{1}{m\pi}\sin m\pi\eta, \\
G_{1m}^s &= -\frac{2+2A}{m^4\pi^4}\eta + \frac{2+2A}{m^4\pi^4} + \left[\frac{4+3A}{12m^3\pi^3} - \frac{12+11A}{8m^3\pi^3}\eta\right. \\
&\quad \left. - \left(\frac{A}{12m\pi} - \frac{12+11A}{16m^3\pi^3}\right)\eta^2 + \frac{A}{12m\pi}\eta^3 - \frac{A}{48m\pi}\eta^4\right]\sin m\pi\eta \\
&\quad - \left[\frac{2+2A}{m^4\pi^4} - \left(\frac{\sigma_{1m}^s}{2m^2\pi^2} - \frac{4+3A}{12m^2\pi^2} + \frac{28+27A}{16m^4\pi^4}\right)\eta\right. \\
&\quad \left. - \frac{4+3A}{8m^2\pi^2}\eta^2 + \frac{4+3A}{24m^2\pi^2}\eta^3\right]\cos m\pi\eta.
\end{aligned}
$$

For small $Q$, it follows that

$$G(\eta, Q) \approx G_0(\eta) \tag{24}$$

and

$$s \approx -\frac{\sigma_0}{Q} - \sigma_1. \tag{25}$$

By using (24), (25) to estimate $G$ and $s$ for small $Q$, it shows that the flow is temporally stable as $Q$ is small when $A > 0$. Moreover, the asymptotic results obtained here are in accordance with the numerical results obtained in the previous section. The selected data are shown in Tables 1 and 2.

## 5. The Analysis of $\sigma_1^o$, $\sigma_1^e$ for the Type $I$ with Large $Q$

In [2], it was pointed out that the type $II$ or type $III$ solutions with small $|\beta|$ are very similar to the type $I$ ones, when $Q$ is sufficiently large, when $1 \le A < 2$ and the problem (4), (7) possesses only the type $I$ solutions for $A \ge 2$. Hence, we need only consider the asymptotic behavior of the eigenvalues for the type $I$ solutions for large $Q$. We shall verify that $\sigma_1^o > 0$ for large $Q$ as $1 \le A < 2$ and it tends to zero as $Q$ increases for $A \ge 1$.

For large $Q$, the outer solution of (4) satisfies

$$Aff'' - (f')^2 \approx \frac{\beta}{Q} \tag{26}$$

The antisymmetric modes at $Q = 0.1$

| | | $\sigma_1^a$ | $\sigma_2^a$ | $\sigma_3^a$ |
|---|---|---|---|---|
| $A$=1 | numerical result | -25.1609 | -222.1567 | -616.8835 |
| | analytical result | -25.1299 | -222.1168 | -616.8685 |
| $A$=5/4 | numerical result | -25.2217 | -222.1633 | -616.8794 |
| | analytical result | -25.1933 | -222.1238 | -616.8711 |
| $A$=3/2 | numerical result | -25.2824 | -222.1698 | -616.8752 |
| | analytical result | -25.2566 | -222.1308 | -616.8736 |
| $A$=2 | numerical result | -25.4039 | -222.1828 | -616.8669 |
| | analytical result | -25.3833 | -222.1399 | -616.8787 |

The symmetric modes at $Q = 0.1$

| | | $\sigma_1^s$ | $\sigma_2^s$ | $\sigma_3^s$ |
|---|---|---|---|---|
| $A$=1 | numerical result | -98.8434 | -394.8431 | -888.2993 |
| | analytical result | -98.8100 | -394.8127 | -888.2770 |
| $A$=5/4 | numerical result | -98.8572 | -394.8411 | -888.2963 |
| | analytical result | -98.8290 | -394.8166 | -888.2788 |
| $A$=3/2 | numerical result | -98.8711 | -394.8390 | -888.2932 |
| | analytical result | -98.8416 | -394.8205 | -888.2906 |
| $A$=2 | numerical result | -98.8988 | -394.8350 | -888.2870 |
| | analytical result | -98.8734 | -394.8285 | -888.2841 |

subject to the outer boundary conditions

$$f(1) = 0, \quad f''(1) = 0.$$

Since $f(1) = 0$, from (26), it follows that

$$f'(1)^2 \approx -\frac{\beta}{Q}.$$

It was verified, in [2], that $f'(1) < 0$ and $\beta < 0$ for large $Q$. For convenience, we set $f'(1) = \alpha$, $\alpha < 0$ and $\alpha^2 = -\beta/Q$. Substituting $f(1) = 0$, $f'(1) = \alpha$ and $f''(1) = 0$ into (4), we obtain $f'''(1) \approx 0$. Then, $f(1) = 0$, $f''(1) = 0$ and $f'''(1) \approx 0$ yield that $f''''(1) \approx 0$. By applying a similar argument, we get $f^{(k)}(1) \approx 0$ for all $k > 2$. It implies that, by the Taylor's expansion,

$$f(\eta) \approx \alpha(\eta - 1).$$

Also, from (11) at $\eta = 1$, we may expect that the eigenvalue $s$ has the asymptotic behavior $s = O(\alpha)$ for large $Q$ at the odd mode. In fact, it is also true for the even mode and will be verified later in this section.

### 5.1. The limits of $\alpha$

We shall first verify that $\alpha$ tends to zero for large $Q$. By applying the transformation as in [2], let $y = b(1 - \eta)$ and $g(y) = \frac{Q}{b} f(\eta)$ for any nonzero $Q$ and positive $b$. Then (4), (7) is equivalent to

$$g''' + (g')^2 - Agg'' = -\frac{Q\beta}{b^4}, \quad \left(' = \frac{d}{dy}\right), \tag{27}$$

subject to

$$g'(0) = g(b) = g''(0) = g''(b) + \frac{Q}{b^3} = 0.$$

Let $g(y; B, E)$ be the solution of (27) subject to

$$g(0) = g'(0) - B = g''(0) = 0, \tag{28}$$

where $A$ is given and $E = -Q\beta/b^4$. Suppose $g(y; B, E)$ meets the $y$-axis at some positive $y_*$. Then, by setting $b = y_*$, the problem (4), (7) has a solution when

$$\begin{aligned} Q(B, E) &= -y_*^3 g''(y_*; B, E), \\ \beta(B, E) &= \frac{E y_*}{g''(y_*; B, E)}. \end{aligned}$$

As given in [2], it is known that $g(y; B, E)$ has a unique positive zero $y_*$ for $(B, E) \in D = \{(B, E)|B > 0,\ \ E < B^2\}$ and it leads to the type $I$ solution of (4), (7) with $Q(B, E) > 0$. In fact, $g(y_*; B, E)$ has the following property.

**Property 1** [2]. *For $A \geq 1$ and $(B, E) \in D$, $g(y; B, E) > 0$, $g''(y; B, E) < 0$, $g'''(y; B, E) < 0$ on $(0, y_*)$ and $g'(y_*; B, E) < 0$.*

Moreover, by the homogeneity of $g$, the set $\Gamma$ of $(Q, \beta)$ on which (4), (7) possesses the type $I$ solution can be rewritten as

$$\Gamma = \{(Q(1, r), \beta(1, r))| - \infty < r < 1\}.$$

Then, by the continuous dependence, $\Gamma$ is a connected subset in the half plane $Q > 0$, $\beta \in \mathbb{R}$. Also, $Q$ and $y_*$ have the following properties.

**Property 2** [2]. *For $A \geq 1$,*
(a) $\lim_{r \to 1^-} y_*(1, r) = \infty$;
(b) $\lim_{r \to 1^-} Q(1, r) = \infty$.

From Property 2 (a), there exists a $\delta > 0$ such that $y_*(1, r) > 1$ for all $r$, $0 < 1 - \delta < r < 1$. By Property 1 and (27), we obtain that

$$(g'(y_*(1, r); 1, r))^2 > g'''(y_*(1, r); 1, r) + (g'(y_*(1, r); 1, r))^2 = r > 0.$$

Then

$$g'(y_*(1, r); 1, r) < -\sqrt{r}.$$

Since $g'(0; 1, r) = 1$, we then have

$$\begin{aligned} -\sqrt{r} - 1 &> g'(y_*(1, r); 1, r) - g'(0; 1, r) \\ &= \textstyle\int_0^{y_*} g''(t; 1, r) dt \\ &\geq y_*(1, r) g''(y_*(1, r); 1, r). \end{aligned}$$

For $0 < 1 - \delta < r < 1$, it is clear that

$$\frac{ry_*(1,r)}{g''(y_*(1,r);1,r)} > \frac{r}{-\sqrt{r}-1}.$$

Hence,

$$\beta(1,r) > \frac{r}{-\sqrt{r}-1}$$

and

$$\lim_{r\to 1^-} \beta(1,r) > -\frac{1}{2}.$$

On the other hand, integrating (27), we get that

$$g''(y_*(1,r);1,r) - ry_*(1,r) = \int_0^{y_*} (A+1)g(t;1,r)g''(t;1,r)dt.$$

By Property 1, we further have

$$1 - \frac{ry_*(1,r)}{g''(y_*(1,r);1,r)} = \int_0^{y_*} \frac{(A+1)g(t;1,r)g''(t;1,r)}{g(y_*(1,r);1,r)}dt > 0.$$

Since $\beta(1,r) = ry_*(1,r)/g''(y_*(1,r);1,r)$, this implies that

$$\lim_{r\to 1^-} \beta(1,r) < 1.$$

Hence, $\lim_{Q\to\infty} \alpha = 0$ is obtained since $\alpha^2 = -\beta/Q$ and $\beta(1,r)/Q(1,r)$ tends to 0 as $r$ tends to $1^-$. We also obtain the feature that $\lim_{r\to 1^-} |\alpha|Q = \infty$.

### 5.2. The asymptotes of $\sigma_1^o$ and $\sigma_1^e$

Now, we proceed with the study of the asymptotic behavior of $\sigma_1^o$, $\sigma_1^e$ for large $Q$. Replacing $f(\eta)$ by $\alpha(\eta - 1)$, and letting $z = 1 - \eta$ and $H(z,Q) = G(\eta,Q)$, we can rewrite (11) in the form

$$H'''' - QA|\alpha|zH''' - Q(s - (2-A)|\alpha|)H'' = 0, \tag{29}$$

where $H(z,Q)$ denotes the inviscid representation of $G(\eta,Q)$. The boundary conditions at $z = 0$ are $H = H'' = 0$ for the even, and $H = H''' = 0$ for the odd modes, respectively. Set

$$H''(z,Q) = u(z,Q)v(z,Q), \tag{30}$$

where $v(z,Q) = e^{A|\alpha|Qz^2/4}$. Substituting (30) into (29) yields

$$u'' - \left(\frac{1}{4}(QA|\alpha|z)^2 + Qs - \frac{4-A}{2}Q|\alpha|\right)u = 0.$$

Setting $u(z,Q) = w(\tilde{z},Q)$, $\tilde{z} = (A|\alpha|Q)^{1/2}z$, and $\tilde{s} = s/(A|\alpha|) - (4-A)/(2A)$, we get that

$$w'' - \left(\frac{1}{4}z^2 + \tilde{s}\right) w = 0 \tag{31}$$

with the general solution

$$w(\tilde{z},Q) = c_1 U(\tilde{s},\tilde{z}) + c_2 V(\tilde{s},\tilde{z}),$$

where $c_1$, $c_2$ are constants and $U$, $V$ are the parabolic cylinder functions. Hence, we obtain

$$H''(z,Q) = e^{\frac{\tilde{z}^2}{4}}(c_1 U(\tilde{s},\tilde{z}) + c_2 V(\tilde{s},\tilde{z})).$$

The asymptotic behaviors of $U$, $V$, for large $Q$ are given by

$$\begin{aligned} U(\tilde{s},\tilde{z}) &= e^{-\frac{\tilde{z}^2}{4}} \tilde{z}^{-\tilde{s}-\frac{1}{2}} \left(1 - \frac{(\tilde{s}+\frac{1}{2})(\tilde{s}+\frac{3}{2})}{2\tilde{z}^2} + \cdots\right), \\ V(\tilde{s},\tilde{z}) &= \sqrt{\frac{2}{\pi}} e^{\frac{\tilde{z}^2}{4}} \tilde{z}^{\tilde{s}-\frac{1}{2}} \left(1 + \frac{(\tilde{s}+\frac{1}{2})(\tilde{s}+\frac{3}{2})}{2\tilde{z}^2} + \cdots\right), \end{aligned}$$

for $\tilde{z} >> |\tilde{s}|$, and

$$\begin{aligned} U(\tilde{s},0) &= \frac{\sqrt{\pi}}{2^{\frac{1}{2}(\tilde{s}+\frac{1}{2})}\Gamma(\frac{1}{2}\tilde{s}+\frac{3}{4})}, \\ U'(\tilde{s},0) &= \frac{\sqrt{\pi}}{2^{\frac{1}{2}(\tilde{s}-\frac{1}{2})}\Gamma(\frac{1}{2}\tilde{s}+\frac{1}{4})}. \end{aligned}$$

We assume here that the normalization of the eigenfunctions is $O(1)$. Note that $H''$ is exponentially large for $z \neq 0$ unless $c_2$ is exponentially small. It follows that

$$H''(0,Q) = \frac{c_1\sqrt{\pi}}{2^{\frac{1}{2}(\tilde{s}+\frac{1}{2})}\Gamma(\frac{1}{2}\tilde{s}+\frac{3}{4})}$$

and

$$H'''(0,Q) = \frac{c_1\sqrt{\pi}}{2^{\frac{1}{2}(\tilde{s}-\frac{1}{2})}\Gamma(\frac{1}{2}\tilde{s}+\frac{1}{4})}.$$

Thus, we obtain the following asymptotic limits

$$\sigma_n^e \sim 2|\alpha|\left(1 - A + (1-n)A\right), \quad n = 1, 2, \cdots, \tag{32}$$

for the even modes, and

$$\sigma_n^o \sim 2|\alpha|\left(\frac{2-A}{2}+(1-n)A\right), \quad n=1,2,\cdots, \tag{33}$$

for the odd modes. From (32) and (33), we know that,

$$\sigma_1^e \sim 2|\alpha|(1-A) \le 0 \tag{34}$$

when $A \ge 1$,

$$\sigma_1^o \sim 2|\alpha|\left(\frac{2-A}{2}\right) > 0 \tag{35}$$

when $1 \le A < 2$, and

$$\sigma_1^o \sim 2|\alpha|\left(\frac{2-A}{2}\right) \le 0 \tag{36}$$

when $A \ge 2$. Therefore, $\sigma_1^e$, $\sigma_1^o$ both tend to zero as $Q$ tends to infinity, and, from (34) and (36), the type $I$ solution for large $Q$ is stable when $A \ge 2$.

## 6. Concluding Remarks

By means of a small perturbation, we have imposed the even and odd boundary conditions for the eigenfunctions to analyze the local stability of the symmetric steady flows. Numerical results for various $A$ indicate that the type $I$ steady flows are stable for $A \ge 2$, $Q > 0$, and $1 \le A < 2$ as $Q$ is small, while it is unstable for $1 \le A < 2$ if $Q$ is large. In fact, our mathematical result verified parts of the numerical observation.

Beyond the symmetric flow, the asymmetric flows are also found numerically by imposing an odd eigenfunction on the type $I$ flows. In fact, the asymmetric flows are bifurcated from a type $I$ flows with a zero eigenvalue and our result indeed indicates that the existence of zero eigenvalues, $\sigma_1^o = 0$, at some $Q$ along the type $I$ branch in Figures 11-13. Hence, it can be conjectured that the problem (4) - (6) possesses asymmetric solutions at some $Q > 0$ when $1 \le A < 2$, although neither the mathematical verification nor the stability study is yet found elsewhere.

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Ching-An Wang
Department of Mathematics, National Chung Cheng University
Minghsiung, Taiwan 621, R.O.C.

Tu-Cheng Wu
Nan-Jeon Junior College of Technology and Commerce
Yen-shui, Taiwan 737, R.O.C.