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FINITE MATRICES SIMILAR TO IRREDUCIBLE ONES

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Abstract. In this paper, we prove that an $n\times n$ $(n \geq 3)$ complex matrix T is similar to an irreducible matrix if and only if T is not quadratic and rank $(T-\lambda I) \geq n/2$ for every complex number λ. As an application, we prove that: for any integers n and k with $3 \leq k < n$, there exists an $n \times n$ irreducible nilpotent matrix of index k. This answers a question posed by P. R. Halmos

1. Introduction

A matrix (or an operator) is said to be *irreducible* if it commutes with no (orthogonal) projection other than 0 and I, and is said to be *reducible* otherwise.

Every operator on a nonseperable Hilbert space is reducible. On infinite-dimensional seperable Hilbert spaces, Gilfeather [4] proved that every normal operator without eigenvalue is similar to an irreducible operator. Later on, Fong and Jiang [3] improved Gilfeather's work by allowing the presence of eigenvalues. The aim of this paper is to completely characterize those matrices which are similar to irreducible ones.

Let T be a 2×2 matrix. Because T is similar to its Jordan form, T is similar to one of the following matrices $\begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix}$, $\begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}$ or $\begin{bmatrix} \alpha & 1 \\ 0 & \alpha \end{bmatrix}$, where α, β are distinct complex numbers. Since $\begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}$ is similar to the irreducible matrix $\begin{bmatrix} \alpha & 1 \\ 0 & \beta \end{bmatrix}$, we see that a 2×2 matrix is similar to an irreducible matrix if and only if it is not a scalar matrix. We have thus characterized those 2×2

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matrices which are similar to irreducible ones. From now on, we consider $n \times n$ complex matrices T, where $n \geq 3$. They are said to be *quadratic* if $T^2 + \alpha T + \beta I = 0$ for some complex numbers α, β. We shall prove the following *Main Theorem* :

Main Theorem. *An $n \times n$ $(n \geq 3)$ matrix T is similar to an irreducible matrix if and only if T is not quadratic and* rank $(T - \lambda I) \geq n/2$ *for every complex number λ.*

We use $M_{n\times m}$ to denote the set of all $n \times m$ complex matrices, and $M_n = M_{n\times n}$. Also, we use diag $[a_1, a_2, \cdots, a_n]$ to denote the diagonal matrix with entries $a_1, a_2, \cdots, a_n$ along the diagonal. For $T \in M_n$, we say that T has property $(*)$ if

$(*)$ T is not quadratic and rank $(T - \lambda I_n) \geq n/2$ for every complex number λ.

We now prove the necessity part of the Main Theorem.

Proposition 1.1. *Let $T \in M_n$ $(n \geq 3)$. If T is similar to an irreducible matrix, then T has property $(*)$.*

Proof. We first show that if T is quadratic, then STS^{-1} is reducible for any invertible matrix $S \in M_n$. Because T is quadratic, so is STS^{-1}. Thus it suffices to show that every quadratic matrix T is reducible. Gilfeather [4] used the structure theory of binormal operators (defined in [1]) to prove this. Here we give an alternative proof. We know that any quadratic matrix T is unitarily equivalent to a matrix of the form

$$\alpha_1 I_m \oplus \alpha_2 I_\ell \oplus \begin{bmatrix} \alpha_1 I_k & T_1 \\ 0_k & \alpha_2 I_k \end{bmatrix},$$

where $\alpha_1, \alpha_2 \in \mathbb{C}$, and T_1 is a $k \times k$ positive definite matrix [6]. Therefore, it suffices to consider T of the form $\begin{bmatrix} \alpha_1 I_k & T_1 \\ 0_k & \alpha_2 I_k \end{bmatrix}$. Since T_1 is positive definite, there exists a $k \times k$ unitary U such that $U^* T_1 U$ is a diagonal matrix T_2. Let A be the matrix $\begin{bmatrix} \alpha_1 I_k & T_2 \\ 0_k & \alpha_2 I_k \end{bmatrix}$. Then A is unitarily equivalent to T. Let P be the projection diag $[1, 0, \cdots, 0] \oplus$ diag $[1, 0, \cdots, 0]$ on $\mathbb{C}^k \oplus \mathbb{C}^k$. Since $PA = AP$, A is reducible, and so is T.

To complete the proof, we suppose that $0 < \text{rank}(T - \lambda I_n) < n/2$ for some $\lambda \in \mathbb{C}$, and show that $S^{-1}TS$ is reducible for any invertible matrix $S \in M_n$. Let $\mathcal{M}$ be the linear span of the ranges of $S^{-1}(T - \lambda I_n)S$ and

$(S^{-1}(T-\lambda I_n)S)^*$, and P be the projection from $\mathbb{C}^n$ onto the subspace $\mathcal{M}$. Since $\mathcal{M}$ is a reducing subspace of $S^{-1}TS$, P commutes with $S^{-1}TS$. Since $0 < \text{rank } (T-\lambda I_n) < n/2$, it follows that $1 \leq \dim \mathcal{M} \leq n-1$. Thus P is neither 0_n nor I_n. Therefore, $S^{-1}TS$ is reducible. This proves our assertion.∎

The rest of this paper aims to prove the sufficiency part of the Main Theorem. Since the Jordan form of T is similar to T and property $(*)$ is preserved under similarity, we may consider the Jordan form directly. If T has exactly one eigenvalue, then we list all the possible cases of T and prove the Main Theorem in Section 2. However, if T has at least two distinct eigenvalues, to avoid messy computations, we will not consider T directly. Rather, we will show that there exists a matrix S in the double commutant of T (defined in Section 3) which is similar to an irreducible matrix. This will imply that T is also similar to an irreducible matrix [3, Lemma 2.1].

In other words, we will divide the proof of the sufficiency part of the Main Theorem into the following two propositions.

Proposition 1.2. *Let $T \in M_n$ $(n \geq 3)$. If T has exactly one eigenvalue and T has property $(*)$, then T is similar to an irreducible matrix.*

Proposition 1.3. *Let $T \in M_n$ $(n \geq 3)$. If T has at least two distinct eigenvalues and T has property $(*)$, then T is similar to an irreducible matrix.*

It is clear that Propositions 1.1, 1.2, and 1.3 will lead to the Main Theorem. We will prove Propositions 1.2 and 1.3 in Sections 2 and 3 respectively.

The following notations will appear frequently. Throughout this paper, any unspecified entry of a matrix is 0. For a square matrix T, let $\sigma(T)$ denote its spectrum, $J(\gamma)$ denote the direct sum of all the Jordan blocks associated to the eigenvalue γ, and $J_n(\gamma)$ denote the $n \times n$ Jordan block associated to the eigenvalue γ. That is, $J_n(\gamma)$ is the the following $n \times n$ matrix

$$\begin{bmatrix} \gamma & 1 & & \\ & \gamma & \ddots & \\ & & \ddots & 1 \\ & & & \gamma \end{bmatrix}.$$

Moreover, if $\mathcal{N}$ is a subspace of a finite-dimensional Hilbert space $\mathcal{M}$, we use $\mathcal{M} \ominus \mathcal{N}$ to denote the space of all vectors in $\mathcal{M}$ which are perpendicular to $\mathcal{N}$.

2. Case of One Eigenvalue

The purpose of this section is to prove Proposition 1.2. Therefore, throughout this section, we always assume that T is an $n \times n$ matrix with exactly one eigenvalue, and has property $(*)$. We want to prove that T is similar to an irreducible matrix. As mentioned in Section 1, we may consider the Jordan form of T directly. That is, $T = J(\gamma)$, where γ is the eigenvalue of T. Let

$$J = \sum_{i=1}^{m} \oplus J_{n_i}(\gamma) \text{ with } n_1 \geq n_2 \geq \cdots \geq n_m \geq 2.$$

Then T is given by

$$T = J \quad \text{or} \quad T = \gamma I_k \oplus J. \tag{2.1}$$

It suffices to prove Proposition 1.2 for the two situations in (2.1). These will be handled by Propositions 2.4 and 2.6 respectively. In these two propositions, we will construct an upper-triangular matrix C which is similar to T. In order to prove that C is irreducible, we need the following two lemmas to help our computation.

Let

$$E = \begin{bmatrix} \gamma I_{m_1} & T_1 & X_3 & \cdots & X_n \\ & \gamma I_{m_2} & T_2 & & \\ & & \gamma I_{m_3} & \ddots & \\ & & & \ddots & T_{n-1} \\ & & & & \gamma I_{m_n} \end{bmatrix}. \tag{2.2}$$

Lemma 2.1. (1) *Let E be defined as in* (2.2). *If for each $1 \leq i \leq n-1$, T_i is one-to-one, then any hermitian matrix commuting with E is of the form $\sum_{i=1}^{n} \oplus F_i$.*

(2) *Suppose in addition that $T_1 = \operatorname{diag}[t_1, t_2, \cdots, t_{m_1}]$ with $|t_i| \neq |t_j|$ if $i \neq j$, $m_1 = m_2 \geq m_3 \geq \cdots \geq m_n$, $T_i = \begin{bmatrix} X_i \\ 0_{(m_i - m_{i+1}) \times m_{i+1}} \end{bmatrix}$ with $X_i \in M_{m_{i+1}}$ diagonal for $i = 2, 3, \cdots, n-1$. Then there exist $a_1, a_2, \cdots, a_{m_1} \in \mathbb{R}$ such that for each i with $1 \leq i \leq n$, $F_i = \operatorname{diag}[a_1, a_2, \cdots, a_{m_j}]$.*

Proof. We first prove part (1). Let $F = [F_{ij}]_{i,j=1}^{n}$ be a hermitian matrix which commutes with E. By comparing the entries of FE and EF, we see that $F_{ij} = 0$ for $i > j$. Since F is hermitian, F must be of the form $\sum_{i=1}^{n} \oplus F_i$ as asserted.

We next prove part (2). Since $FE = EF$, we have

$$F_j T_j = T_j F_{j+1} \tag{2.3}$$

for all $j = 1, 2, \cdots, n-1$. Note that F_1 and F_2 are hermitian. So setting $j = 1$ in (2.3) gives $F_1 = F_2 = \text{diag}\,[a_1, a_2, \cdots, a_{m_1}]$ for some $a_1, a_2, \cdots, a_{m_1} \in \mathbb{R}$. By (2.3), $F_j = \text{diag}\,[a_1, a_2, \cdots, a_{m_j}]$ for each $j = 3, 4, \cdots, n$. This completes the proof. ∎

Lemma 2.2. *Let E be defined as in* (2.2). *If $T_1 = 0$ and every T_i is one-to-one for $i = 2, 3, \cdots, n-1$, then any hermitian matrix commuting with E is of the form*

$$\begin{bmatrix} F_1 & F_0 \\ F_0^* & F_2 \end{bmatrix} \oplus \sum_{i=3}^{n} \oplus F_i.$$

Proof. Let $F = [F_{ij}]_{i,j=1}^n$ be a hermitian matrix which commutes with E. By comparing the $(i-1, 1)$, $(i-1, 2)$ entries of FE and EF, we get $F_{i1} = 0$ and $F_{i2} = 0$ for all $3 \le i \le n$. Because F is hermitian, we may assume that

$$F = \begin{bmatrix} F_1 & F_0 \\ F_0^* & F_2 \end{bmatrix} \oplus [F_{ij}]_{i,j=3}^n.$$

Thus $[F_{ij}]_{i,j=3}^n$ commutes with

$$\begin{bmatrix} \gamma I_{m_3} & T_3 & & \\ & \gamma I_{m_4} & \ddots & \\ & & \ddots & T_{n-1} \\ & & & \gamma I_{m_n} \end{bmatrix}.$$

By Lemma 2.1 (1), we may assume that $[F_{ij}]_{i,j=3}^n = \sum_{i=3}^{n} \oplus F_i$. This completes the proof. ∎

Before we prove Proposition 2.4, we need the following decomposition structure.

Remark 2.3. Let $J = \sum_{i=1}^{m} \oplus J_{n_i}(\gamma)$ with $n_1 \ge n_2 \ge \cdots \ge n_m \ge 2$. Let $\mathcal{M}_j = \ker\,(J - \gamma I)^j \ominus \ker\,(J - \gamma I)^{j-1}$, and $m_j = \dim \mathcal{M}_j$ for $j = 1, 2, \cdots, n_1$. Thus J is unitarily equivalent to

$$\begin{bmatrix} \gamma I_{m_1} & T_1 & & \\ & \gamma I_{m_2} & \ddots & \\ & & \ddots & T_{n_1-1} \\ & & & \gamma I_{m_{n_1}} \end{bmatrix} \text{ on } \sum_{j=1}^{n_1} \oplus \mathcal{M}_j,$$

where $T_j = \begin{bmatrix} I_{m_{j+1}} \\ 0_{(m_j - m_{j+1})\times m_{j+1}} \end{bmatrix} \in M_{m_j \times m_{j+1}}$. We note that $m_1 = m_2 = m$ and so $T_1 = I_m$. Let $X_0 = \text{diag}\ [1, 2, \cdots, m] \in M_m$, and $X = X_0 \oplus I$ on $\mathcal{M}_1 \oplus \left(\sum_{j=2}^{n_1} \oplus \mathcal{M}_j \right)$. Then X is invertible and so J is similar to

(2.4)

$$A = XJX^{-1} = \left[\begin{array}{c|ccccc} \gamma I_{m_1} & X_0 & 0 & \cdots & & 0 \\ \hline & \gamma I_{m_2} & T_2 & & & \\ & & \gamma I_{m_3} & \ddots & & \\ & & & & \ddots & T_{n_1-1} \\ & & & & & \gamma I_{m_{n_1}} \end{array}\right] \text{ on } \sum_{j=1}^{n_1} \oplus \mathcal{M}_j.$$

Throughout this section, A is defined as in (2.4).

Proposition 2.4. *Let $T = \sum_{i=1}^{m} \oplus J_{n_i}(\gamma) \in M_n$ with $n_1 \geq n_2 \geq \cdots \geq n_m \geq 2$. If T has property $(*)$, then T is similar to an irreducible matrix.*

Proof. By Remark 2.3, T is similar to A of (2.4). Let $X_1 \in M_m$ be the matrix whose entries are all equal to 1, and let

$$Y = \left[\begin{array}{c|ccccc} I_{m_1} & X_1 & 0 & \cdots & 0 \\ \hline & I_{m_2} & & & \\ & & I_{m_3} & & \\ & & & \ddots & \\ & & & & I_{m_{n_1}} \end{array}\right].$$

Then Y is invertible. Since T is not quadratic, we have $n_1 \geq 3$. Moreover, T is similar to

$$C = YAY^{-1} = \left[\begin{array}{c|cccccc} \gamma I_{m_1} & X_0 & X_1 T_2 & 0 & \cdots & & 0 \\ \hline & \gamma I_{m_2} & T_2 & & & & \\ & & \gamma I_{m_3} & T_3 & & & \\ & & & \gamma I_{m_4} & \ddots & & \\ & & & & & \ddots & T_{n_1-1} \\ & & & & & & \gamma I_{m_{n_1}} \end{array}\right]. \qquad (2.5)$$

It suffices to show that C is irreducible. Let $P = [P_{ij}]_{i,j=1}^{n_1}$ be a projection commuting with C. By Lemma 2.1, $P = \sum_{j=1}^{n_1} \oplus P_j$, and there exist $a_1, a_2, \cdots,$

$a_m \in \mathbb{R}$ such that $P_j = \text{diag}\,[a_1, a_2, \cdots, a_{m_j}]$ for each j. Since $PC = CP$, we have

$$P_1 X_1 T_2 = X_1 T_2 P_3.$$

A simple computation shows that $a_1 = a_2 = \cdots = a_m$. Therefore, $P = 0_n$ or I_n, and hence C is irreducible. This proves our assertion. ∎

By Proposition 2.4, we have proved Proposition 1.2 for the case $T = J$ in (2.1). The other case of (2.1), namely, $T = \gamma I_k \oplus J$, will be handled in Proposition 2.6. Before that, we shall consider T to be similar to another matrix of a special form as in the following remark.

Remark 2.5. Let $T = \gamma I_k \oplus \sum_{i=1}^{m} \oplus J_{n_i}(\gamma) \in M_n$ with $n_1 \geq n_2 \geq \cdots \geq n_m \geq 2$. Let $J = \sum_{i=1}^{m} \oplus J_{n_i}(\gamma)$. Then $T = \gamma I_k \oplus J$ on $\mathbb{C}^k \oplus \mathbb{C}^{n-k}$. By Remark 2.3, J is similar to A of (2.4). Let $X_1 \in M_{k \times m_1}$ be the matrix whose entries are all equal to 1, and let

$$X = \left[\begin{array}{c|ccccc} I_k & X_1 & 0 & \cdots & 0 \\ \hline & I_{m_1} & & & \\ & & I_{m_2} & & \\ & & & \ddots & \\ & & & & I_{m_{n_1}} \end{array}\right].$$

Then X is invertible and so T is similar to

$$B = X(\gamma I_k \oplus A)X^{-1} = \left[\begin{array}{c|cccccc} \gamma I_k & 0 & X_1 X_0 & & & & \\ \hline & \gamma I_{m_1} & X_0 & & & & \\ & & \gamma I_{m_2} & T_2 & & & \\ & & & \gamma I_{m_3} & \ddots & & \\ & & & & \ddots & T_{n_1-1} & \\ & & & & & \gamma I_{m_{n_1}} & \end{array}\right], \tag{2.6}$$

where $X_0 = \text{diag}\,[1, 2, \cdots, m] \in M_m$. Throughout this section, B is defined as in (2.6).

Proposition 2.6. *Let* $T = \gamma I_k \oplus \sum_{i=1}^{m} \oplus J_{n_i}(\gamma) \in M_n$ *with* $n_1 \geq n_2 \geq \cdots \geq n_m \geq 2$. *If* T *has property* $(*)$, *then* T *is similar to an irreducible matrix.*

Proof: By Remark 2.5, T is similar to B of (2.6). We will construct matrices $X_2, X_3, \cdots, X_{n_1}$ in two different cases. We will use them to obtain an invertible matrix Y, followed by an irreducible matrix C similar to T.

Case (1): Suppose that $k < m_3$. Let

$$X_2 = \left[\begin{array}{cccc|c} 1 & & & & \\ & 2 & & & \\ & & \ddots & & 0_{k\times(m_2-k)} \\ & & & k & \end{array}\right] \in M_{k\times m_2},$$

and $X_j = 0_{k\times m_j}$ for each $j = 3, 4, \cdots, n_1$.

Case (2): Suppose that $k \geq m_3$. Since $\sum_{j=2}^{n_1} m_j = \operatorname{rank} T \geq n/2 = (k + \sum_{j=1}^{n_1} m_j)/2$, it follows that $k \leq \sum_{j=3}^{n_1} m_j$. So there exists $3 \leq \ell \leq m$ such that $\sum_{j=3}^{\ell-1} m_j < k \leq \sum_{j=3}^{\ell} m_j$. For each $j = 2, 3, \cdots, \ell-2$, we let $r_j = \sum_{i=3}^{j} m_i$ and $s_j = k - r_j - m_{j+1}$, and let

$$X_j = \left[\begin{array}{ccccccc} \multicolumn{7}{c}{0_{r_j\times m_j}} \\ \hline j+1 & & & & 0 & \cdots & 0 \\ & j+2 & & & 0 & \cdots & 0 \\ & & \ddots & & \vdots & \cdots & \vdots \\ & & & j+m_{j+1} & 0 & \cdots & 0 \\ \hline \multicolumn{7}{c}{0_{s_j\times m_j}} \end{array}\right] \in M_{k\times m_j}.$$

In addition, let

$$X_{\ell-1} = \left[\begin{array}{ccccccc} \multicolumn{7}{c}{0_{(k-m_\ell)\times m_{\ell-1}}} \\ \hline \ell+1 & & & & 0 & \cdots & 0 \\ & \ell+2 & & & 0 & \cdots & 0 \\ & & \ddots & & \vdots & \cdots & \vdots \\ & & & \ell+m_\ell & 0 & \cdots & 0 \end{array}\right] \in M_{k\times m_{\ell-1}},$$

and $X_j = 0_{k\times m_j}$ for each $j = \ell, \ell+1, \cdots, n_1$.

So far, we have constructed the appropriate X_j in both Cases (1) and (2). Define

$$Y = \left[\begin{array}{c|cccc} I_k & 0 & X_2 & \cdots & X_{n_1} \\ \hline & I_{m_1} & & & \\ & & \ddots & & \\ & & & & I_{m_{n_1}} \end{array}\right].$$

Then Y is invertible. Since T is not quadratic, we have $n_1 \geq 3$. Because T is similar to B, T is also similar to

$$C = YBY^{-1} = \left[\begin{array}{c|ccccc} \gamma I_k & 0 & X_1X_0 & X_2T_2 & \cdots & X_{n_1-1}T_{n_1-1} \\ \hline & \gamma I_{m_1} & X_0 & & & \\ & & \gamma I_{m_2} & T_2 & & \\ & & & \gamma I_{m_3} & \ddots & \\ & & & & \ddots & T_{n_1-1} \\ & & & & & \gamma I_{m_{n_1}} \end{array}\right]. \tag{2.7}$$

It suffices to show that C is irreducible. Let $P = [P_{ij}]_{i,j=0}^{n_1}$ be a projection commuting with C. By Lemma 2.2, we may assume that

$$P = \begin{bmatrix} P_0 & Q \\ Q^* & P_1 \end{bmatrix} \oplus \sum_{j=2}^{n_1} \oplus P_j.$$

Since

$$Q^* \, [X_2T_2 \;\, X_3T_3 \;\, \cdots \;\, X_{n_1-1}T_{n_1-1}] = [0 \;\, 0 \;\, \cdots \;\, 0]$$

and $[X_2T_2 \;\, X_3T_3 \;\, \cdots \;\, X_{n_1-1}T_{n_1-1}]$ is surjective, we have $Q^* = 0_{m\times k}$. Hence $P = \sum_{j=0}^{n_1} \oplus P_j$ and so $\sum_{j=1}^{n_1} \oplus P_j$ commutes with A of (2.4). By Lemma 2.1(2), there exist $a_1, a_2, \cdots, a_m \in \mathbb{R}$ such that for each j, $1 \leq j \leq n_1$, we have $P_j = \mathrm{diag}\,[a_1, a_2, \cdots, a_{m_j}]$. Since $PC = CP$, we have

$$P_0X_iT_i = X_iT_iP_{i+1} \tag{2.8}$$

for all $i = 2, 3, \cdots, n_1 - 1$, and

$$P_0X_1X_0 = X_1X_0P_2. \tag{2.9}$$

By (2.8), P_0 is also diagonal, with diagonal terms in $\{a_1, a_2, \cdots, a_m\}$. Finally, it follows from (2.9) that $a_1 = a_2 = \cdots = a_m$, and so $P = 0_n$ or I_n. Hence C is irreducible and so we complete the proof. ∎

It is obvious that Proposition 1.2 follows from Propositions 2.4 and 2.6.

Note that the preceeding discussions lead to an affirmative answer to a problem posed by P. R. Halmos. An $n \times n$ matrix T is said to be *nilpotent of index* k if $T^k = 0$ but $T^{k-1} \neq 0$. Halmos constructed 4×4 and 5×5 irreducible nilpotent matrices of index 3 [5, Problem 164]. The following corollary answers his problem for the general case.

Corollary 2.7. *For any integers k and n satisfying $3 \leq k < n$, there exists an irreducible nilpotent $n \times n$ matrix of index k.*

Proof. Let $n = km + \ell$, where m and ℓ are integers with $0 \le \ell < k$, and let $T = \underbrace{J_k(0) \oplus \cdots \oplus J_k(0)}_{m} \oplus J_\ell(0)$. It is easy to see that T has property $(*)$. By Propositions 2.4 and 2.6, T is similar to the irreducible matrix C of (2.5) or (2.7), depending on whether $\ell \neq 1$ or $\ell = 1$. Since T is nilpotent of index k, the same is true for C. Therefore, the matrix C is the required irreducible nilpotent matrix. ■

3. Case of Multiple Eigenvalues

The purpose of this section is to prove Proposition 1.3. Therefore, throughout this section, we always assume that T is an $n \times n$ matrix with at least two distinct eigenvalues, and has property $(*)$. We want to prove that T is similar to an irreducible matrix. Before that, we need the following definitions and lemmas. For $T \in M_n$, let $\{T\}' = \{S' \in M_n \mid S'T = TS'\}$ be the *commutant* of T, and $\{T\}'' = \{S \in M_n \mid SS' = S'S$ for every $S' \in \{T\}'\}$ be the *double commutant* of T. In [3], Fong and Jiang proved the following.

Lemma 3.1 [3, Lemma 2.1]. *If there exists a matrix $S \in \{T\}''$ which is similar to an irreducible matrix, then so is T.*

Lemma 3.2 [2, Lemma 1.2]. *Let $T = \sum_{i=1}^{h} \oplus J(\gamma_i)$ on $\sum_{i=1}^{h} \oplus \mathbb{C}^{k_i}$ with all γ_i distinct, and let $\ell = \sum_{i=2}^{h} k_i$. The for all $\alpha_1, \alpha_2, \cdots, \alpha_h \in \mathbb{C}$, both $J(\gamma_1) \oplus \alpha_1 I_\ell$ and $\sum_{i=1}^{h} \oplus \alpha_i I_{k_i}$ are in $\{T\}''$.*

As in Section 2, we may consider the Jordan form of T directly. Let $\sigma(T) = \{\gamma_1, \gamma_2, \cdots, \gamma_h\}$ with all γ_i distinct. For convenience, we may assume that

$$T = \sum_{i=1}^{h} \oplus J(\gamma_i) \quad \text{on} \quad \sum_{i=1}^{h} \oplus \mathbb{C}^{k_i}, \quad \text{where } k_1 \ge k_2 \ge \cdots \ge k_h. \tag{3.1}$$

The following two propositions are crucial to the proof of Proposition 1.3.

Proposition 3.3. *Let $T = \sum_{i=1}^{h} \oplus J(\gamma_i)$ on $\sum_{i=1}^{h} \oplus \mathbb{C}^{k_i}$, where all γ_i are distinct and $k_1 \ge k_2 \ge \cdots \ge k_h$. Let $\ell = \sum_{i=2}^{h} k_i > k_1$ and $S = \sum_{i=1}^{h} \oplus i I_{k_i}$. If S has property $(*)$, then S is similar to an irreducible matrix.*

Proposition 3.4. *Let* $T = \sum_{i=1}^{h} \oplus J(\gamma_i)$ *on* $\sum_{i=1}^{h} \oplus \mathbb{C}^{k_i}$, *where all* γ_i *are distinct and* $k_1 \geq k_2 \geq \cdots \geq k_h$. *Let* $\ell = \sum_{i=2}^{h} k_i \leq k_1$ *and* $S = J(\gamma_1) \oplus \gamma_2 I_\ell$ *on* $\mathbb{C}^{k_1} \oplus \mathbb{C}^{\ell}$. *If* S *has property* $(*)$, *then* S *is similar to an irreducible matrix.*

We shall prove Propositions 3.3 and 3.4 later. For the moment, we assume that they are valid, and show that they lead to Proposition 1.3.

Proof of Proposition 1.3. Without loss of generality, we may assume that T is defined as in (3.1). Let $\ell = \sum_{i=2}^{h} k_i$. If $\ell > k_1$, then we let $S = \sum_{i=1}^{h} \oplus i I_{k_i}$. Otherwise, if $\ell \leq k_1$, then we let $S = J(\gamma_1) \oplus \gamma_2 I_\ell$ on $\mathbb{C}^{k_1} \oplus \mathbb{C}^{\ell}$. By Lemma 3.2, S is always in $\{T\}''$. It is easy to see that since T has property $(*)$, so does S. By Propositions 3.3 and 3.4, S is similar to an irreducible matrix. Finally, by Lemma 3.1, T is also similar to an irreducible matrix. ∎

To prove Propositions 3.3 and 3.4, we need the following lemma, which follows from a direct computation.

Lemma 3.5. (1) *Let* $A \in M_n$ *and* $B \in M_m$. *If* $\sigma(A)$ *and* $\sigma(B)$ *are disjoint, then* $A \oplus B$ *is similar to* $C = \begin{bmatrix} A & X \\ 0_{m\times n} & B \end{bmatrix}$ *for any matrix* $X \in M_{n\times m}$. *In addition, any projection* P *commuting with* C *is of the form* $P_1 \oplus P_2$ *on* $\mathbb{C}^n \oplus \mathbb{C}^m$.

(2) *Moreover, if* A *is irreducible and* $n \geq m$, *then we may choose* X *to be one-to-one in which case* C *is irreducible.*

Proof of Proposition 3.3. By using Lemma 3.5(1) inductively, we may construct an upper-triangular matrix C which is similar to S such that

$$C = \begin{bmatrix} I_{k_1} & X_2 & X_3 & \cdots & X_h \\ & 2I_{k_2} & Y_3 & \cdots & Y_h \\ & & 3I_{k_3} & & \\ & & & \ddots & \\ & & & & hI_{k_h} \end{bmatrix} \tag{3.2}$$

for some $X_i \in M_{k_1\times k_i}$ and $Y_i \in M_{k_2\times k_i}$. Let us describe X_i and Y_i more clearly.

Since $\sum_{i=2}^{h} k_i > k_1$, there exists g, $2 < g \leq h$, such that $\sum_{i=2}^{g-1} k_i < k_1 \leq \sum_{i=2}^{g} k_i$.

For $2 \le i \le g-1$, we let $r_i = \sum_{j=2}^{i-1} k_j$ and $s_i = k_1 - \sum_{j=2}^{i} k_j$, and let

$$X_i = \left[\begin{array}{cccc} \multicolumn{4}{c}{0_{r_i \times k_i}} \\ \hline 1 & & & \\ & 2 & & \\ & & \ddots & \\ & & & k_i \\ \hline \multicolumn{4}{c}{0_{s_i \times k_i}} \end{array}\right] \in M_{k_1 \times k_i}.$$

Also, let

$$X_g = \left[\begin{array}{cccc} \multicolumn{4}{c}{0_{(k_1-k_g) \times k_g}} \\ \hline 1 & & & \\ & 2 & & \\ & & \ddots & \\ & & & k_g \end{array}\right] \in M_{k_1 \times k_g}.$$

For each i, $g+1 \le i \le h$, let

$$X_i = \left[\begin{array}{cccc} 1 & & & \\ & 2 & & \\ & & \ddots & \\ & & & k_i \\ \hline \multicolumn{4}{c}{0_{(k_1-k_i) \times n_i}} \end{array}\right] \in M_{k_1 \times k_i}.$$

Also, let $Y_i \in M_{k_2 \times k_i}$ be the matrix whose entries are all equal to 1 for each $3 \le i \le h$. Since S is not quadratic, we have $h \ge 3$. It suffices to show that C is irreducible. Let $P = [P_{ij}]_{i,j=1}^{h}$ be a projection commuting with C. By Lemma 3.5(1) again, $P = \sum_{i=1}^{h} \oplus P_i$. By the equality of the $(1,i)$ entries of PC and CP, where $2 \le i \le h$, we see that P_i is diagonal for each $1 \le i \le h$. Similarly, by the equialty of the $(2,i)$ entries of PC and CP, where $3 \le i \le h$, we see that all the diagonal entries are equal and so $P = 0_n$ or I_n. Therefore, C is irreducible. ∎

For Proposition 3.4, we consider the matrix

$$S = J(\gamma_1) \oplus \gamma_2 I_\ell \quad \text{on } \mathbb{C}^{k_1} \oplus \mathbb{C}^{\ell}, \text{ where } \gamma_1 \ne \gamma_2 \text{ and } k_1 \ge \ell. \tag{3.3}$$

Let

$$J = \sum_{j=1}^{m} \oplus J_{n_j}(\gamma_1) \quad \text{with } n_1 \ge n_2 \ge \cdots \ge n_m \ge 2. \tag{3.4}$$

Then $J(\gamma_1)$ is given by the following cases:

$$\text{Case (A):}\quad J(\gamma_1) = J \in M_{k_1}, \tag{3.5}$$

$$\text{Case (B):}\quad J(\gamma_1) = \gamma_1 I_k \oplus J \in M_{k_1}, \text{ with } k \geq \ell, \tag{3.6}$$

$$\text{Case (C):}\quad J(\gamma_1) = \gamma_1 I_k \oplus J \in M_{k_1}, \text{ with } k < \ell. \tag{3.7}$$

It suffices to prove Proposition 3.4 for these three cases. They will be handled by Lemmas 3.6, 3.7 and 3.8 respectively.

We now consider Case (A). Recall that S and J are defined as in (3.3) and (3.4) respectively. Since $\ell \leq k_1$, it is easy to see that $k_1 = \sum_{j=1}^{m} n_j \geq \ell$.

Lemma 3.6. *Let $S = \sum_{j=1}^{m} \oplus J_{n_j}(\gamma_1) \oplus \gamma_2 I_\ell \in M_n$, where $\gamma_1 \neq \gamma_2$, $n_1 \geq n_2 \geq \cdots \geq n_m \geq 2$, and $\sum_{j=1}^{m} n_j \geq \ell$. Then S is similar to an irreducible matrix.*

Proof. Clearly S has property $(*)$ already. Let $k_1 = \sum_{j=1}^{m} n_j$ and $J = \sum_{j=1}^{m} \oplus J_{n_j}(\gamma_1)$. Then $J \in M_{k_1}$. If $n_1 \geq 3$, then J is similar to an irreducible matrix by Proposition 2.4. In addition, we know that $k_1 \geq \ell$, and that the spectra of J and $\gamma_2 I_\ell$ are disjoint. By Lemma 3.5(2), S is similar to an irreducible matrix. So the remaining condition is that $n_1 = n_2 = \cdots n_m = 2$. Next we construct X_1, X_2 and X_3 in different cases. They will be used to obtain an irreducible matrix C which is similar to S.

Case (1): Suppose that $\ell \leq m$. By Remark 2.3, J is similar to the matrix A of (2.4). Namely,

$$A = \begin{bmatrix} \gamma_1 I_m & X_0 \\ 0_m & \gamma_1 I_m \end{bmatrix},$$

where $X_0 = \text{diag}\,[1, 2, \cdots, m] \in M_m$. Let

$$X_1 = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & \ell \\ \hline & 0_{(m-\ell)\times\ell} & \end{bmatrix} \in M_{m\times\ell},$$

and $X_2 \in M_{m\times\ell}$ be the matrix with all entries equal to 1. Since the spectra of A and $\gamma_2 I_\ell$ are disjoint, by Lemma 3.5(1), S is similar to

$$C = \begin{bmatrix} \gamma_1 I_m & X_0 & X_1 \\ & \gamma_1 I_m & X_2 \\ & & \gamma_2 I_\ell \end{bmatrix}.$$

By a direct computation, we see that C is irreducible.

Case (2): Suppose that $m < \ell$. We notice that $\ell \leq 2m$. Let

$$X_0 = \begin{bmatrix} 1 & 1 & & \\ & 1 & \ddots & \\ & & \ddots & 1 \\ & & & 1 \end{bmatrix} \in M_m,$$

and let $Y = X_0 \oplus I_m \in M_{2m}$. Then J is similar to

$$J' = YJY^{-1} = \begin{bmatrix} \gamma_1 I_m & X_0 \\ 0_m & \gamma_1 I_m \end{bmatrix}.$$

In addition, let

$$X_1 = \left[\begin{array}{ccc|c} 1 & & & \\ & \ddots & & 0_{m\times(\ell-m)} \\ & & m & \end{array}\right] \in M_{m\times\ell},$$

and

$$X_2 = \left[\begin{array}{c|ccc} & 1 & & \\ 0_{m\times(\ell-m)} & & \ddots & \\ & & & m \end{array}\right] \in M_{m\times\ell}.$$

Since the spectra of J' and $\gamma_2 I_\ell$ are disjoint, by Lemma 3.5(1), S is similar to

$$C = \begin{bmatrix} \gamma_1 I_m & X_0 & X_1 \\ & \gamma_1 I_m & X_2 \\ & & \gamma_2 I_\ell \end{bmatrix}.$$

By a direct computation, we see that C is irreducible. This completes the proof. ∎

We now prove Proposition 3.4 for Case (B) given by (3.6). Recall that J is defined as in (3.4),

$$J(\gamma_1) = \gamma_1 I_k \oplus J \in M_{k_1},$$

and S is defined as in (3.3). Since $\ell \leq k_1$, we have $k_1 = \sum_{j=1}^{m} n_j + k \geq \ell$.

Lemma 3.7. *Let* $S = \gamma_1 I_k \oplus J \oplus \gamma_2 I_\ell \in M_n$, *where* $\gamma_1 \neq \gamma_2$, $J = \sum_{j=1}^{m} \oplus J_{n_j}(\gamma_1)$ *with* $n_1 \geq n_2 \geq \cdots \geq n_m \geq 2$, *and* $\sum_{j=1}^{m} n_j + k \geq \ell$. *If* $k \geq \ell$, *and* S *has property* $(*)$, *then* S *is similar to an irreducible matrix.*

Proof. By Remark 2.5, $\gamma_1 I_k \oplus J$ is similar to

$$B = \left[\begin{array}{c|cccccc} \gamma_1 I_k & 0 & X_1X_0 & & & & \\ \hline & \gamma_1 I_{m_1} & X_0 & & & & \\ & & \gamma_1 I_{m_2} & T_2 & & & \\ & & & \gamma_1 I_{m_3} & \ddots & & \\ & & & & \ddots & T_{n_1-1} & \\ & & & & & \gamma_1 I_{m_{n_1}} \end{array}\right], \tag{3.8}$$

where $X_0 = \text{diag}\,[1, 2, \cdots, m] \in M_m$ and $X_1 \in M_{k\times m}$ whose entries are all equal to 1. Next we construct matrices $X_2, X_3, \cdots, X_{n_1}$ in different cases. They will be used to obtain an invertible matrix Y, which leads to an irreducible matrix C similar to S.

Case (1): Suppose that $k < \ell + m_3$. Let

$$X_2 = \begin{cases} \left[\begin{array}{ccccccc} & & 0_{(k-m_3)\times m_2} & & & & \\ \hline 1 & & & & 0 & \cdots & 0 \\ & 2 & & & 0 & \cdots & 0 \\ & & \ddots & & \vdots & \cdots & \vdots \\ & & & m_3 & 0 & \cdots & 0 \end{array}\right] \in M_{k\times m_2} & \text{if } k \ge m_3, \\[2ex] \left[\begin{array}{cccc|c} 1 & & & & \\ & 2 & & & \\ & & \ddots & & 0_{k\times(m_2-k)} \\ & & & k & \end{array}\right] \in M_{k\times m_2} & \text{if } k < m_3, \end{cases}$$

and $X_j = 0_{k\times m_j}$ for $j = 3, 4, \cdots, n_1$.

Case (2): Suppose that $k \ge \ell + m_3$. Since $\ell + \sum_{j=2}^{n_1} m_j = \text{rank } S \ge n/2 = (k+\ell+\sum_{j=1}^{n_1} m_j)/2$, it follows that $k \le \sum_{j=3}^{n_1} m_j + \ell$. Thus there exists $3 \le r \le n_1$, such that $\sum_{j=3}^{r-1} m_j + \ell < k \le \sum_{j=3}^{r} m_j + \ell$. For each $2 \le j \le r-2$, we let

$r_j = \sum_{i=3}^{j} m_i + \ell$, $s_j = k - r_j - m_{j+1}$, and let

$$X_j = \left[\begin{array}{cccccc} \multicolumn{6}{c}{0_{r_j \times m_j}} \\ \hline j+1 & & & 0 & \cdots & 0 \\ & j+2 & & 0 & \cdots & 0 \\ & & \ddots & \vdots & \cdots & \vdots \\ & & j+m_{j+1} & 0 & \cdots & 0 \\ \hline \multicolumn{6}{c}{0_{s_j \times m_j}} \end{array}\right] \in M_{k \times m_j}.$$

Also, let

$$X_{r-1} = \left[\begin{array}{cccccc} \multicolumn{6}{c}{0_{(k-m_r) \times m_{r-1}}} \\ \hline r+1 & & & 0 & \cdots & 0 \\ & r+2 & & 0 & \cdots & 0 \\ & & \ddots & \vdots & \cdots & \vdots \\ & & r+m_r & 0 & \cdots & 0 \end{array}\right] \in M_{k \times m_{r-1}}.$$

For each $j = r, r+1, \cdots, n_1$, let $X_j = 0_{k \times m_j}$.

So far, we have constructed the appropriate X_j in both Cases (1) and (2). Define

$$Y = \left[\begin{array}{c|cccc} I_k & 0 & X_2 & \cdots & X_{n_1} \\ \hline & I_{m_1} & & & \\ & & I_{m_2} & & \\ & & & \ddots & \\ & & & & I_{m_{n_1}} \end{array}\right].$$

Then Y is invertible. Since S is not quadratic, we have $n_1 \geq 2$. Since $\gamma_1 I_k \oplus J$ is similar to B of (3.8), $\gamma_1 I_k \oplus J$ is similar to

$$D = YBY^{-1} = \left[\begin{array}{c|ccccc} \gamma_1 I_k & 0 & X_1 X_0 & X_2 T_2 & \cdots & X_{n_1-1} T_{n_1-1} \\ \hline & \gamma_1 I_{m_1} & X_0 & & & \\ & & \gamma_1 I_{m_2} & T_2 & & \\ & & & \gamma_1 I_{m_3} & \ddots & \\ & & & & \ddots & T_{n_1-1} \\ & & & & & \gamma_1 I_{m_{n_1}} \end{array}\right].$$

Let

$$X_{n_1+1} = \left[\begin{array}{cccc} n_1+1 & & & \\ & n_1+2 & & \\ & & \ddots & \\ & & & n_1+\ell \\ \hline \multicolumn{4}{c}{0_{(k-\ell)\times\ell}} \end{array}\right] \in M_{k\times\ell}.$$

Since the spectra of $\gamma_2 I_\ell$ and D are disjoint, by Lemma 3.5 (1), S is similar to

$$C = \left[\begin{array}{c|cccccc|c} \gamma_1 I_k & 0 & X_1X_0 & X_2T_2 & \cdots & X_{n_1-1}T_{n_1-1} & & X_{n_1+1} \\ \hline & \gamma_1 I_{m_1} & X_0 & & & & & \\ & & \gamma_1 I_{m_2} & T_2 & & & & \\ & & & \gamma_1 I_{m_3} & \ddots & & & \\ & & & & \ddots & T_{n_1-1} & & \\ & & & & & \gamma_1 I_{m_{n_1}} & & \\ \hline & & & & & & & \gamma_2 I_\ell \end{array}\right].$$

It suffices to show that C is irreducible. Let $P = [P_{ij}]_{i,j=0}^{n_1+1}$ be a projection commuting with C. By Lemma 3.5 (1), $P = [P_{ij}]_{i,j=0}^{n_1} \oplus P_{n_1+1}$. It follows that $[P_{ij}]_{i,j=o}^{n_1}$ is a projection commuting with D. By Lemma 2.2, we may further assume that

$$P = \begin{bmatrix} P_0 & Q \\ Q^* & P_1 \end{bmatrix} \oplus \sum_{j=2}^{n_1} \oplus P_j.$$

Since

$$Q^*[X_2T_2 \ X_3T_3 \ \cdots \ X_{n_1-1}T_{n_1-1} \ X_{n_1+1}] = [0 \ 0 \ \cdots \ 0 \ 0]$$

and $[X_2T_2 \ X_3T_3 \ \cdots \ X_{n_1-1}T_{n_1-1} \ X_{n_1+1}]$ is surjective, $Q^* = 0_{m\times k}$, and so $P = \sum_{j=0}^{n_1+1} \oplus P_j$. Since $\sum_{j=1}^{n_1} \oplus P_j$ commutes with

$$\left[\begin{array}{c|cccc} \gamma_1 I_{m_1} & X_0 & 0 & \cdots & 0 \\ \hline & \gamma_1 I_{m_2} & T_2 & & \\ & & \gamma_1 I_{m_3} & \ddots & \\ & & & \ddots & T_{n_1-1} \\ & & & & \gamma_1 I_{m_{n_1}} \end{array}\right],$$

by Lemma 2.1 (2), there exist $a_1, a_2, \cdots, a_m \in \mathbb{R}$ such that each $P_j = \text{diag}\,[a_1, a_2, \cdots, a_{m_j}]$. Since $PC = CP$, we have

$$P_0 X_i T_i = X_i T_i P_{i+1} \qquad (3.9)$$

for all $i = 2, 3, \cdots, n_1 - 1$,

$$P_0 X_{n_1+1} = X_{n_1+1} P_{n_1+1} \tag{3.10}$$

and

$$P_0 X_1 X_0 = X_1 X_0 P_2. \tag{3.11}$$

By (3.9) and (3.10), there exist $b_1, b_2, \cdots, b_\ell \in \mathbb{R}$ such that $P_{n_1+1} = \text{diag}\,[b_1, b_2, \cdots, b_\ell]$ and $P_0 = P_{n_1+1} \oplus R$ for some diagonal matrix $R \in M_{k-\ell}$ with entries in $\{a_1, a_2, \cdots, a_m\}$. Finally, it follows from (3.11) that $a_i = b_j$ for all i and j, and so $P = 0_n$ or I_n. Hence C is irreducible and so we complete the proof. ∎

By Lemma 3.7, we have proved Proposition 3.4 for Case (B) given by (3.6). This leaves only the final Case (C) given by (3.7). As in Case (B) (or Lemma 3.7), we have $k_1 = \sum_{j=1}^{m} n_j + k \geq \ell$.

Lemma 3.8. *Let $S = \gamma_1 I_k \oplus J \oplus \gamma_2 I_\ell \in M_n$, where $\gamma_1 \neq \gamma_2$, $J = \sum_{j=1}^{m} \oplus J_{n_j}(\gamma_1)$ with $n_1 \geq n_2 \geq \cdots \geq n_m \geq 2$, and $\sum_{j=1}^{m} n_j + k \geq \ell$. If $k < \ell$ and S has property $(*)$, then S is similar to an irreducible matrix.*

Proof. By Remark 2.5, $\gamma_1 I \oplus J$ is similar to

$$B = \left[\begin{array}{c|cccccc} \gamma_1 I_k & 0 & X_1 X_0 & & & & \\ \hline & \gamma_1 I_{m_1} & X_0 & & & & \\ & & \gamma_1 I_{m_2} & T_2 & & & \\ & & & \gamma_1 I_{m_3} & \ddots & & \\ & & & & \ddots & T_{n_1-1} & \\ & & & & & \gamma_1 I_{m_{n_1}} & \end{array}\right], \tag{3.12}$$

where $X_0 = \text{diag}\,[1, 2, \cdots, m] \in M_m$ and $X_1 \in M_{k \times m}$ whose entries are all equal to 1. By Lemma 3.5(1), we may construct an upper-triangular matrix C which is similar to S, such that

$$C = \left[\begin{array}{c|cccccc|c} \gamma_1 I_k & 0 & X_1 X_0 & 0 & \cdots & 0 & & Y_0 \\ \hline & \gamma_1 I_{m_1} & X_0 & & & & & Y_1 \\ & & \gamma_1 I_{m_2} & T_2 & & & & Y_2 \\ & & & \gamma_1 I_{m_3} & \ddots & & & Y_3 \\ & & & & \ddots & T_{n_1-1} & & \vdots \\ & & & & & \gamma_1 I_{m_{n_1}} & & Y_{n_1} \\ \hline & & & & & & & \gamma_2 I_\ell \end{array}\right] \tag{3.13}$$

for some $Y_i \in M_{m_i\times\ell}$ for $1 \le i \le n_1$ and $Y_0 \in M_{k\times\ell}$. We now construct Y_i as follows. Let

$$Y_0 = \left[\begin{array}{cccc|c} 1 & & & & \\ & 2 & & & \\ & & \ddots & & 0_{k\times(\ell-k)} \\ & & & k & \end{array}\right] \in M_{k\times\ell}.$$

For $Y_1, Y_2, \cdots, Y_{n_1}$, consider the following two cases.

Case (1): Suppose that $m > \ell$. Let $Y_1 = 0_{m\times\ell}$ and

$$Y_2 = \left[\begin{array}{cccc} 1 & & & \\ & 2 & & \\ & & \ddots & \\ & & & \ell \\ \hline & 0_{(m-\ell)\times\ell} & & \end{array}\right] \in M_{m\times\ell}.$$

Also, let $Y_j = 0_{m_j\times\ell}$ for each $j = 3, 4, \cdots, n_1$.

Case (2): Suppose that $m \le \ell$. For each $1 \le j \le n_1$, we will construct Y_j depending on whether $\ell \le k + m$ or not. We first set $u = m + k + 2$.

Case (2a): Suppose that $\ell \le k + m$. Let

$$Y_1 = \left[\begin{array}{c|cccc} & u+1 & & & \\ & & u+2 & & \\ 0_{(\ell-k)\times k} & & & \ddots & \\ & & & & u+(\ell-k) \\ \hline 0_{(m-\ell-k)\times k} & & 0_{(m-\ell-k)\times(\ell-k)} & & \end{array}\right] \in M_{m\times\ell},$$

and $Y_j = 0_{m_j\times\ell}$ for $j = 2, 3, \cdots, n_1$.

Case (2b): Suppose that $\ell > k + m$. Let

$$Y_1 = \left[\begin{array}{c|cccc|c} & u+1 & & & & \\ & & u+2 & & & \\ 0_{m\times k} & & & \ddots & & 0_{m\times(\ell-m-k)} \\ & & & & u+m & \end{array}\right] \in M_{m\times\ell}.$$

Since $k + \sum_{i=1}^{m} n_i \ge \ell$, there exists $1 < r \le n_1$ such that $k + \sum_{j=1}^{r-1} m_j < \ell \le k + \sum_{j=1}^{r} m_j$. For each $j = 2, 3, \cdots, r-1$, we let $r_j = k + \sum_{i=1}^{j-1} m_i$ and $s_j = \ell - m_j - r_j$, and let

$$Y_j = \left[\begin{array}{c|cccc|c} & j+1 & & & & \\ & & j+2 & & & \\ 0_{m_j\times r_j} & & & \ddots & & 0_{m_j\times s_j} \\ & & & & j+m_j & \end{array}\right] \in M_{m_j\times \ell}.$$

Also, let

$$Y_r = \left[\begin{array}{c|cccc} & r+1 & & & \\ & & r+2 & & \\ 0_{m_r\times(\ell-m_r)} & & & \ddots & \\ & & & & r+m_r \end{array}\right] \in M_{m_r\times \ell}.$$

For $j = r+1, r+2, \cdots, n_1$, let $Y_j = 0_{m_j\times\ell}$.

We have constructed the appropriate Y_j in both Cases (1) and (2). Since S is not quadratic, we have $n_1 \geq 2$. It suffices to show that C is irreducible. Let $P = [P_{ij}]_{i,j=0}^{n_1+1}$ be a projection commuting with C. By Lemma 3.5(1), we may assume that that $P = [P_{ij}]_{i,j=0}^{n_1} \oplus P_{n_1+1}$, and so $[P_{ij}]_{i,j=0}^{n_1}$ commutes with B. By Lemma 2.2, we may assume that

$$P = \begin{bmatrix} P_0 & Q \\ Q^* & P_1 \end{bmatrix} \oplus \sum_{j=2}^{n_1+1} \oplus P_j.$$

By the $(0, n_1+1)$ and $(1, n_1+1)$ entries of $PC = CP$, we have

$$P_0Y_0 + QY_1 = Y_0P_{n_1+1}, \tag{3.14}$$

and

$$Q^*Y_0 + P_1Y_1 = Y_1P_{n_1+1}. \tag{3.15}$$

By (3.14) and (3.15), $Q = 0_{k\times m}$, $P_0 = \operatorname{diag}[b_1, b_2, \cdots, b_k]$ for some $b_1, b_2, \cdots, b_k \in \mathbb{R}$, and $P_{n_1+1} = P_0 \oplus R$ for some $R \in M_{\ell-k}$. By computing $PC = CP$ entry by entry, we see that $P = 0_n$ or I_n. Therefore, C is irreducible. This proves our assertion. ∎

The above lemmas complete the proof of Proposition 3.4 and hence that of the Main Theorem.

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