TAIWANESE JOURNAL OF MATHEMATICS
Vol. 4, No. 3, pp. 447-456, September 2000

HARDY-TYPE INEQUALITIES

R. Radha

Abstract. Hardy-type inequalities are proved for n-dimensional Hermite and special Hermite expansions. Paley-type theorems for these expansions are also deduced.

1. Introduction

It was observed by Hardy and Littlewood as well as many others that there are many results in Fourier analysis that hold for $L^p(\mathbb{T})$, $1 < p < \infty$, fail to be true for $L^1(\mathbb{T})$ and yet remain true for $\mathrm{Re}H^1$, where $\mathrm{Re}H^1$ is the real Hardy space consisting of the boundary values of the real parts of the functions in the Hardy space H^1 on the unit disk in the plane. As an example a well-known result of Paley shows that

$$\sum_{-\infty}^{\infty\prime} |c_k|^p |k|^{p-2} \ < \ \infty,$$

where $\sum_{-\infty}^{\infty} c_k e^{ik\theta}$ denotes the Fourier series and $\sum'$ is the sum which runs over nonzero k's. This result is false when $p = 1$. However, Hardy has shown that if $f \in \mathrm{Re}H^1$, we have

$$\sum_{-\infty}^{\infty\prime} \frac{|c_k|}{|k|} < \infty.$$

Kanjin in [2] has proved Hardy's inequalities for the one-dimensional Hermite and Laguerre expansions. Our aim of this paper is to obtain similar type of inequalities for n-dimensional Hermite and special Hermite expansions.

[0] Received January 30, 1999; revised May 8, 1999.
Communicated by S.-Y. Shaw.
2000 *Mathematics Subject Classification*: Primary 42C10, 33C50; Secondary 42A16.
Key words and phrases: Hardy's inequality, Hardy space, Hermite expansion, Paley theorem.

2. Notations and Prelimineries

The Hermite functions $\tilde{h}_k$ on the real line are defined by

$$\tilde{h}_k(x) = H_k(x)e^{-\frac{1}{2}x^2}, k = 0, 1, 2, \dots ,$$

where $H_k(x)$ denotes the Hermite polynomial. These are eigenfunctions of the Hermite operator (harmonic oscillator) $-\Delta + x^2$ with the eigenvalues $2k+1$. The normalised Hermite functions $h_k(x)$ are defined by

$$h_k(x) = (2^k k! \sqrt{\pi})^{-\frac{1}{2}} \tilde{h}_k(x),$$

which form a complete orthonormal family in $L^2(\mathbb{R}, dx)$.

Let μ be a multiindex and $x \in \mathbb{R}^n$. Then the n-dimensional Hermite functions $\Phi_\mu(x)$ are defined by taking the product of the one-dimensional normalised Hermite functions $h_{\mu_j}(x_j)$:

$$\Phi_\mu(x) = \prod_{j=1}^{n} h_{\mu_j}(x_j).$$

Then they form a complete orthonormal system for $L^2(\mathbb{R}^n, dx)$ and they are eigen functions of the Hermite operator $H = -\Delta + |x|^2$ on $\mathbb{R}^n$ with eigenvalues $(2|\mu| + n)$, where $|\mu| = \mu_1 + \mu_2 + ... + \mu_n$.

The special Hermite functions, which occupy a central place in the study of Hermite and Laguerre expansions, are defined by

$$\Phi_{\mu\nu}(x + iy) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\xi} \Phi_\mu \left(\xi + \frac{1}{2}y\right) \Phi_\nu \left(\xi - \frac{1}{2}y\right) d\xi.$$

These functions appear as the entry functions of the Schrödinger representation of the Heisenberg group. They form a complete orthonormal system in $L^2(\mathbb{C}^n)$. Let

$$L = -\Delta_z + \frac{1}{4}|z|^2 - iN,$$

where

$$N = \sum_{j=1}^{n} \left(x_j \frac{\partial}{\partial y_j} - y_j \frac{\partial}{\partial x_j} \right).$$

Then $\Phi_{\mu\nu}$ are eigenfunctions of L, with eigenvalue $2|\nu| + n$ and L is called the special Hermite operator. For various results concerning these expansions, we refer to [4].

The study of Hardy spaces over $\mathbb{R}^n$ provides basic insights into such topics as maximal functions, singular integrals and L^p-spaces. For definitions and their importance in analysis, we refer to [1] and [3] except that we state here the atomic decomposition of H^1 space which will be used in the course of our discussions.

A function a is an H^1-atom (associated to a ball B) if (i) a is supported in B, (ii) $|a| \leq |B|^{-1}$ a.e., and (iii) $\int a dx = 0$. Then we have

Theorem 2.1. *$f \in H^1$ if and only if f can be written as a sum of H^1-atoms, $\{a_k\}$,*

$$f = \sum_k \lambda_k a_k,$$

where $\{\lambda_k\}$ is a sequence of complex numbers with $\sum |\lambda_k| < \infty$, and one has

$$c_1 \|f\|_{H^1} \leq \sum |\lambda_k| \leq c_2 \|f\|_{H^1}.$$

3. Results for Hermite Expansions

Proposition 3.1. *Let $\epsilon > 0$ be fixed. Choose $\delta > -(1+\epsilon)/2$. Let $\{\phi_\mu\}$ be an orthonormal basis in $L^2(\mathbb{R}^n)$ such that $|\nabla \phi_\mu| \leq c n^{\frac{1}{2}} \mu_1^\delta.....\mu_j^\delta$, where $\mu_1, ..\mu_j$ $(1 \leq j \leq n)$ are the nonzero indices of μ. Let $\sigma = ((n+1)(1+\epsilon)+n\delta)/(2+n)$ and $\hat{f}(\mu) = \int_{\mathbb{R}^n} f(x)\phi_\mu(x)dx$. Then for every $f \in H^1(\mathbb{R}^n)$ we have*

$$\sum_{\mu \in \overline{\mathbb{N}}^n} \frac{|\hat{f}(\mu)|}{[(\mu_1+1)(\mu_2+1)...(\mu_n+1)]^\sigma} \leq c(n,\epsilon)\|f\|_{H^1(\mathbb{R}^n)},$$

where $\overline{\mathbb{N}} = \mathbb{N} \bigcup \{0\}$, $c(n,\epsilon)$ is a constant depending on the dimension n and ϵ only.

Proof. When $\mu = 0$, each μ_j in $\mu = (\mu_1, ..., \mu_n)$ is zero. Thus $|\phi_\mu(x)| \leq c(n)$. By the atomic decomposition of H^1, it follows that

$$|\hat{f}(\mu)| \ \leq \ c(n)\|f\|_{H^1}.$$

Let a be an H^1-atom supported in a ball $B = B(x_0, r)$. Then

$$\hat{a}(\mu) = \int_B a(x)[\phi_\mu(x) - \phi_\mu(x_0)]dx.$$

By applying the mean value theorem and the Schwarz inequality, we get

$$|\hat{a}(\mu)| \leq c n^{\frac{1}{2}} \mu_1^\delta ... \mu_j^\delta \|a\|_2^{\frac{-2}{n}}. \tag{1}$$

To prove the result, we need only prove the following:

$$\begin{aligned}(2)\qquad & \sum_{\mu_1....\mu_j\le\nu}\frac{|\hat{a}(\mu)|}{[(\mu_1+1)(\mu_2+1)...(\mu_n+1)]^{\sigma}}\\ & +\sum_{\mu_1....\mu_j>\nu}\frac{|\hat{a}(\mu)|}{[(\mu_1+1)(\mu_2+1)...(\mu_n+1)]^{\sigma}}\\ & = S_1+S_2\le c(n,\epsilon).\end{aligned}$$

But

$$\begin{aligned} S_1 &\le \sum_{\mu_1.....\mu_j\le\nu}\frac{|\hat{a}(\mu)|}{\mu_1^{\sigma}....\mu_j^{\sigma}}\\ &\le cn^{\frac{1}{2}}\|a\|_2^{\frac{-2}{n}}\sum_{\mu_1.....\mu_j\le\nu}\mu_1^{\delta-\sigma}...\mu_j^{\delta-\sigma}\\ &= cn^{\frac{1}{2}}\|a\|_2^{\frac{-2}{n}}\sum_{m\le\nu}d_j(m)m^{(\delta-\sigma)}\\ &\le c(n,\epsilon)\|a\|_2^{\frac{-2}{n}}\nu^{\delta-\sigma+1+\epsilon},\end{aligned}$$

$$\begin{aligned} S_2 &= \sum_{\mu_1.....\mu_j>\nu}\frac{|\hat{a}(\mu)|}{(\mu_1....\mu_j)^{\sigma}}\\ &\le \|a\|_2\left\{\sum_{\mu_1.....\mu_j>\nu}\frac{1}{(\mu_1....\mu_j)^{2\sigma}}\right\}^{\frac{1}{2}}\\ &= \|a\|_2\left\{\sum_{m>\nu}\frac{d_j(m)}{m^{2\sigma}}\right\}^{\frac{1}{2}}\\ &\le c\|a\|_2\nu^{\frac{-2\sigma+1+\epsilon}{2}},\end{aligned}$$

where $d_j(m)$ denotes the number of representations of m as a product of j integers. $d_j(m)$ satisfies the following: There exists a constant c such that $d_j(m)\le cm^{\epsilon}$. We choose $\nu=\|a\|_2^q$ where $q=2(2+n)/n(1+\epsilon+2\delta)$ and we get (2). ∎

In the following theorem, we obtain a Hardy-type inequality for Hermite expansions.

Theorem 3.1. *If* $\{\Phi_\mu\}_{\mu\in\mathbb{N}^n}$ *is the collection of Hermite functions on* $\mathbb{R}^n$ *and if* $\hat{f}(\mu)=\int_{\mathbb{R}^n}f(x)\Phi_\mu(x)dx$*, then there exists a constant* $c(n,\epsilon)$ *such that*

$$\sum_{\mu\in\overline{\mathbb{N}}^n}\frac{|\hat{f}(\mu)|}{[(\mu_1+1)(\mu_2+1)....(\mu_n+1)]^{\frac{5n+12(n+1)(1+\epsilon)}{12(2+n)}}}\le c(n,\epsilon)\|f\|_{H^1(\mathbb{R}^n)}$$

for $f \in H^1(\mathbb{R}^n)$ and $\epsilon > 0$ is any fixed real number.

Proof. We know that $|h_k(x)| \leq ck^{\frac{-1}{12}}$ for $k = 1, 2, ...$ and $|h_0(x)| \leq c$. Let $A_k = \frac{-\partial}{\partial x_k} + x_k$, $A_k^* = \frac{\partial}{\partial x_k} + x_k$. Then, using the identities

$$A_k \Phi_\mu = (2\mu_k + 2)^{\frac{1}{2}} \Phi_{\mu+\epsilon_k},$$

$$A_k^* \Phi_\mu = (2\mu_k)^{\frac{1}{2}} \Phi_{\mu-\epsilon_k},$$

where $\epsilon_1, \epsilon_2, ..., \epsilon_n$ is the standard basis for $\mathbb{R}^n$, we get

$$\frac{\partial}{\partial x_k} \Phi_\mu = \left(\frac{\mu_k}{2}\right)^{\frac{1}{2}} \Phi_{\mu-\epsilon_k} - \left(\frac{\mu_k + 1}{2}\right)^{\frac{1}{2}} \Phi_{\mu+\epsilon_k},$$

from which we get

$$|\frac{\partial}{\partial x_k} \Phi_\mu| \leq c\mu_1^{\frac{-1}{12}} ......\mu_k^{\frac{5}{12}} ...\mu_j^{\frac{-1}{12}}$$

for $1 \leq k \leq j$ and

$$|\frac{\partial}{\partial x_l} \Phi_\mu| \leq c\mu_1^{\frac{-1}{12}} ...\mu_j^{\frac{-1}{12}}$$

for $j + 1 \leq l \leq n$, where $\mu_1, ..., \mu_j$ are the nonzero indices of μ. Then

$$|\nabla \Phi_\mu| \leq cn^{\frac{1}{2}} \mu_1^{\frac{5}{12}} .....\mu_j^{\frac{5}{12}}$$

and the result follows from Proposition 3.1. ∎

Now as in [2] we deduce a Paley-type theorem for $\{\Phi_\mu\}$, which will be a sharper inequality for $n = 2$.

Theorem 3.2.

1. *If $1 < p \leq 2$, then there exists a constant $c(n, \epsilon)$ such that*

$$\sum_\mu |\hat{f}(\mu)|^p [(\mu_1 + 1)....(\mu_n + 1)]^{(p-2)\sigma} \leq c(n, \epsilon) \|f\|^p_{L^p(\mathbb{R}^n)}$$

for $f \in L^p(\mathbb{R}^n)$, $\sigma = (5n + 12(n + 1)(1 + \epsilon))/12(2 + n)$, $\epsilon > 0$ a fixed real number.

2. *If $2 \leq q < \infty$, and if $\{b(\mu)\}_{\mu \in \mathbb{N}^n}$ satisfies*

$$\sum_\mu |b(\mu)|^q [(\mu_1 + 1)....(\mu_n + 1)]^{(q-2)\sigma} < \infty,$$

then

$$\|f\|^q_{L^q(\mathbb{R}^n)} \leq c(n,\epsilon)\sum_{\mu}|b(\mu)|^q[(\mu_1+1).....(\mu_n+1)]^{(q-2)\sigma},$$

where $f \sim \sum_\mu b(\mu)\Phi_\mu \in L^q(\mathbb{R}^n)$.

Proof. Define $l^p_k(\mathbb{N}^n)$, $k>0$, $1\leq p<\infty$, to be the collection $\{b(\mu)\}$ for which $[\sum_\mu \frac{|b(\mu)|^p}{[(\mu_1+1)...(\mu_n+1)]^{2k}}]^{\frac{1}{p}} = \|b(\mu)\|_{l^p_k} < \infty$. Define $T_k f = \hat{f}(\mu)[(\mu_1+1)...(\mu_n+1)]^k$ for f. Take $k=\sigma$. If $f\in H^1(\mathbb{R}^n)$, then by Theorem 3.1, we get $T_k f \in l^1_k$ as

$$\|T_k f\|_{l^1_k} \leq c(n,\epsilon)\|f\|_{H^1(\mathbb{R}^n)}.$$

As $\|T_k f\|_{l^2_k} = \|f\|_2$, we see that T_k is both weak type $(H^1(\mathbb{R}^n), l^1_k)$ and (L^2, l^2_k). Then by interpolation theorem, we get T_k is bounded from L^p to l^p_k and we obtain (1) for $1<p\leq 2$. By standard duality argument we get (2). ∎

4. Results for Special Hermite Expansions

Theorem 4.1. *Let* $\{\Phi_{\mu,\nu}\}$ *denote the collection of special Hermite functions. Define* $\hat{f}(\mu,\nu) = \int_{\mathbb{R}^{2n}} f(x,y)\Phi_{\mu,\nu}(x,y)dxdy$. *Then we have the following inequality for the special Hermite expansions*:

$$\sum_{\mu,\nu} \frac{|\hat{f}(\mu,\nu)|}{[(\mu_1+1)\dots(\mu_n+1)(\nu_1+1)\dots(\nu_n+1)]^{\frac{(2n+1)(1+\epsilon)+n}{2(1+n)}}} \leq C(n,\epsilon)\|f\|_{H^1(\mathbb{R}^{2n})},$$

where $\epsilon>0$ *is a fixed real number.*

Proof. $Z_j = \frac{\partial}{\partial z_j} + \frac{1}{2}\bar{z}_j$, $\overline{Z_j} = \frac{\partial}{\partial \bar{z}_j} - \frac{1}{2}z_j$, $j=1,2,\dots n$. Then using the identities

(3) $$Z_j(\Phi_{\mu\nu}) = i(2\nu_j)^{\frac{1}{2}}\Phi_{\mu,\nu-\epsilon_j},$$

(4) $$\bar{Z}_j(\Phi_{\mu\nu}) = i(2\nu_j+2)^{\frac{1}{2}}\Phi_{\mu,\nu+\epsilon_j},$$

we get

$$\frac{\partial}{\partial x_j}\Phi_{\mu,\nu} = iy_j\Phi_{\mu,\nu} + i(2\nu_j)^{\frac{1}{2}}\Phi_{\mu,\nu-\epsilon_j} + i(2\nu_j+2)^{\frac{1}{2}}\Phi_{\mu,\nu+\epsilon_j}.$$

$$\begin{aligned}|\Phi_{\mu,\nu}(z)| &= (2\pi)^{\frac{-n}{2}} \left| \int e^{ix\xi} \Phi_\mu \left(\xi + \frac{1}{2}y\right) \Phi_\nu \left(\xi - \frac{1}{2}y\right) d\xi \right| \\ &\leq C \int \left|\Phi_\mu \left(\xi + \frac{1}{2}y\right)\right| \left|\Phi_\nu \left(\xi - \frac{1}{2}y\right)\right| d\xi \\ &\leq C\|\Phi_\mu\|_2 \|\Phi_\nu\|_2 = C.\end{aligned}$$

$$\begin{aligned}|y_j\Phi_{\mu,\nu}(z)| &\leq C\Pi_{\substack{k=1\\k\neq j}}^n \left| \int e^{ix_k\xi_k} h_{\mu_k} \left(\xi_k + \frac{1}{2}y_k\right) h_{\nu_k} \left(\xi_k - \frac{1}{2}y_k\right) d\xi \right| \\ &\quad \left| y_j \int e^{ix_j\xi_j} h_{\mu_j} \left(\xi_j + \frac{1}{2}y_j\right) h_{\nu_j} \left(\xi_j - \frac{1}{2}y_j\right) d\xi_j \right| \\ &\leq C|y_j\Phi_{\mu_j,\nu_j}(z_j)| \quad \text{(by applying Schwarz inequality for } n-1 \text{ terms in the product).}\end{aligned}$$

As

$$\begin{aligned}iy_j\Phi_{\mu_j,\nu_j}(z_j) = i(2\pi)^{-\frac{1}{2}} \Bigg\{ \int e^{ix_j\xi_j} &\left(\left(\xi_j + \frac{1}{2}y_j\right) - \left(\xi_j - \frac{1}{2}y_j\right) \right) \\ &\times h_{\mu_j} \left(\xi_j + \frac{1}{2}y_j\right) h_{\nu_j} \left(\xi - \frac{1}{2}y_j\right) d\xi_j \Bigg\},\end{aligned}$$

we get

$$\begin{aligned}\left|y_j\Phi_{\mu_j,\nu_j}(z_j)\right| &\leq C \int \left| \left(\xi_j + \frac{1}{2}y_j\right) h_{\mu_j} \left(\xi_j + \frac{1}{2}y_j\right) \right| \left| h_{\nu_j} \left(\xi_j - \frac{1}{2}y_j\right) \right| d\xi_j \\ &+ C \int \left| \left(\xi_j - \frac{1}{2}y_j\right) h_{\nu_j} \left(\xi_j - \frac{1}{2}y_j\right) \right| \left| h_{\mu_j} \left(\xi_j + \frac{1}{2}y_j\right) \right| d\xi_j \\ &\leq C \left[\int |\xi_j h_{\mu_j}(\xi_j)|^2 d\xi_j \right]^{\frac{1}{2}} \\ &+ C \left[\int |\xi_j h_{\nu_j}(\xi_j)|^2 d\xi_j \right]^{\frac{1}{2}} \text{(by Schwarz inequality and making change of variables).}\end{aligned}$$

Using

$$\left(-\frac{d}{dx} + x\right) \tilde{h}_k(x) = \tilde{h}_{k+1}(x), \tag{5}$$

$$\left(\frac{d}{dx} + x\right) \tilde{h}_k(x) = 2k\tilde{h}_{k-1}(x), \tag{6}$$

it follows that

$$xh_k(x) = \left(\frac{k+1}{2}\right)^{\frac{1}{2}} h_{k+1}(x) + \left(\frac{k}{2}\right)^{\frac{1}{2}} h_{k-1}(x).$$

Squaring this and using the fact that $\{h_k\}$ is an orthonormal basis for $L^2(\mathbb{R})$, we obtain

$$\begin{aligned}\left[\int |\xi_j h_{\mu_j}(\xi_j)|^2 d\xi_j\right]^{\frac{1}{2}} &= \left(\frac{2\mu_j+1}{2}\right)^{\frac{1}{2}},\\ \left[\int |\xi_j h_{\nu_j}(\xi_j)|^2 d\xi_j\right]^{\frac{1}{2}} &= \left(\frac{2\nu_j+1}{2}\right)^{\frac{1}{2}}.\end{aligned}$$

Thus, we get

$$\sum_{j=1}^{n}\Big|\frac{\partial}{\partial x_j}\Phi_{\mu,\nu}\Big|^2 \le C(n)\mu_1\ldots\mu_j\nu_1\ldots\nu_k,$$

where $\mu_1,\ldots,\mu_j,\nu_1,\ldots,\nu_k$ are the nonzero indices of (μ,ν).
Again by (3) and (4), we have

$$\frac{\partial}{\partial y_j}\Phi_{\mu,\nu} = -ix_j\Phi_{\mu,\nu} - (2\nu_j)^{\frac{1}{2}}\Phi_{\mu,\nu-\epsilon_j} + (2\nu_j+2)^{\frac{1}{2}}\Phi_{\mu,\nu+\epsilon_j},$$

and $|x_j\Phi_{\mu,\nu}| \le C|x_j\Phi_{\mu_j,\nu_j}(z_j)|$. But

$$\begin{aligned}x_j\Phi_{\mu_j,\nu_j}(z_j) = & i(2\pi)^{\frac{-1}{2}}\left\{\int e^{ix_j\xi_j}h'_{\mu_j}\left(\xi_j+\frac{1}{2}y_j\right)h_{\nu_j}\left(\xi_j-\frac{1}{2}y_j\right)d\xi_j\right.\\ &\left.+\int e^{ix_j\xi_j}h_{\mu_j}\left(\xi+\frac{1}{2}y_j\right)h'_{\nu_j}\left(\xi_j-\frac{1}{2}y_j\right)d\xi_j\right\}.\end{aligned}$$

From (5) and (6), we get

$$h'_k(x) = \left(\frac{k}{2}\right)^{\frac{1}{2}} h_{k-1}(x) - \left(\frac{k+1}{2}\right)^{\frac{1}{2}} h_{k+1}(x). \tag{7}$$

Furthermore,

$$\begin{aligned}|x_j\Phi_{\mu_j,\nu_j}(z_j)| &\le C\left\{\int\left|h'_{\mu_j}\left(\xi_j+\frac{1}{2}y_j\right)\right|^2 d\xi_j\right\}^{\frac{1}{2}}\\ &+ C\left\{\int\left|h'_{\nu_j}\left(\xi_j-\frac{1}{2}y_j\right)\right|^2 d\xi_j\right\}^{\frac{1}{2}}.\end{aligned} \tag{8}$$

Squaring (7), then making change of variables in (8), we get

$$|x_j \Phi_{\mu_j,\nu_j}(z_j)| \le C\left(\frac{2\mu_j+1}{2}\right) + C\left(\frac{2\nu_j+1}{2}\right).$$

Thus, we get

$$\sum_{j=1}^{n} \left|\frac{\partial}{\partial y_j}\Phi_{\mu\nu}\right|^2 \le Cn\mu_1\ldots\mu_j\nu_1\ldots\nu_k,$$

which shows that

$$|\nabla\Phi_{\mu,\nu}| \le C(2n)^{\frac{1}{2}}\mu_1^{\frac{1}{2}}\ldots\mu_j^{\frac{1}{2}}\nu_1^{\frac{1}{2}}\ldots\nu_k^{\frac{1}{2}}.$$

Hence if we take $\delta = 1/2$, by Proposition 3.1, we obtain the required result. ■

Now, if we define $\ell_k^p(\mathbb{N}^{2n}), k > 0, 1 \le p < \infty$, by

$$\left\{\{b(\mu,\nu)\}\left|\left\{\sum_{\mu,\nu}\frac{|b(\mu,\nu)|^p}{[(\mu_1+1)\ldots(\mu_{n+1})(\nu_1+1)\ldots(\nu_{n+1})]^{2k}}\right\}^{\frac{1}{p}} = \|b(\mu)\|_{\ell_k{}^p} < \infty\right.\right\}$$

and

$$T_k f \;=\; \hat{f}(\mu,\nu)\,[(\mu_1+1)\ldots(\mu_{n+1})(\nu_1+1)\ldots(\nu_n+1)]^k,$$

using the Parseval's formula for special Hermite expansions, we deduce a Paley-type theorem for special Hermite expansions.

Theorem 4.2. *For the special Hermite expansions, we have the following:*

1. *If* $1 < p \le 2$, *then there exists a constant* $C(n,\epsilon)$ *such that*

$$\sum_{\mu,\nu}|\hat{f}(\mu,\nu)|^p[(\mu_1+1)\ldots(\mu_n+1)(\nu_1+1)\ldots(\nu_n+1)]^{(p-2)\sigma} \le C(n,\epsilon)\|f\|^p_{L^p(\mathbb{R}^{2n})},$$

where $\sigma = ((2n+1)(1+\epsilon)+n)/2(1+n)$, $\epsilon > 0$ *a fixed real number.*

2. *If* $2 \le q < \infty$, *and if* $\{b(\mu,\nu)|(\mu,\nu)\in\mathbb{N}^{2n}\}$ *satisfies*

$$\sum_{\mu,\nu}|b(\mu,\nu)|^q[(\mu_1+1)\ldots(\mu_n+1)(\nu_1+1)\ldots(\nu_n+1)]^{(q-2)\sigma} < \infty,$$

then

$$\|F\|^q_{L^q(\mathbb{R}^{2n})} \le C(n,\epsilon)\sum_{\mu,\nu}|b(\mu,\nu)|^q[(\mu_1+1)\ldots(\mu_n+1)(\nu_1+1)\ldots(\nu_n+1)]^{(q-2)\sigma} \quad for\ F \sim \sum_{\mu,\nu} b(\mu,\nu)\Phi_{\mu,\nu}.$$

Acknowledgements

The author wishes to thank Prof.S.Thangavelu for initiating her into this problem. She extends her thanks to Prof. R. Balasubramanian for his valuable suggestions in improving the inequality of Proposition 3.1 to the present stage. Her thanks are due to Council of Scientific and Industrial Research, India, for its grant when the author was in Indian Institute of Technology, Madras. She also thanks the referee for his useful suggestions in revising the manuscript.

References

1. R. R. Coifman and G. Weiss, Extensions of Hardy spaces and their use in analysis, *Bull. Amer. Math. Soc.* **83** (1977), 569-645.
2. Y. Kanjin, Hardy's inequalities for Hermite and Laguerre expansions, *Bull. London Math. Soc.* **29** (1997), 331-337.
3. E. M. Stein, *Harmonic Analysis*: *Real Variable Methods*, *Orthogonality and Oscillatory Integrals*, Princeton Univ. Press, 1993.
4. S. Thangavelu, *Lectures on Hermite and Laguerre expansions*, Mathematical Notes 42, Princeton Univ. Press, 1993.

Department of Mathematics, Anna University, Madras - 600 025, India
E-mail: radharam@annauniv.edu, radharam@imsc.ernet.in