TAIWANESE JOURNAL OF MATHEMATICS
Vol. 4, No. 3, pp. 321-363, September 2000

FU JEN LECTURES IN HARDY SPACES

Der-Chen Chang

Dedicated to Professor Fon-Che Liu on his 60th Birthday

Abstract. In this paper, we survey some multiplier theorems and their applications to estimates for wave equation in Hardy spaces $H^p(\mathbb{R}^n)$. We also prove H^p boundedness for Calderón-Zygmund operators of type σ which can be considered as a generalization of classical singular integral operators. Using these results, we discuss some recent progress in H^p regularity properties of the solving operators for hyperbolic equations.

1. Introduction

Based on a series of lectures presented by the author during the 1999 annual meeting of the Mathematical Society of the Republic of China, held at Fu Jen Catholic University from December 11 to December 16, 1999, this article attempts to present some aspects of the recent progress in Hardy spaces and the applications of Hardy spaces to partial differential equations. The exposition is therefore divided into two concentrations: (1) Hardy space and (2) estimates of Calderón-Zygmund operators in H^p spaces.

It is well-known that the Lebesgue L^p spaces play an important role in Fourier analysis. However, many important classes of operators are not well-behaved on L^1 and L^∞ spaces. Many of these operators are unbounded on L^1 and so L^1 is too large to be the domain of such operators. By the same token, the target space of many important operators exceeds L^∞. Hence, L^∞ is too small to be the range of such operators resulting in a duality between these

[0] Received March 3, 2000.
Communicated by P. Y. Wu.
2000 *Mathematics Subject Classification*: Primary 35J25, 42B25; Secondary 46E15, 42B30.
Key words and phrases: Hardy space, atom, molecule, multiplier, Calderón-Zygmund operator, hyperbolic equation.
The author is supported in part by the NSF grant DMS9622249, a William Fulbright Scholarship, and a competitive research grant from Georgetown University.

two deficiencies. We are then motivated to find substitutes for the spaces of L^1 and L^∞ to alleviate this dual deficiency.

The Hardy space H^1 (derived from complex function theoretic considerations in the early part of the last century) and the space BMO of functions of bounded mean oscillation (discovered in the 1960s in the context of partial differential equations) turn out to be more appropriate spaces to study instead of L^1, L^∞ respectively. In fact, many of the operators that we wish to study, and which are ill-behaved on L^1 and L^∞, are bounded both on H^1 and on BMO. These two new spaces lead to deep insights concerning complex analysis, singular integrals, Cauchy integrals on Lipschitz curves, weighted norm inequalities, and partial differential equations.

The classical theory of Hardy spaces derives, as has already been noted, from complex function theory. Let $0 < p < \infty$. The original definition for $H^p(\mathbb{D})$, in the context of the unit disc $\mathbb{D}$, is as follows:

$$\left\{ f \text{ holomorphic on } \mathbb{D} : \sup_{0<r<1} \left(\int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right)^{\frac{1}{p}} < \infty \right\}.$$

On the upper half plane $\mathbb{R}^2_+ \subset \mathbb{C}$, the definition becomes

$$\left\{ f \ \text{ holomorphic on } \ \mathbb{R}^2_+ : \sup_{y>0} \left(\int_{-\infty}^{+\infty} |f(x+iy)|^p dx \right)^{\frac{1}{p}} < \infty \right\}.$$

(Note that these two definitions give rise to isomorphic spaces of functions, but not canonically so. The matter is treated in detail in Hoffman [22].)

Later, the theory of Hardy spaces was generalized to the upper half-space $\mathbb{R}^{n+1}_+$ for $(n-1)/n < p \leq 1$ by Stein and Weiss; see [37]. The methods developed in that paper rely heavily on the harmonicity of functions and on the Euclidean structure of $\mathbb{R}^n$. It was ten years before methods were developed to free the Hardy space theory from specifics of Euclidean analysis.

In 1971, Burkholder, Gundy and Silverstein [4] discovered an important fact: in order to show that a harmonic function u, defined on the unit disc, is the real part of some holomorphic function in $H^p(\mathbb{D})$, it is both necessary and sufficient to see that the auxiliary function

$$u^*(e^{i\theta}) \equiv \sup_{0<r<1} |u(re^{i\theta})|$$

lies in $L^p(\partial\mathbb{D})$. Notice that the Cauchy-Riemann equations no longer play any role in this new characterization of H^p. Not even the notion of the harmonic conjugate function need be invoked. However, lurking in the background is the fact that $u(re^{i\theta})$ is the Poisson integral of its (putative) boundary function

$\tilde{u}(e^{i\theta})$. Thus the Poisson kernel plays a tacit role in the characterization given by Burkholder, Gundy and Silverstein.

Later, Fefferman and Stein [18] removed the maximal function characterization of the Hardy spaces from any dependence on the Poisson kernel. In order to describe this development, we pass to the upper half plane $\mathbb{R}^2_+ \subset \mathbb{C}$. Let $\varphi \in C_0^\infty$ be a testing function, nonnegative, and with total mass 1. For $y > 0$, we set $\varphi_y(x) = y^{-1}\varphi(x/y)$. Now define, for $u(x+iy) = f * \varphi_y(x)$ on $\mathbb{R}^2_+$,

$$u^*(x) = \sup_{y>0} |u(x+iy)|.$$

The theorem of Fefferman and Stein is that f is the boundary value of the real part of an H^p function in $\mathbb{R}^2_+$ if and only if u^* lies in $L^p(\mathbb{R})$. As a result of this new characterization, and accompanying theorems in their paper, the theory of Hardy spaces is now a "real variable" theory.

Today, the real variable theory of Hardy spaces is well-developed. There are now maximal function, area integral (in the sense of Lusin), "atomic decomposition" and "molecular decomposition" characterizations of H^p. We refer the reader to the books of Stein [34], Garcia-Cuerva and Rubio de Francia [21], and, of course, Fefferman and Stein [18] for detailed treatment of these ideas.

The second part of this article is concentrated on estimates of Calderón-Zygmund operators in H^p spaces. The theory of Calderón-Zygmund on singular integral operators has become extensively studied by many mathematicians in the past 40 years. Since then, this theory became one of the most powerful tools attacking problems in analysis, e.g., elliptic boundary value problems, Cauchy integrals on Lipschitz curves, $\bar{\partial}$-Neumann problem, Fourier integral operators etc. Readers can refer to books and papers by, e.g., Chang-Nagel-Stein [8], Chang-Krantz-Stein [9], Chang-Dafni-Stein [10], Chang-Li [11], Christ [13], Coifman-Meyer [16], Journé [25], Sadosky [29], Stein [33, 34, 35] etc. and references therein.

Let $f \in \mathcal{S}(\mathbb{R}^n)$ be a Schwartz function. It is well-known that the Fourier transform

$$\mathcal{F}(f)(\xi) = \hat{f}(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)dx$$

is also a Schwartz function (see Chapter 1 in Stein and Weiss [36]). Therefore, we may define the inverse Fourier transform

$$\mathcal{F}^{-1}(f)(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \hat{f}(\xi)d\xi$$

of $\hat{f}$. Moreover, for a partial differential operator with constant coefficients,

$$P(D) = \sum_{|\mathbf{k}|=0}^{m} a_{\mathbf{k}} \frac{\partial^{|\mathbf{k}|}}{\partial x_1^{k_1} \cdots \partial x_n^{k_n}} = \sum_{|\mathbf{k}|=0}^{m} a_{\mathbf{k}} D^{\mathbf{k}},$$

we have

$$P(D)f(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} P(i\xi)\hat{f}(\xi)d\xi = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \psi(\xi)\hat{f}(\xi)d\xi.$$

In fact, we may treat $P(D)f(x)$ as a multiplier operator, i.e.,

$$T_\psi(f)(x) = \mathcal{F}^{-1}(\psi\hat{f})(x).$$

Let us first recall the famous Marcinkiewicz multiplier theorem (see Chapter 4 in Stein [33] and Chapter 6 in Stein [34]):

Theorem 1.1. *Let $\psi \in L^\infty(\mathbb{R}^n)$ and $\psi \in C^{n+2}(\mathbb{R}^n \setminus \{0\})$. Suppose that*

$$\left| D_\xi^{\mathbf{k}} \psi(\xi) \right| \le C_{\mathbf{k}} (1 + |\xi|)^{-\lambda - |\mathbf{k}|}$$

for some $\lambda > 0$ and all $|\mathbf{k}| \le n + 2$. Then

$$\|T_\psi(f)\|_{L^p_{s+\lambda}} \le C_p \|f\|_{L^p_s}$$

for all $f \in L^2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$, $1 < p < \infty$. Here $L^p_s(\mathbb{R}^n)$ is the L^p-Sobolev space of order s.

Basically, the above theorem rounds out the picture of L^p-estimates for constant coefficient elliptic differential operators (see Hörmander [23] and Stein [33]). It is very interesting to generalize the above result to the case $0 < p \le 1$. In order to do that, we have to consider Hardy spaces as our domain and target. As we have mentioned in the beginning of this section, the Hardy space $H^p(\mathbb{R}^n)$ is defined as the set of all distribution f whose maximal function

$$f^*(x) = \sup_{0<\varepsilon<\infty} |f * \varphi_\varepsilon(x)| \in L^p(\mathbb{R}^n).$$

Here $\varphi \in \mathcal{S}(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \varphi(x)dx \ne 0$ and $\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(x/\varepsilon)$ (see Fefferman and Stein [18]). One of the purposes of this article is to discuss H^p estimates for solving operators of hyperbolic equations, i.e., H^p estimates for the operator

$$T(f)(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{i\Phi(x,\xi)} \psi(\xi)\hat{f}(\xi)d\xi.$$

Roughly speaking, the above operator is very close to a multiplier operator, i.e., $\Phi(x,\xi) = x\cdot\xi$. Before we go further, let us review some results of multiplier theory on Hardy spaces. A function ψ is called a multiplier on H^p space, i.e., $\psi \in \mathcal{M}(H^p)$, if ψ is a measurable function and for each $f \in H^p(\mathbb{R}^n)$, we have $\mathcal{F}^{-1}(\psi\hat{f}) \in H^p(\mathbb{R}^n)$. Moreover,

$$f \mapsto T_\psi(f) = \mathcal{F}^{-1}(\psi\hat{f})$$

is a bounded linear operator on $H^p(\mathbb{R}^n)$. In this case, the operator norm will be called the norm of the multiplier ψ, i.e.,

$$\|\psi\|_{\mathcal{M}(H^p)} = \|T_\psi\|_{op} = \sup_{f\in H^p, f\neq 0} \frac{\|T_\psi(f)\|_{H^p}}{\|f\|_{H^p}}.$$

Now the question is: does this operator make sense? Of course, we can treat f as a tempered distribution, i.e., $f \in \mathcal{S}'(\mathbb{R}^n)$. It follows that $\hat{f} \in \mathcal{S}'(\mathbb{R}^n)$. However, can we define Fourier transform on the product of a measurable function with a tempered distribution? Therefore, we have to study Fourier transform on H^p spaces more carefully. Now let us consider H^p space by using atomic decomposition.

Definition 1.2. *Let $0 < p \le 1 \le q \le \infty$ with $p \neq q$. A (p,q,s)-atom $a(x)$ centered at the origin is a function in $L^q(\mathbb{R}^n)$ which is supported on a ball $B(0;r)$ for some $r > 0$, and satisfies*

- (*size condition*):
$$\|a\|_{L^q} \le |B|^{\frac{1}{q}-\frac{1}{p}};$$
- (*moment condition*):
$$\int_B a(x)x^{\mathbf{k}}dx = 0$$

for all monomials $x^{\mathbf{k}}$ with $|\mathbf{k}| \le s$ with $s \ge n_p = \left[n\left(\frac{1}{p}-1\right)\right]$.

A (p,q,s)-atom centered at $x_0 \in \mathbb{R}^n$ is defined to be a $L^q(\mathbb{R}^n)$ function a on $\mathbb{R}^n$ such that $\eta_{x_0}(a)(x) = a(x-x_0)$ is a (p,q,s)-atom centered at the origin. Now we may give another definition of Hardy spaces as follows (see Coifman [14] and Garnett-Latter [20]):

$$\begin{aligned} &H^p_{(q)}(\mathbb{R}^n) \\ &= \left\{ f \in L^2(\mathbb{R}^n) : f = \sum_{k=1}^{\infty} \lambda_k a_k, \text{where } a_k \text{ are } (p,q,s)\text{-atoms}, \sum_{k=1}^{\infty} |\lambda_k|^p < \infty \right\}, \end{aligned}$$

equipped with a "norm" as follows:

$$\|f\|^p_{H^p_{(q)}} = \inf \left\{ \sum_{k=1}^{\infty} |\lambda_k|^p \right\},$$

where the infimum is taken over all possible (p, q, s)-atomic decompositions of f. Let $r \leq q$, $r \geq 1$, $q > p$; then (p, q, s)-atoms satisfy

$$\left(\frac{1}{|B|} \int_B |a(y)|^r dy \right)^{\frac{1}{r}} \leq \left(\frac{1}{|B|} \int_B |a(y)|^q dy \right)^{\frac{1}{q}} \leq |B|^{\frac{1}{q} - \frac{1}{p}} \leq |B|^{\frac{1}{r} - \frac{1}{p}}.$$

This implies that for $1 \leq r < q$, every (p, q, s)-atom is also a (p, r, s)-atom. Moreover, there exist two universal constants C_1 and C_2 such that

$$C_1 \|f\|_{H^p_{(q)}} \leq \|f\|_{H^p_{(\infty)}} \leq C_2 \|f\|_{H^p_{(q)}} \quad \text{for all} \quad f \in H^p_{(q)}(\mathbb{R}^n).$$

Hence, we may use $H^p(\mathbb{R}^n)$ to represent $H^p_{(q)}(\mathbb{R}^n)$ for all $1 \leq q \leq \infty$ with $p > q$ (see Chang [7]). Now, with the help of the moment condition, we have the following result:

Lemma 1.3. *Let $0 < p \leq 1$ and let a be a $(p, 2, s)$-atom supported on a ball B centered at the origin. Then*

1.
$$|D^{\mathbf{k}} \hat{a}(\xi)| \leq \frac{C |\xi|^{n_p + 1 - |\mathbf{k}|}}{\|a\|_{L^2}^{d\left(\frac{n_p+1}{n} + \frac{1}{2}\right) - 1}};$$

2.
$$\|(D^{\mathbf{k}} \hat{a})^2\|_{L^{q'}} \leq \frac{C}{\|a\|_{L^2}^{d\left(\frac{2|\mathbf{k}|}{n} + \frac{1}{q}\right) - 2}}$$

for $0 \leq |\mathbf{k}| \leq s$. Here $(1/q) + (1/q') = 1$, $1 \leq q' \leq \infty$, $d = 1/[(1/p) - (1/2)]$ and C is a constant independent of a.

Proof. (1) By the size condition of a, we know that

$$|B| \leq \|a\|_{L^2}^{-d}.$$

Let $P(x)$ be the Taylor polynomial expansion of $e^{-ix\cdot\xi}$ at $\xi = 0$ of degree $n_p - |\mathbf{k}|$. Then

$$\begin{aligned} D^{\mathbf{k}} \hat{a}(\xi) &= (2\pi)^{-\frac{n}{2}} \int_B a(x) (-ix)^{\mathbf{k}} e^{-ix\cdot\xi} dx \\ &= (2\pi)^{-\frac{n}{2}} \int_B a(x) (-ix)^{\mathbf{k}} \left[e^{-ix\cdot\xi} - P(x) \right] dx. \end{aligned}$$

Therefore,

$$\begin{aligned}|D^{\mathbf{k}}\hat{a}(\xi)| &\leq C|\xi|^{n_p+1-|\mathbf{k}|}\int_B |a(x)|\cdot|x|^{n_p+1}dx\\ &\leq C|\xi|^{n_p+1-|\mathbf{k}|}|B|^{\frac{n_p+1}{n}+\frac{1}{2}}\|a\|_{L^2}\\ &\leq C|\xi|^{n_p+1-|\mathbf{k}|}\|a\|_{L^2}^{-d\left(\frac{n_p+1}{n}+\frac{1}{2}\right)+1}.\end{aligned}$$

(2) When $q' = 1$, we have $q = \infty$ and

$$\begin{aligned}\int_{\mathbb{R}^n}|D^{\mathbf{k}}\hat{a}(\xi)|^2 d\xi &= C\int_B |x^{\mathbf{k}}|^2|a(x)|^2dx\\ &\leq C|B|^{\frac{2|\mathbf{k}|}{n}}\|a\|_{L^2}^2 \leq C\|a\|_{L^2}^{-\frac{2d|\mathbf{k}|}{n}+2}.\end{aligned}$$

When $q' = \infty$, we have $q = 1$ and

$$\begin{aligned}|D^{\mathbf{k}}\hat{a}(\xi)|^2 &= C\left(\int_B |x^{\mathbf{k}}|\cdot|a(x)|dx\right)^2\\ &\leq C|B|^{\frac{2|\mathbf{k}|}{n}+1}\|a\|_{L^2}^2 \leq C\|a\|_{L^2}^{-d\left(\frac{2|\mathbf{k}|}{n}+1\right)+2}.\end{aligned}$$

Now we may use results for $q' = 1$ and $q' = \infty$ to obtain estimates for $1 < q' < \infty$. The proof of the lemma is complete. ∎

Remark 1. When B is centered at other point $x_0 \neq 0$, the above lemma is still true since the difference of the Fourier transform of a between $\mathrm{supp}(a) \subset B(0;r)$ and $\mathrm{supp}(a) \subset B(x_0;r)$ is only a complex number with absolute value 1.

Using the atomic decomposition of $H^p(\mathbb{R}^n)$, it is easy to obtain the following result (see also Coifman [15] and Stein [34, p. 128]):

Proposition 1.4. *Let $f \in H^p(\mathbb{R}^n)$, $0 < p \leq 1$. Then $\hat{f}$ is a continuous function. Moreover,*

$$|\hat{f}(\xi)| \leq C\cdot\|f\|_{H^p}\cdot|\xi|^{n\left(\frac{1}{p}-1\right)}.$$

2. Multiplier Theory on H^p Spaces

In order to study multiplier theory on $H^p(\mathbb{R}^n)$, we need to study atomic decomposition for H^p one step further. Roughly speaking, showing an operator T is bounded from H^p into L^p, it is enough to show that the L^p norm of $T(a)$ is uniformly bounded for all (p,q,s)-atoms a. However, we cannot use this method to show that T is bounded from H^p into itself since $T(a)$ is, in general, no longer a (p,q,s)-atom. This is because $T(a)$ does not necessarily satisfy the size condition.

Let us start with a simple example. Suppose $n = 1$ and $p = 1$. It is well-known that the Hilbert transform $\mathcal{H}$ is bounded from $H^1(\mathbb{R})$ into $H^1(\mathbb{R})$. It is natural for us to see how the operator $\mathcal{H}$ acts on a 1-atom.

Let a be a $(1,2,0)$-atom supported on an interval I which is centered at the origin. Then we have

$$\|\mathcal{H}(a)\|_{L^2(\mathbb{R})} \le C\|a\|_{L^2(\mathbb{R})} \le C\cdot|I|^{-\frac{1}{2}}. \tag{2.1}$$

We also know that when $|x| > 2|I|$,

$$\begin{aligned}|\mathcal{H}(a)(x)| &= \left|\int_{\mathbb{R}} \frac{a(y)}{x-y}dy\right| = \left|\int_{\mathbb{R}}\left[\frac{1}{x-y} - \frac{1}{x}\right]a(y)dy\right| \\ &\le \frac{C}{|x|^2}\int_I |ya(y)|dy \le C\frac{|I|}{|x|^2},\end{aligned}$$

and therefore,

$$\int_{|x|>2|I|} |x|^2|\mathcal{H}(a)(x)|^2dx \le C\int_{|x|>|I|}|x|^2\cdot\frac{|I|^2}{|x|^4}dx \le C|I|.$$

On the other hand,

$$\int_{|x|\le 2|I|} |x|^2|\mathcal{H}(a)(x)|^2dx \le C\cdot|I|^2\cdot|I|^{-1} \le C|I|.$$

It follows that

$$\int_{\mathbb{R}} |x|^2|\mathcal{H}(a)(x)|^2dx \le C|I|. \tag{2.2}$$

Combining (2.1) and (2.2), we conclude that

$$\left(\int_{\mathbb{R}}|\mathcal{H}(a)(x)|^2dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}}|x|^2|\mathcal{H}(a)(x)|^2dx\right)^{\frac{1}{2}} \le C.$$

We will use this property of $\mathcal{H}(a)$ to study molecular decomposition for $H^p(\mathbb{R}^n)$. Inspired by the above discussion, we may define an $H^1(\mathbb{R})$ "molecule" $m(x)$ (centered at x_0) as follows:

•

$$\left(\int_{\mathbb{R}} |m(x)|^2 dx\right)^{\frac{1}{4}} \left(\int_{\mathbb{R}} |x-x_0|^2 |m(x)|^2 dx\right)^{\frac{1}{4}} < \infty, \tag{2.3}$$

•

$$\int_{\mathbb{R}} m(x)dx = 0.$$

The number on the left-hand side of (2.3) is called the "norm" of the molecule $m(x)$, denoted as $\mathcal{N}(m)$. Apparently, $m \in H^1(\mathbb{R})$ and a $(1,2,0)$-atom is an H^1 molecule. Therefore, we may conclude that $f \in H^1(\mathbb{R})$ if and only if

$$f(x) = \sum_{k=1}^{\infty} \mu_k m_k(x), \qquad \text{a.e.} \quad x \in \mathbb{R}.$$

Here $m_k(x)$ are H^1-molecules for all $k \in \mathbb{N}$ with

$$\sum_{k=1}^{\infty} \mathcal{N}(m_k) < \infty \quad \text{and} \quad \sum_{k=1}^{\infty} |\mu_k| < \infty.$$

Moreover, from the above example, we know that the image of a 1-atom under the action of Hilbert transform $\mathcal{H}$ is an H^1-molecule.

Now we may extend the above discussion to general n and p. As we may expect, this will be much more complicated than the case for $n = p = 1$.

Definition 2.1. *Let $0 < p \le 1 \le q \le \infty$ with $p \neq q$, $s \ge n_p$, and $\varepsilon > \max\{s/n, (1/p)-1\}$. Denote $\alpha = 1-(1/p)+\varepsilon$ and $\beta = 1-(1/q)+\varepsilon$. An $L^q(\mathbb{R}^n)$ function m is called a (p,q,s,ε)-molecule centered at x_0 if $|x|^{n\beta} m(x) \in L^q(\mathbb{R}^n)$ and*

- (*size condition*) :

$$\mathcal{N}(m) = \left(\int_{\mathbb{R}^n} |m(x)|^q dx\right)^{\frac{\alpha}{q\beta}} \cdot \left(\int_{\mathbb{R}^n} |x-x_0|^{qn\beta} |m(x)|^q dx\right)^{\frac{1}{q}-\frac{\alpha}{q\beta}} < \infty;$$

- (*moment condition*):

$$\int_{\mathbb{R}^n} m(x) x^{\mathbf{k}} dx = 0, \qquad 0 \le |\mathbf{k}| \le s.$$

According to Definition 2.1, it is easy to see that the molecule we had discussed at the beginning of this section is in fact a $(1,2,0,1/2)$-molecule.

Remark 2. The conditions $p < q$ and $\varepsilon > (1/p)-1$ guarantee $\alpha > 0$ and $0 < \alpha/\beta < 1$ and ε is an index which shows how fast the decay of $m(x)$ at infinity is. It is easy to show that if $m(x)$ is a (p,q,s,ε)-molecule, then it is a (p,q,s,ε')-molecule whenever $\varepsilon' < \varepsilon$ with $\varepsilon' > \max\{s/n, (1/p)-1\}$.

We can define $H^p(\mathbb{R}^n)$ as follows (see Taibleson and Weiss [28]): $f \in H^p(\mathbb{R}^n)$ if and only if

$$f = \sum_{k=1}^{\infty} \lambda_k m_k,$$

where m_k's are (p,q,s,ε)-molecules with $\mathcal{N}(m_k) < C$ for all k and $\sum_{k=1}^{\infty} |\lambda_k|^p < \infty$. Moreover,

$$\|f\|^p_{H^p(\mathbb{R}^n)} = \inf \left\{ \sum_{k=1}^{\infty} |\lambda_k|^p \right\}.$$

Here the infimum is taken over all possible molecular decompositions of f.

Proposition 2.2. *Let ψ be an $H^p(\mathbb{R}^n)$ multiplier, $0 < p \le 1$, with norm A. Then there exists a constant C independent of ψ such that*

$$|\psi(\xi)| \le CA$$

for all $\xi \neq 0$. Moreover, ψ is a continuous function defined on $\mathbb{R}^n \setminus \{0\}$.

Proof. Since $\psi \in \mathcal{M}(H^p)$ with norm A, we have

$$\|T_\psi(f)\|_{H^p} = \|\mathcal{F}^{-1}(\psi \hat{f})\|_{H^p} \le A \cdot \|f\|_{H^p}. \tag{2.4}$$

Let $f_t(x) = t^{-n/p} f(x/t), t > 0$. Then $f \longmapsto f_t$ is a bounded operator on $H^p(\mathbb{R}^n)$ with

$$\|f_t\|_{H^p} = \|f\|_{H^p}. \tag{2.5}$$

This is because if a is a (p,∞,s)-atom supported on the ball B, then

$$\|a_t\|_{L^\infty} \le t^{-\frac{n}{p}} \|a\|_{L^\infty} \le t^{-\frac{n}{p}} |B|^{-\frac{1}{p}} = |B_t|^{-\frac{1}{p}},$$

where $B_t = \{x/t : x \in B\}$. It follows that a_t is also a (p,∞,s)-atom. By Proposition 1.4, (2.4) and (2.5), we have

$$|\psi(\xi)\hat{f}_t(\xi)| \le CA \cdot \|f\|_{H^p} |\xi|^{n\left(\frac{1}{p}-1\right)}. \tag{2.6}$$

Notice that

$$\hat{f}_t(\xi) = t^{n(1-\frac{1}{p})} \hat{f}(t\xi).$$

For $\xi \neq 0$, denote $\xi' = \xi/|\xi|$, $t = |\xi|^{-1}$. Then (2.6) implies that

$$|\psi(\xi)\hat{f}_{|\xi|^{-1}}(\xi)| = |\psi(\xi) \cdot |\xi|^{n\left(\frac{1}{p}-1\right)} \hat{f}(\xi')| \leq CA \cdot \|f\|_{H^p} \cdot |\xi|^{n\left(\frac{1}{p}-1\right)}.$$

Hence, we have

$$|\psi(\xi)\hat{f}(\xi')| \leq CA \cdot \|f\|_{H^p}.$$

Pick $\phi \in C^\infty(\mathbb{R})$ with $\mathrm{supp}(\phi) \subset \left[2^{-2}, 2^2\right]$ and $\phi \equiv 1$ on $\left(2^{-1}, 2\right)$. Let $\hat{f}_0(\xi) = \phi(|\xi|)$. Then by Plancherel's formula, it is easy to show that f_0 is a $(p, 2, s, \varepsilon)$-molecule. Therefore, $f_0 \in H^p(\mathbb{R}^n)$. Moreover, $\hat{f}_0(\xi') = 1$. It follows that

$$|\psi(\xi)| \leq CA \cdot \|f_0\|_{H^p} = CA.$$

From the above discussion, we know that $\psi(\xi)$ is the Fourier transform of an H^p distribution when $\xi \neq 0$. Hence, we may conclude that ψ is a continuous function by Proposition 1.4. ∎

Lemma 2.3. *Let N be an integer with $N > n/2$. Assume that*

$$R^{2|\mathbf{k}|-n} \int_{R \leq |x| \leq 2R} |D^{\mathbf{k}}\psi(x)|^2 dx \leq A^2 \tag{2.7}$$

for all $0 \leq |\mathbf{k}| \leq N$ and $R > 0$. Then there exists a constant C independent of ψ such that

1. *when $q = 1$ or $n/q > 2(|\mathbf{k}| - N) + n$, we have*

$$\left[\int_{R \leq |x| \leq 2R} |D^{\mathbf{k}}\psi(x)|^{2q} dx\right]^{\frac{1}{q}} \leq C^2 A^2 R^{\frac{n}{q} - 2|\mathbf{k}|}; \tag{2.8}$$

2. *when $2(|\mathbf{k}|-N)+n < 0$, we have $|x|^{|\mathbf{k}|}|D^{\mathbf{k}}\psi(x)| \leq CA$. Moreover, $D^{\mathbf{k}}\psi(x)$ is a continuous function on $\mathbb{R}^n \setminus \{0\}$.*

Remark 3. When $|\mathbf{k}| = 0$, we have $n - 2N < 0$. By the above lemma, we know that ψ is a bounded continuous function on $\mathbb{R}^n \setminus \{0\}$ with a bound depending on A.

Proof. Pick a smooth, nonnegative cut-off function η with $0 \leq \eta \leq 1$,

$$\mathrm{supp}(\eta) \subset \left\{x \in \mathbb{R}^n : \frac{1}{2} \leq |x| \leq 4\right\} \quad \text{and} \quad \eta \equiv 1 \quad \text{on} \quad \{x \in \mathbb{R}^n : 1 \leq |x| \leq 2\}.$$

Let

$$f(x) = R^{|\mathbf{k}|}\eta\left(\frac{x}{R}\right) D^{\mathbf{k}}\psi(x),$$

and

$$g(x) = f(Rx) = R^{|\mathbf{k}|}\psi(x)D^{\mathbf{k}}\psi(Rx).$$

By (2.7), we know that $\|D^{\mathbf{m}}g\|_{L^2} \leq C'A$ whenever $0 \leq |\mathbf{m}| \leq k = N - |\mathbf{k}|$, where C' is a constant independent of $\psi, \mathbf{m}, N$ and R. It follows that $g \in L^2_k(\mathbb{R}^n)$. By the Sobolev embedding theorem, we know that $L^2_k(\mathbb{R}^n) \subset C_0(\mathbb{R}^n)$ if $k > n/2$, i.e., g is a continuous function which tends to zero whenever $|x| \to \infty$. When $k > (n/2) - (n/r) \geq 0$, $L^2_k(\mathbb{R}^n) \subset L^r(\mathbb{R}^n)$. In particular, $\|g\|_{L^r} \leq CA$, where C is a constant independent of ψ and R. Denote $2q = r$. Then $k > (n/2) - (n/r)$ can be rewritten as $n/q > 2(|\mathbf{k}| - N) + n$. Now we have

$$\begin{aligned}\left[\int_{R\leq|x|\leq 2R} |D^{\mathbf{k}}\psi(x)|^{2q}dx\right]^{\frac{1}{q}} &\leq R^{-2|\mathbf{k}|}\left(\int_{\mathbb{R}^n} |f(x)|^{2q}dx\right)^{\frac{1}{q}} \\ &= R^{\frac{n}{q}-2|\mathbf{k}|}\left(\int_{\mathbb{R}^n} |g(x)|^{2q}dx\right)^{\frac{1}{q}} \\ &= R^{\frac{n}{q}-2|\mathbf{k}|} \cdot \|g\|^2_{L^{2q}} \leq C^2A^2R^{\frac{n}{q}-2|\mathbf{k}|}.\end{aligned}$$

This is (2.8) and the proof of the lemma is therefore complete. ∎

Theorem 2.4 (H^p multiplier theorem). *Let* $0 < p \leq 1$ *with* $(1/p)-(1/2) < N/n$ *and let* $\lambda \geq 0$. *Assume the function* ψ *satisfies*

$$R^{2(\lambda+|\mathbf{k}|)-n}\int_{R\leq|x|\leq 2R} |D^{\mathbf{k}}\psi(x)|^2dx \leq A^2$$

whenver $0 \leq |\mathbf{k}| \leq N$. *Then*

$$\|T_\psi(f)\|_{H^p_{s+\lambda}} \leq C\|f\|_{H^p_s}$$

for all $f \in H^p_s(\mathbb{R}^n)$. *Here*

$$\|f\|_{H^p_s} = \left\|\mathcal{F}^{-1}((1+|\xi|^2)^{\frac{s}{2}}\mathcal{F}(f)(\xi))\right\|_{H^p}$$

is the Hardy-Sobolev space of order s. *Moreover*, $\|T_\psi\|_{op} \leq CA$.

Proof. Without loss of generality, we may assume that $\lambda = 0$. For $\lambda > 0$, it can be proved by the result of $\lambda = 0$ and Plancherel's theorem. Let a be a $(p, 2, s_1)$-atom with $s \leq s_1$. Let ε be a constant satisfying the condition of molecules. Then a will be a $(p, 2, s, \varepsilon)$ molecule with $\mathcal{N}(a) \le C$. Here C is a constant independent of a. In particular, if a is supported on a ball centered at the origin and $N = n[(1/2)+\varepsilon] = n\beta$ is a positive integer, then by Plancherel's theorem, we know that

$$\left\{ \|\hat{a}\|_{L^2}^{\left(\frac{1}{2}-\frac{1}{p}+\frac{N}{n}\right)} \cdot \|D^{\mathbf{m}}\hat{a}\|_{L^2}^{\frac{1}{p}-\frac{1}{2}} \right\}^{\frac{n}{N}} \le C, \quad \text{where} \quad |\mathbf{m}| = N.$$

Next, let a be a $(p, 2, N-1)$-atom. We want to show that $\mathcal{F}^{-1}(\psi\hat{a})$ is a $(p, 2,\, n_p,\, (N/n) - (1/2))$ molecule supported on a ball centered at the origin. Moreover,

$$\mathcal{N}\left(\mathcal{F}^{-1}(\psi\hat{a})\right) \le CA,$$

with C a constant depending only on p, N and n. We first check the size condition for $\mathcal{F}^{-1}(\psi\hat{a})$. By Plancherel's theorem, we just need to show that

$$\left(\|\psi\hat{a}\|_{L^2}^{\frac{1}{2}-\frac{1}{p}+\frac{N}{n}} \|C^{\mathbf{m}}(\psi\hat{a})\|_{L^2}^{\frac{1}{p}-\frac{1}{2}} \right)^{\frac{n}{N}} \le CA \tag{2.9}$$

whenever $|\mathbf{m}| = N$. Denote $d = ((1/p)-(1/2))^{-1}$. Then (2.9) can be rewritten as

$$\|D^{\mathbf{m}}(\psi\hat{a})\|_{L^2} \le CA^{\frac{Nd}{n}} \|\psi\hat{a}\|_{L^2}^{1-\frac{Nd}{n}}.$$

Since $1 - (Nd/n) \le 0$, by Lemma 2.3, $|\psi(x)| \le CA$. Therefore, it reduces to show that

$$\|D^{\mathbf{m}}(\psi\hat{a})\|_{L^2} \le CA\|a\|_{L^2}^{1-\frac{Nd}{n}}.$$

By Leibniz formula, we just need to check

$$\|(D^{\mathbf{m}_1}\hat{a})(D^{\mathbf{m}_2}\psi)\|_{L^2} \le CA\|a\|_{L^2}^{1-\frac{Nd}{n}}, \quad \text{for all} \quad |\mathbf{m}_1| + |\mathbf{m}_2| = N. \tag{2.10}$$

In fact, if $|\mathbf{m}_2| = 0$ and $|\mathbf{m}_1| = N$, then by Lemma 1.3 (2) (with $q' = 1$), we have result (2.10). If $0 < |\mathbf{m}_2| < N$, by Lemma 1.3 (1), we have

$$\begin{aligned}
&\|(D^{\mathbf{m}_1}\hat{a})(D^{\mathbf{m}_2}\psi)\|_{L^2}^2 \\
&= \sum_{\ell\in\mathbb{Z}} \int_{2^\ell<|x|\le 2^{\ell+1}} |D^{\mathbf{m}_1}\hat{a}(\xi)|^2 |D^{\mathbf{m}_2}\psi(\xi)|^2 d\xi \\
&\le C \sum_{\ell=-\infty}^{M} \|a\|_{L^2}^{2\left(1-\frac{Nd}{n}-\frac{d}{2}\right)} 2^{2\ell(N-|\mathbf{m}_1|)} \int_{2^\ell<|x|\le 2^{\ell+1}} |D^{\mathbf{m}_2}\psi(\xi)|^2 d\xi \\
&\quad + \sum_{\ell=M}^{\infty} \left(\int_{2^\ell<|x|\le 2^{\ell+1}} |D^{\mathbf{m}_1}\hat{a}(\xi)|^{2q'} d\xi \right)^{\frac{1}{q'}} \left(\int_{2^\ell<|x|\le 2^{\ell+1}} |D^{\mathbf{m}_2}\psi(\xi)|^{2q} d\xi \right)^{\frac{1}{q}} \\
&= I + II.
\end{aligned}$$

By the hypothesis of the theorem, we know that

$$I \leq CA^2 \sum_{\ell=-\infty}^{M} \|a\|_{L^2}^{2\left(1-\frac{Nd}{n}-\frac{d}{2}\right)} 2^{\ell n} \leq CA^2 2^{nM} \|a\|_{L^2}^{2\left(1-\frac{Nd}{n}-\frac{d}{2}\right)}.$$

Choosing M such that $2^n M \sim \|a\|_{L^2}^d$, we have

$$I \leq CA^2 \|a\|_{L^2}^{2\left(1-\frac{Nd}{n}\right)}.$$

It remains to estimate II. Choose $q = 1$ when $|\mathbf{m}_2| > n/2$ and $q = \infty$ when $0 < |\mathbf{m}_2| \leq n/2$ and $0 < |\mathbf{m}_2| < N - (n/2)$. When $N - (n/2) \leq |\mathbf{m}_2| \leq n/2$, choose q such that $1 < q < \infty$ with $0 < 2|\mathbf{m}_2| - (n/q) < 2N - n$. Therefore, we can always pick q such that $2|\mathbf{m}_2| - (n/q) > 0$ and q, $\mathbf{m}_2$ satisfying the hypotheses of Lemma 2.3. It follows that estimate (2.8) holds. Now we may apply Lemma 1.4 (2) to obtain the following:

$$\begin{aligned} II &\leq CA^2 \sum_{\ell=M}^{\infty} \|a\|_{L^2}^{-d\left(\frac{2|\mathbf{m}_1|}{n}+\frac{1}{q}\right)+2} 2^{\left(\frac{n}{q}-2|\mathbf{m}_2|\right)\ell} \\ &\leq CA^2 \|a\|_{L^2}^{-d\left(\frac{2|\mathbf{m}_1|}{n}+\frac{1}{q}\right)+2} 2^{M\left(\frac{n}{q}-2|\mathbf{m}_2|\right)}. \end{aligned}$$

Since $2^n M \sim \|a\|_{L^2}^d$, we may conclude that

$$II \leq CA^2 \|a\|_{L^2}^{2\left(1-\frac{Nd}{n}\right)}.$$

This completes the proof of (2.10) and hence (2.9).

Next, we want to show that $\mathcal{F}^{-1}(\psi\hat{a})$ satisfies the moment condition. By (2.9), it is easy to see that $D^{\mathbf{m}}(\psi\hat{a})$ is a continuous function on $\mathbb{R}^n \setminus \{0\}$ whenever $|\mathbf{m}| \leq n_p$. By Lemma 1.3 (1) and Lemma 2.3, we know that $\psi(\xi)$ is bounded and $\hat{a}(\xi) = O(|\xi|^{N+1})$ when $|\xi| \to 0$. Therefore, $(\psi\hat{a})(0) = 0$, i.e., $\psi\hat{a}$ satisfies the zero moment condition. In general, if $0 < |\mathbf{m}| \leq n_p \leq N-1$, we have

$$D^{\mathbf{m}}(\psi\hat{a})(0) = \lim_{h\to 0} h^{-|\mathbf{m}|} \Delta_h^{\mathbf{m}}(\psi\hat{a})(0),$$

where $\Delta_h^{\mathbf{m}}$ is the difference operator of order $\mathbf{m}$. Therefore,

$$|\Delta_h^{\mathbf{m}}(\psi\hat{a})(0)| \leq C|h|^{\mathbf{m}}.$$

It follows that $D^{\mathbf{m}}(\psi\hat{a})(0) = 0$, i.e., $\mathcal{F}^{-1}(\psi\hat{a})$ satisfies the necessary moment conditions. Since $a \mapsto \mathcal{F}^{-1}(\psi\hat{a})$ is translation invariant, the above result also holds for atoms which are supported on balls centered at arbitrary points in $\mathbb{R}^n$. Finally, for general $f \in H^p(\mathbb{R}^n)$, we know that

$$f = \sum_{j=1}^{\infty} \lambda_j a_j,$$

where a_j's are $(p, 2, N-1)$-atoms and $\sum_{j=1}^{\infty} |\lambda_j|^p \leq 2\|f\|_{H^p}^p$. Then by the above result, we have

$$\mathcal{F}^{-1}(\psi \hat{f}) = \sum_{j=1}^{\infty} \lambda_j \mathcal{F}^{-1}(\psi \hat{a}).$$

It follows that

$$\begin{aligned} \|\mathcal{F}^{-1}(\psi \hat{f})\|_{H^p} &\leq \sum_{j=1}^{\infty} |\lambda_j| \cdot \|\mathcal{F}^{-1}(\psi \hat{a})\|_{H^p} \leq C \sum_{j=1}^{\infty} |\lambda_j| \cdot \mathcal{N}(\mathcal{F}^{-1}(\psi \hat{a})) \\ &\leq CA \sum_{j=1}^{\infty} |\lambda_j| \leq CA \left(\sum_{j=1}^{\infty} |\lambda_j|^p \right)^{\frac{1}{p}} \leq CA\|f\|_{H^p}. \end{aligned}$$

The proof of this theorem is therefore complete. ∎

From the above theorem, we can conclude the following result immediately:

$$\left\| \frac{\partial^2 \mathbf{G}}{\partial x_j \partial x_k}(f) \right\|_{H_s^p} \leq C\|f\|_{H_s^p}, \qquad \text{for} \quad j, k = 1, \ldots, n,$$

for all $f \in H^p(\mathbb{R}^n)$, $0 < p \leq 1$ and $s \geq 0$. Here $\mathbf{G}$ is the Newtonian potential for the Laplacian on $\mathbb{R}^n$. Here we would like to mention another application of Theorem 2.4 (see Miyachi [26]).

Example. Consider the following multiplier:

$$\psi(\xi) = (1 + |\xi|^2)^{-\frac{\lambda}{2}} e^{it|\xi|^2}, \quad \xi \in \mathbb{R}^n.$$

By Theorem 2.4, $\psi \in \mathcal{M}(H^p)$ and

$$\|\psi\|_{\mathcal{M}(H^p)} \leq C(1 + |t|)^{n\left(\frac{1}{p} - \frac{1}{2}\right)}$$

when $0 \leq (1/p) - (1/2) \leq \lambda/2n$. This means that the solution $u(t, x)$, ($x \in \mathbb{R}^n$, $t \in \mathbb{R}$), of the Schrödinger equation

$$\begin{cases} i\frac{\partial u}{\partial t} = \triangle u & \text{in} \quad \mathbb{R}^n \times \mathbb{R}^+ \\ u(0, x) = f(x) & \text{on} \quad \mathbb{R}^n \end{cases}$$

satisfies the estimates

$$\|u(\cdot,t)\|_{H^p_s} \le C(1+|t|)^{n\left(\frac{1}{p}-\frac{1}{2}\right)}\|f\|_{H^p_{s+\lambda}}$$

for $0 \le (1/p)-(1/2) \le \lambda/2n$, $s \in \mathbb{R}$. This extends the result given by Brenner [2], where the estimate is restricted to the case:

$$0 < \frac{1}{p} - \frac{1}{2} < \frac{\lambda}{2n}, \qquad p > 1.$$

The above estimate is sharp since it does not hold if $(1/p)-(1/2) > \lambda/2n$ as shown by Brenner. The estimate above is sharp in another way: the factor $(1+|t|)^{n((1/p)-(1/2))}$ cannot be improved. More generally, in the estimates

$$\left\|(1+|\xi|^2)^{-\frac{\lambda}{2}}e^{it|\xi|^\mu}\right\|_{\mathcal{M}(H^p)} \le C(1+|t|)^{n\left(\frac{1}{p}-\frac{1}{2}\right)},$$

with $0 \le (1/p)-(1/2) \le \lambda/n\mu$, which can be shown by Theorem 2.4, the factor $(1+|t|)^{n((1/p)-(1/2))}$ cannot be improved if $\mu > 1$. In fact,

$$\left|\mathcal{F}^{-1}[\Phi(|\xi|)r(|\xi|)\exp(it\nu(|\xi|))](x)\right| \ge C_1 \frac{|x|^{\frac{1}{2}(1-n)}r(s)s^{\frac{1}{2}(n-1)}}{\sqrt{t\nu''(s)}}$$

for $t \ge 1$ and $|x| \ge C_2 t$, where $r(x) = (1+x^2)^{\lambda/2}$, $\nu(x) = x^\mu$, $\mu > 1$. Here $s = s_{t,x}$ is the solution of the equation: $t\nu''(s) - |x| = 0$.

3. Estimates of Calserón-Zygmund Operators in H^p

In this section, we continue our discussion on estimates of Calderón-Zygmund operators in Hardy spaces by using molecular decompositions. We are going to deal with a slightly bigger class of standard singular integral operators. We also replace the usual Lebesgue measure dx by measures $w(x)dx$, where w is an A_1-weight, i.e., $w \in A_1$ if and only if

$$M(w)(x) = \sup_{x\in B}\frac{1}{|B|}\int_B w(y)dy \le A\cdot w(x)$$

for almost every $x \in B$ (see Stein [34, p. 197]). Here $M(w)$ is the Hardy-Littlewood maximal function of w. Following results of Coifman, Taibleson and Weiss [17], we may define weighted Hardy spaces $H^p(w(x)dx)$ by weighted atomic and molecular decompositions. We will not repeat similar definitions here. Readers can consult a forthcoming work by the author.

Definition 3.1. *Let $T : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ be a linear operator and let $K(x, y)$ be the kernel of T, i.e.,*

$$T(f)(x) = \int_{\mathbb{R}^n} K(x, y) f(y) dy \tag{3.1}$$

for all $f \in C_0^\infty(\mathbb{R}^n)$ and for almost all $x \notin \operatorname{supp}(f)$. We assume that

$$\|T(f)\|_{L^2(\mathbb{R}^n)} \le C_1 \|f\|_{L^2(\mathbb{R}^n)}.$$

We also assume that the kernel K is smooth away from the diagonal $\Delta = \{(x, y) \in \mathbb{R}^n : x = y\}$ and satisfies

$$\int_{|x-y|>2|z-y|^\sigma} (|K(x, y) - K(x, z)| + |K(y, x) - K(z, x)|) dx \le C \tag{3.2}$$

with $0 < \sigma \le 1$. Then T is called a Calderón-Zygmund operator of type σ.

Remark 4. Let T be an operator given by (3.1). Assume that $T : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ can be extended as a bounded operator from $L^2(\mathbb{R}^n)$ into itself. We assume further that the kernel $K(x, y)$ satisfies the following condition:

$$|K(x, y) - K(x, z)| + |K(y, x) - K(z, x)| \le C \cdot \Theta\left(\frac{|z - y|}{|x - y|}\right) \cdot |x - y|^{-n}$$

for $|x - y| > 2|z - y|^\sigma$, $0 < \sigma \le 1$. Here $\Theta : \mathbb{R}^+ \to \mathbb{R}^+$ is a nondecreasing function such that $\Theta(0) = 0$, $\Theta(2t) \le C \cdot \Theta(t)$, and

$$\int_0^1 \frac{\Theta(t)}{t} dt < \infty.$$

Then it is easy to see that T is a Calderón-Zygmund operator of type σ. In particular, if $\sigma = 1$ and $\Theta(t) = t^\delta$ with $\delta > 0$, then T is a standard Calderón-Zygmund operator (see Chang [6] and Journé [25]). Therefore, all results in this section will also hold for standard singular integral operators.

For $0 < \sigma \le 1$, we denote B_j either the set $\{x \in \mathbb{R}^n : 2^j |z - y|^\sigma \le |x - y| < 2^{j+1}|z - y|^\sigma\}$ or the set $\{y \in \mathbb{R}^n : 2^j |x - z|^\sigma \le |y - x| < 2^{j+1}|x - z|^\sigma\}$. Now we may state our results as follows.

Theorem 3.2. *Let T be a Calderón-Zygmund operator of type σ which satisfies the following conditions: there exists a convergent series $\sum_{j=1}^\infty C_j$ with positive terms such that*

$$\left(\int_{B_j} |K(x, y) - K(x, z)|^q dx\right)^{\frac{1}{q}} \le C_j \cdot |B_j|^{-\frac{1}{q'}} \tag{3.3}$$

and

$$\left(\int_{B_j} |K(y,x)-K(z,x)|^q dx\right)^{\frac{1}{q}} \le C_j \cdot |B_j|^{-\frac{1}{q'}}, \tag{3.4}$$

where $(1/q)+(1/q')=1$ *with* $1<q'<2$. *We also assume further the reverse Hölder inequality holds for* q', *i.e.*,

$$\left(\frac{1}{|Q|}\int_Q w^{q'}(x)dx\right)^{\frac{1}{q'}} \le C\cdot \frac{1}{|Q|}\int_Q w(x)dx. \tag{3.5}$$

Then we have

$$\int_{\mathbb{R}^n} |T(f)(x)|^p dx \le C\int_{\mathbb{R}^n} |M_{q'}(f)(x)|^p w(x)dx \tag{3.6}$$

for $1<p<\infty$ *and*

$$w\{x\in\mathbb{R}^n : |T(f)(x)|>\lambda\} \le \frac{c}{\lambda}\int_{\mathbb{R}^n}|f(x)|w(x)dx.$$

Here

$$M_{q'}(f)(x)=\left(\sup_{x\in B}\frac{1}{|B|}\int_B |f(x)|^{q'}dx\right)^{\frac{1}{q'}}.$$

From (3.6), we know that

$$\begin{aligned}\int_{\mathbb{R}^n}|T(f)(x)|^p w(x)dx &\le C\int_{\mathbb{R}^n}|M_{q'}(f)(x)|^p w(x)dx\\ &= C\int_{\mathbb{R}^n}\left(\sup_{x\in B}\frac{1}{|B|}\int_B|f(x)|^{q'}dx\right)^{\frac{p}{q'}} w(x)dx\\ &\le C_{p,w}\int_{\mathbb{R}^n}|f(x)|^{\frac{q'p}{q'}}w(x)dx = \|f\|^p_{L^p(w(x)dx)}\end{aligned}$$

for $q'<p<\infty$ with $q>2$. Hence we have the following corollary:

Corollary 3.3. *Let* $q'<p<\infty$, $q>2$ *and let* T *be a Calderón-Zygmund operator of type* σ *which satisfies the hypotheses in Theorem* 3.2. *Then* T, *originally defined on* $C_0^\infty(\mathbb{R}^n)$, *can be extended as a bounded operator from* $L^p(w(x)dx)$ *into itself.*

In order to prove the above theorem and Theorems 3.6 to 3.8 below, we need two auxiliary lemmas.

Lemma 3.4. *Let T be a Calderón-Zygmund operator of type σ which satsifies the hypotheses in Theorem* 3.2. *Then T can be extended as a bounded operator from $L^p(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$ for $1 < p < \infty$ and of weak type* (1,1).

The proof of Lemma 3.4 is standard. One may use Calderón-Zygmund decomposition and the Marcinkiewicz interpolation theorem to conclude the result. Readers can read Stein [34, Chapter 2] or Sadosky [29, Chapter 6] for detailed discussions. The proof of the following lemma can be imitated line by line of Stein [34, pp. 206-209] and so the details will not be explained here.

Lemma 3.5. *Let $w \in A_\infty$ and let T be a Calderón-Zygmund operator of type σ which satsifies the hypotheses in Theorem* 3.2. *Then there exist two universal constants C and c such that for all $\eta > 0$ and $\lambda > 0$,*

$$w\{x \in \mathbb{R}^n : |T(f)(x)| > 4\lambda,\, M_{q'}(f)(x) \le \eta\lambda\} \le C\eta^c w\{x \in \mathbb{R}^n : |T(f)(x)| > \lambda\}$$

for all $f \in C_0^\infty(\mathbb{R}^n)$.

Now we may use Lemmas 3.4 and 3.5 to prove our main theorems.

Proof of Theorem 3.2. By Lemma 3.4, it is easy to prove

$$\int_{\mathbb{R}^n} |T(f)(x)|^p w(x)dx \le C \int_{\mathbb{R}^n} |M_{q'}(f)(x)|^p w(x)dx.$$

We will not go through the details here. Now we turn to the proof of the following:

$$w\{x \in \mathbb{R}^n : |T(f)(x)| > \lambda\} \le \frac{C}{\lambda} \int_{\mathbb{R}^n} |f(x)|w(x)dx.$$

In order to do that, we consider the Calderón-Zygmund decomposition of the function $f \in C_0^\infty(\mathbb{R}^n)$ at level $\gamma > 0$ and the Whitney decomposition of the open set $\mathcal{Q} = \{x \in \mathbb{R}^n : M(f)(x) > \gamma\}$. We have the following:

- $\mathbb{R}^n = \mathcal{Q} \cup F$ with $\mathcal{Q} \cap F = \emptyset$;
- $|f(x)| \le \gamma$ for almost every $x \in F$;
- $\mathcal{Q} = \cup_{\ell=1}^\infty Q_\ell$, $\gamma < \frac{1}{|Q_\ell|}\int_{Q_\ell} |f(y)|dy \le 2^n\gamma$.

Let $Q_\ell^o =$ interior of Q_ℓ. It follows that $Q_\ell^o \cap Q_j^o = \emptyset$ for $\ell \ne j$. Denote

$$g(x) = \begin{cases} f(x), & x \in F \\ \frac{1}{|Q_\ell|}\int_{Q_\ell} |f(y)|dy, & x \in Q_\ell,\ \ell \in \mathbb{N} \end{cases}$$

and

$$b(x) = f(x) - g(x).$$

It follows that $g \in L^2(w(x)dx)$ and

$$\begin{aligned}\int_{\mathbb{R}^n} |g(x)|^2 w(x)dx &= \int_F |f(x)|^2 w(x)dx + \sum_{\ell=1}^{\infty} \int_{Q_\ell} \left(\frac{1}{|Q_\ell|} \int_{Q_\ell} |f(y)|dy \right)^2 w(x)dx \\ &\leq \gamma \|f\|_{L^1(w(x)dx)} + 4^n \gamma^2 \int_{\mathcal{Q}} w(x)dx \leq (4^n + 1)\gamma \|f\|_{L^1(w(x)dx)}.\end{aligned}$$

By Lemma 3.4, we have

$$w\{x \in \mathbb{R}^n : |T(g)(x)| > \gamma\} \leq C \cdot \left(\frac{\|g\|_{L^2(w(x)dx)}}{\gamma} \right)^2 \leq \frac{C}{\gamma} \|f\|_{L^1(w(x)dx)}.$$

It remains to prove

$$w\{x \in \mathbb{R}^n : \, |T(b)(x)| > \gamma\} \leq \frac{C}{\gamma} \|f\|_{L^1(w(x)dx)}.$$

Let $Q_\ell^* = 27\sqrt{n} Q_\ell$ and $\mathcal{Q}^* = \cup_{\ell=1}^{\infty} Q_\ell^*$. Obviously, we know that $w(\mathcal{Q}^*) \leq C \cdot w(\mathcal{Q})$. For any fixed $z_\ell \in Q_\ell$, we have

$$\begin{aligned}&\int_{(\mathcal{Q}^*)^c} |K(x,y) - K(x,z_\ell)| w(x)dx \\ &\leq \sum_{j=1}^{\infty} \left(\int_{B_j} |K(x,y) - K(x,z_\ell)|^q dx \right)^{\frac{1}{q}} \left(\int_{B_j} w(x)^{q'} dx \right)^{\frac{1}{q'}} \\ &\leq \sum_{j=1}^{\infty} C_j \left(\frac{1}{|B_j|} \int_{B_j} w(x)^{q'} dx \right)^{\frac{1}{q'}} \leq \sum_{j=1}^{\infty} C_j \left(\frac{1}{|B_j|} \int_{B_j} w(x)^{q'} dx \right) \\ &\leq \left(\sum_{j=1}^{\infty} C_j \right) M(w)(y) \leq C \cdot w(y).\end{aligned}$$

Note that $\int_{Q_\ell} b(x)dx = 0$ for all $\ell \in \mathbb{N}$. Thus, we have

$$\begin{aligned}\int_{(\mathcal{Q}^*)^c} |T(b)(x)| w(x)dx &= \int_{(\mathcal{Q}^*)^c} \left| \sum_{\ell=1}^{\infty} \int_{Q_\ell} [K(x,y) - K(x,z_\ell)] b(y)dy \right| w(x)dx \\ &\leq \sum_{\ell=1}^{\infty} \int_{Q_\ell} |b(y)| \left\{ \int_{(\mathcal{Q}^*)^c} |K(x,y) - K(x,z_\ell)| w(x)dx \right\} dy \\ &\leq C \sum_{\ell=1}^{\infty} \int_{Q_\ell} |b(y)| w(y)dy \leq C \|f\|_{L^1(w(x)dx)}.\end{aligned}$$

By the Chebyshev inequality, we obtain

$$w\left\{x \in \mathbb{R}^n : |T(b)(x)| > \frac{\beta}{2}\right\} \leq \frac{C}{\gamma}\|f\|_{L^1(w(x)dx)}.$$

The proof of Theorem 3.2 is therefore complete. ∎

Theorem 3.6. *Let T be a Calderón-Zygmund operator of type σ which satsifies the hypotheses in Theorem* 3.2 *with $C_j \leq 2^{-jn(2-p)/p}$. We assume further that $0 < p < 1$ with $1/p = q'$ satisfies* (3.5). *Then the operator T can be extended as a bounded operator from $H^p(w(x)dx)$ into $L^p(w(x)dx)$ for $0 < p \leq 1$.*

Proof. In order to prove the operator is bounded from $H^1(w(x)dx)$ into $L^1(w(x)dx)$, we just need to show that $\|T(a)\|_{L^1(w(x)dx)} \leq A$ for all weighted $(1,2,0)$-atoms $a(x)$. By Corollary 3.3, we know that

$$\int_{2B} |T(a)(x)|w(x)dx \leq \|T(a)\|_{L^2(w(x)dx)} \cdot \left(\int_{2B} w(x)dx\right)^{\frac{1}{2}} \leq A. \tag{3.7}$$

On the other hand, by the hypothesis of the theorem, we know that with x_0 a fixed point in B,

$$\begin{aligned}
&\int_{(2B)^c} |T(a)(x)|w(x)dx \\
&\leq \int_B |a(y)| \left(\int_{(2B)^c} |K(x,y) - K(x,x_0)|w(x)dx\right) dy \\
&\leq \int_B |a(y)| \left(\sum_{j=1}^{\infty} \int_{B_j} |K(x,y) - K(x,x_0)|^p dx\right)^{\frac{1}{p}} \left(\int_{B_j} w(x)^{p'} dx\right)^{\frac{1}{p'}} dy \\
&\leq \int_B |a(y)| \sum_{j=1}^{\infty} C_j \left(\frac{1}{|B_j|}\int_{B_j} w(x)^{p'} dx\right)^{\frac{1}{p'}} dy \\
&\leq \int_B |a(y)| \sum_{j=1}^{\infty} C_j \left(\frac{1}{|B_j|}\int_{B_j} w(x) dx\right) dy \\
&\leq C \int_B |a(y)| M(w)(y) dy \leq C \int_B |a(y)| w(y) dy \\
&\leq C \cdot \|a\|_{L^2(w(x)dx)} \cdot \left(\int_{2B} w(x)dx\right)^{\frac{1}{2}} \leq A.
\end{aligned}$$

Combining (3.7) and the above estimate, we conclude that $\|T(a)\|_{L^1(w(x)dx)} \leq A$. Next, we show that T is bounded from $H^p(w(x)dx)$ into $L^p(w(x)dx)$ for $0 < p < 1$. In order to do that, we just need to show that $\|T(a)\|_{L^p(w(x)dx)} \leq A$ for all weighted (p, ∞, s)-atoms $a(x)$. Again, by Corollary 3.3 and the Hölder inequality, we know that

$$
\begin{aligned}
\int_{2B} |T(a)(x)|^p w(x)dx &\leq \|T(a)\|^p_{L^2(w(x)dx)} \cdot \left(\int_{2B} w(x)dx\right)^{1-\frac{p}{2}} \\
&\leq C\|a\|^p_{L^2(w(x)dx)} \cdot \left(\int_{B} w(x)dx\right)^{1-\frac{p}{2}} \\
&\leq \|a\|^p_{L^\infty(\mathbb{R}^n)} \left(\int_{B} w(x)dx\right)^{\frac{p}{2}} \left(\int_{B} w(x)dx\right)^{1-\frac{p}{2}} \leq A.
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
&\int_{B_j} |K(x,y) - K(x,x_0)| w(x)^{\frac{1}{p}} dx \\
&\leq \left(\int_{B_j} |K(x,y) - K(x,x_0)|^q dx\right)^{\frac{1}{q}} \left(\int_{B_j} w(x)^{\frac{q'}{p}} dx\right)^{\frac{1}{q'}} \\
&\leq C_j \left(\frac{1}{|B_j|}\int_{B_j} w(x)^{\frac{q'}{p}} dx\right)^{\frac{1}{q'}} \\
&\leq CC_j \left(\frac{1}{|B_j|}\int_{B_j} w(x) dx\right)^{\frac{1}{p}} \\
&\leq CC_j (M(w)(y))^{\frac{1}{p}} \leq CC_j w(y)^{\frac{1}{p}}.
\end{aligned}
$$

Since the atom a satisfies the moment condition, by the Hölder inequality, we have

$$\begin{aligned}
&\int_{(2B)^c} |T(a)(x)|^p w(x)dx \\
&\leq \int_{(2B)^c} \left| \int_B a(y)[K(x,y) - K(x,x_0)]dy \right|^p w(x)dx \\
&\leq \|a\|^p_{L^\infty(\mathbb{R}^n)} \sum_{j=1}^{\infty} \int_{B_j} \left(\int_B |K(x,y) - K(x,x_0)|dy \right)^p w(x)dx \\
&\leq C \left(\int_B w(x)dx \right)^{-1} \sum_{j=1}^{\infty} |B_j|^{1-p} \left\{ \int_{B_j} \left[\int_B |K(x,y) - K(x,x_0)| w(x)^{\frac{1}{p}} dy \right] dx \right\}^p \\
&\leq C \left(\int_B w(x)dx \right)^{-1} \sum_{j=1}^{\infty} |B_j|^{1-p} \left\{ \int_B \left[\int_{B_j} |K(x,y) - K(x,x_0)| w(x)^{\frac{1}{p}} dx \right] dy \right\}^p \\
&\leq C \left(\int_B w(x)dx \right)^{-1} \sum_{j=1}^{\infty} |B_j|^{1-p} \left(\int_B C_j w(y)^{\frac{1}{p}} dy \right)^p \\
&\leq C \left(\int_B w(x)dx \right)^{-1} \sum_{j=1}^{\infty} C_j^p (2^{jn}|B|)^{1-p} |B|^p \left(\frac{1}{|B|} \int_B w(y)dy \right) \\
&\leq C \sum_{j=1}^{\infty} C_j^p 2^{jn(1-p)} \leq A.
\end{aligned}$$

Apparently, we have $\|T(a)\|_{L^p(\mathbb{R}^n)} \leq A$ and the proof of the theorem is therefore complete. ∎

Theorem 3.7. *Let $w \in A_1$ with $w(x) \geq C > 0$ for almost every $x \in \mathbb{R}^n$ and let T be a Calderón-Zygmund operator of type σ which satsifies the hypotheses in Theorem 3.2 with $C_j \leq 2^{-2jn}$. We assume further that $T^*(1) = 0$. Here T^* is the adjoint operator of T. Then the operator T can be extended as a bounded operator from $H^1(w(x)dx)$ into $H^1(w(x)dx)$.*

Proof. For the proof of this theorem, we just need to show that $T(a)$ is a weighted $(1,2,0,1/2n)$-molecule with $\mathcal{N}_{2,w}(T(a)) \leq A$ for all $(1,2,0)$-atoms a.

By Corollary 3.3 and the assumption $w(x) \geq C > 0$ for almost every $x \in \mathbb{R}^n$, we know that

$$|B| \leq C \int_B w(x)dx = Cw(B).$$

It follows that

$$\int_{2B} |T(a)(x)|^2 |x - x_0|^{n+1} w(x)dx \leq C|B|^{1+\frac{1}{n}} \|T(a)\|^2_{L^2(w(x)dx)} \leq Cw(B)^{\frac{1}{n}}.$$

On the other hand, for $x_0 \in B$ fixed, notice that

$$\begin{aligned}
&\int_{|x-y|>2|x_0-y|} |x-x_0|^{n+1}|K(x,y)-K(x,x_0)|^2 w(x)dx \\
&\le C\sum_{j=1}^{\infty}(2^{j+2}|y-x_0|)^{n+1}\left(\int_{B_j}|K(x,y)-K(x,x_0)|^{2q}dx\right)^{\frac{1}{q}}\left(\int_{B_j}w(x)^{q'}dx\right)^{\frac{1}{q'}} \\
&\le C\sum_{j=1}^{\infty}C_j|B_j|^{\frac{1}{n}}\left(\frac{1}{|B_j|}\int_{B_j}w(x)^{q'}dx\right)^{\frac{1}{q'}} \le C\sum_{j=1}^{\infty}C_j|B_j|^{\frac{1}{n}}\left(\frac{1}{|B_j|}\int_{B_j}w(x)dx\right) \\
&\le CM(w)(y)\sum_{j=1}^{\infty}C_j 2^{jn}|B|^{\frac{1}{n}} \le Cw(y)w(B)^{\frac{1}{n}}.
\end{aligned}$$

Now, as we have done in the proof of Theorem 3.6, we have the following:

$$\begin{aligned}
&\int_{(2B)^c}|x-x_0|^{n+1}|T(a)(x)|^2 w(x)dx \\
&\le \int_{(2B)^c}|x-x_0|^{n+1}\left(\int_B |a(y)|\cdot|K(x,y)-K(x,x_0)|dy\right)^2 w(x)dx \\
&\le \left\{\int_B |a(y)|\left(\int_{|x-y|>2|x_0-y|}|x-x_0|^{n+1}|K(x,y)-K(x,x_0)|^2 w(x)dx\right)^{\frac{1}{2}}dy\right\}^2 \\
&\le Cw(B)^{\frac{1}{n}}\left(\int_B |a(y)|w(y)^{\frac{1}{2}}dy\right)^2 \le Cw(B)^{\frac{1}{n}}\|a\|^2_{L^2(w(x)dx)}|B| \le Aw(B)^{\frac{1}{n}}.
\end{aligned}$$

Therefore,

$$\mathcal{N}_{2,w}(T(a)) = \|T(a)\|^{\frac{1}{n+1}}_{L^2(w(x)dx)}\|T(a)(x)\cdot|x-x_0|^{\frac{n}{2}}\|^{\frac{n}{n+1}}_{L^2(w(x)dx)} \le A.$$

By the assumption $T^*(1)=0$, we also know that $T(a)$ satisfies the moment condition, i.e.,

$$\int_{\mathbb{R}^n} T(a)(x)dx = 0.$$

This completes the proof. ∎

Theorem 3.8. *Let $w \in A_1$ with $w(x) \ge C > 0$ for almost every $x \in \mathbb{R}^n$ and let T be a Calderón-Zygmund operator of type σ which satsifies the hypotheses in Theorem 3.2 with $C_j \le 2^{-2nb(j+2)}$. We assume further that $T^*(x^{\mathbf{k}}) = 0$ with $|\mathbf{k}| \le n_p$. Here T^* is the adjoint operator of T. Then the operator T can be extended as a bounded operator from $H^p(w(x)dx)$ into $H^p(w(x)dx)$ for $0 < p < 1$.*

Proof. In order to prove this theorem, we just need to show that $T(a)(x)$ is a weighted $(p, 2, n_p, \varepsilon)$-molecule satisfying

$$\mathcal{N}_{2,w}(T(a)) = \|T(a)\|_{L^2(w(x)dx)}^{\frac{\alpha}{\beta}} \|T(a)(x)|x - x_0|^{n\beta}\|_{L^2(w(x)dx)}^{1-\frac{\alpha}{\beta}} < \infty \tag{3.8}$$

for all weighted $(p, 2, n_p)$-atoms $a(x)$. Here $\varepsilon > \max\{s/n, (1/p) - 1\}$, and $\alpha = 1 - (1/p) + \varepsilon$, $\beta = (1/2) + \varepsilon$.

By Corollary 3.3, we have

$$\begin{aligned} \int_{2B} |T(a)(x)|^2 |x - x_0|^{2n\beta} w(x)dx &\le C|B|^{2\beta} \|T(a)\|_{L^2(w(x)dx)}^2 \\ &\le Cw(B)^{2\beta+1-\frac{2}{p}}. \end{aligned} \tag{3.9}$$

On the other hand, we know that

$$\begin{aligned} &\int_{|x-y|>2|x_0-y|} |K(x,y) - K(x,x_0)|^2 |x - x_0|^{2n\beta} w(x)dx \\ &\le C \sum_{j=1}^{\infty} (2^{j+2}|y - x_0|)^{2n\beta} \left(\int_{B_j} |K(x,y) - K(x,x_0)|^{2q} dx \right)^{\frac{1}{q}} \left(\int_{B_j} w(x)^{q'} dx \right)^{\frac{1}{q'}} \\ &\le C \sum_{j=1}^{\infty} 2^{2n\beta(j+2)} |B|^{2\beta} C_j |B_j|^{-1-\frac{1}{q'}} \left(\int_{B_j} w(x)^{q'} dx \right)^{\frac{1}{q'}} \\ &\le C \sum_{j=1}^{\infty} C_j 2^{2n\beta(j+2)-jn} |B|^{2\beta-1} \left(\frac{1}{|B_j|} \int_{B_j} w(x)dx \right) \\ &\le CM(w)(y) \sum_{j=1}^{\infty} C_j 2^{2n\beta(j+2)-jn} |B|^{2\beta-1} \le Cw(y)w(B)^{2\beta-1}. \end{aligned}$$

By Minkowski's inequality for integrals, it follows that

$$\begin{aligned} &\int_{(2B)^c} |T(a)(x)|^2 |x - x_0|^{2n\beta} w(x)dx \\ &\le \int_{(2B)^c} |x - x_0|^{2n\beta} \left(\int_B |a(y)||K(x,y) - K(x,x_0)|dy \right)^2 w(x)dx \\ &\le \left\{ \int_B |a(y)| \left(\int_{|x-y|>2|x_0-y|} |K(x,y) - K(x,x_0)|^2 |x - x_0|^{2n\beta} w(x)dx \right)^{\frac{1}{2}} dy \right\}^2 \\ &\le C|B|^{2\beta-1} \left(\int_B |a(y)| \cdot w(y)^{\frac{1}{2}} dy \right)^2 \\ &\le C|B|^{2\beta} \|a\|_{L^2(w(x)dx)}^2 \le Aw(B)^{1+2\beta-\frac{2}{p}}. \end{aligned}$$

Combining (3.9) and the above estimate, we obtain (3.8). By the assumption $T^*(x^{\mathbf{k}}) = 0$ for all $|\mathbf{k}| \le n_p$, we know that $T(a)$ satisfies the moment conditions. The proof of the theorem is therefore complete. ∎

4. Applications to Hyperbolic Equations

In this section, we will apply Theorems 2.4, 3.7 and 3.8 to hyperbolic partial differential equations, mainly hyperbolic equations. Let us consider the Cauchy problem for the wave equation:

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = \Delta u, & t > 0, x \in \mathbb{R}^n, \\ u(0,x) = f(x), & x \in \mathbb{R}^n, \\ \frac{\partial u}{\partial t}(0,x) = g(x), & x \in \mathbb{R}^n. \end{cases}$$

This problem was studied by many mathematicians such as M. Beals [1], Hirshmann [24], Fefferman-Stein [18], Miyachi [27], Seeger-Sogge-Stein [30], Sjöstrand [31] and Sugimoto [32].

We can write the solution $u = u(t,x)$ of the above problem as

$$\begin{aligned} u(t,x) &= \mathcal{F}^{-1}(\cos(t|\xi|)\mathcal{F}(f)(\xi))(x) + \mathcal{F}^{-1}\left(\frac{\sin(t|\xi|)}{|\xi|}\mathcal{F}(g)(\xi)\right)(x) \\ &= (U(t)f)(x) + (V(t)g)(x). \end{aligned}$$

In fact, the operators $U(t)$ and $V(t)$ are multiplier operators:

$$\begin{aligned} U(t) &= T_{\psi_1} \quad \text{with} \quad \psi_1(\xi) = \cos t|\xi|, \\ V(t) &= T_{\psi_2} \quad \text{with} \quad \psi_2(\xi) = \frac{\sin t|\xi|}{|\xi|}. \end{aligned}$$

When $n = 1$, it is known that $U(t)$ and $V(t)$ are bounded operators in $L^p(\mathbb{R})$, $1 < p < \infty$, i.e., the following inequalities hold for all p:

$$\|U(t)f\|_{L^p} \le C_p(t)\|f\|_{L^p} \qquad \text{and} \qquad \|V(t)g\|_{L^p} \le C_p(t)\|g\|_{L^p}. \tag{4.1}$$

In fact, by Theorem 2.4, we know that $U(t)$ and $V(t)$ are also bounded operators in $H^p(\mathbb{R})$, $0 < p \le 1$, i.e.,

$$\|U(t)f\|_{H^p} \le C_p(t)\|f\|_{H^p} \qquad \text{and} \qquad \|V(t)g\|_{H^p} \le C_p(t)\|g\|_{H^p}. \tag{4.2}$$

The situation is quite different if $n \ge 2$:

$$\begin{aligned} &\|U(t)f\|_{L^p} \le C_p(t)\|f\|_{L^p} \iff p = 2; \\ &\|V(t)g\|_{L^p} \le C_p(t)\|g\|_{L^p} \iff \left|\frac{1}{p} - \frac{1}{2}\right| \le \frac{1}{n-1}. \end{aligned}$$

Instead of the simple inequalities (4.1) and (4.2), let us consider the following operators, for $n \geq 2$ and $p \neq 2$, if we take λ and μ sufficiently large:

$$f \mapsto \lambda t^{-\lambda} \int_0^t (t-s)^{\lambda-1} U(s) f ds,$$

$$f \mapsto \lambda t^{-\lambda} \int_0^t (t-s)^{\lambda-1} V(s) f ds,$$

$$(\mathbf{I} - \Delta)^{\frac{\mu}{2}} f \mapsto U(t) f,$$

and

$$(\mathbf{I} - \Delta)^{\frac{\mu}{2}} f \mapsto V(t) f.$$

Then these four operators are also Fourier multiplier operators with

$$\psi_{1,\lambda,t}(\xi) = \lambda t^{-\lambda} \int_0^t (t-s)^{\lambda-1} \cos(s|\xi|) ds, \tag{4.3}$$

$$\psi_{2,\lambda,t}(\xi) = \lambda t^{-\lambda} \int_0^t (t-s)^{\lambda-1} \frac{\sin(s|\xi|)}{|\xi|} ds, \tag{4.4}$$

$$\psi_{3,\mu,t}(\xi) = (1 + |\xi|^2)^{-\frac{\mu}{2}} \cos(t|\xi|), \tag{4.5}$$

and

$$\psi_{4,\mu,t}(\xi) = (1 + |\xi|^2)^{-\frac{\mu}{2}} \frac{\sin(t|\xi|)}{|\xi|}. \tag{4.6}$$

We shall study the problems related to these multipliers. Let Φ be a fixed smooth function on the real line $\mathbb{R}$ such that

$$0 \leq \Phi(x) \leq 1, \qquad \Phi(x) = \begin{cases} 0 & \text{if} \quad x \leq 1, \\ 1 & \text{if} \quad x \geq 2. \end{cases}$$

It is easy to see that

$$\psi_{1,\lambda,t}(\xi) = \lambda \Gamma(\lambda) (t|\xi|)^{-\lambda} \cos\left(t|\xi| - \frac{\lambda\pi}{2} \right) + H(t\xi),$$

where $H(\xi)$ satisfies

$$\left| D^{\mathbf{k}} H(\xi) \right| \leq C_{\mathbf{k}} |\xi|^{-1-|\mathbf{k}|} \quad \text{for } |\xi| > 1$$

and all $|\mathbf{k}| \geq 0$. By Theorems 1.1 and 2.4, we know that $H(t\xi) \in \mathcal{M}(H^p)$ for $0 < p < \infty$. Therefore, H^p estimates for the multiplier $\psi_{1,\lambda,t}$ is equivalent to the H^p estimates for $\Phi(|\xi|)|\xi|^{-\lambda}\cos(|\xi|-(\lambda\pi/2))$. Similar conclusions also hold for $\psi_{2,\lambda,t}$, $\psi_{3,\mu,t}$ and $\psi_{4,\mu,t}$. Immitating the arguments in Miyachi [27], we obtain the following lemma:

Lemma 4.1. *Let $\psi_{j,\lambda,t}$, $j=1,2$, be defined in* (4.3), (4.4) *and let $\psi_{j,\mu,t}$, $j=3,4$, be defined in* (4.5), (4.6), *respectively. Then we have the following*:

1. $\psi_{1,\lambda,t}\in\mathcal{M}(H^p)$ *if and only if*

$$\Phi(|\xi|)|\xi|^{-\lambda}\cos\left(|\xi|-\frac{\lambda\pi}{2}\right)\in\mathcal{M}(H^p).$$

Moreover, if $\psi_{1,\lambda,t}\in\mathcal{M}(H^p)$, then $\|\psi_{1,\lambda,t}\|_{\mathcal{M}(H^p)}$ does not depend on the variable t.

2. $\psi_{2,\lambda,t}\in\mathcal{M}(H^p)$ *if and only if*

$$\Phi(|\xi|)|\xi|^{-\lambda-1}\sin\left(|\xi|-\frac{\lambda\pi}{2}\right)\in\mathcal{M}(H^p).$$

Moreover, if $\psi_{2,\lambda,t}\in\mathcal{M}(H^p)$, then $\|\psi_{2,\lambda,t}\|_{\mathcal{M}(H^p)}/t$ does not depend on the variable t.

3. $\psi_{3,\mu,t}\in\mathcal{M}(H^p)$ *if and only if*

$$\Phi(|\xi|)|\xi|^{-\mu}\cos|\xi|\in\mathcal{M}(H^p).$$

Moreover, if $\psi_{3,\mu,t}\in\mathcal{M}(H^p)$, then $\|\psi_{3,\mu,t}\|_{\mathcal{M}(H^p)}\le C(1+t)^\mu$.

4. $\psi_{4,\mu,t}\in\mathcal{M}(H^p)$ *if and only if*

$$\Phi(|\xi|)|\xi|^{-\mu-1}\sin|\xi|\in\mathcal{M}(H^p).$$

Moreover, if $\psi_{4,\mu,t}\in\mathcal{M}(H^p)$ and $\mu\ge 0$, then $\|\psi_{4,\mu,t}\|_{\mathcal{M}(H^p)}\le Ct(1+t)^\mu$.

Hence, Lemma 4.1 reduces the problem to studying multipliers of the following form:

$$\psi(\xi)=\Phi(|\xi|)|\xi|^{-\lambda}e^{\pm i|\xi|},$$

i.e.,

$$T_\psi(f)(x)=(2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n}\Phi(|\xi|)|\xi|^{-\lambda}e^{i(x\cdot\xi\pm|\xi|)}\hat{f}(\xi)d\xi$$

with $\lambda\ge 0$. Now we may apply Theorem 2.4 to prove the following theorem:

Theorem 4.2. *Let $n \geq 2$ and $\lambda \geq 0$. Then the operator T_ψ is bounded on $H^p(\mathbb{R}^n)$ into itself if and only if*

$$(n-1)\left|\frac{1}{p}-\frac{1}{2}\right| \leq \lambda.$$

Applying the above theorem, we have the following corollary immediately.

Corollary 4.3. *Let $\psi_{j,\lambda,t}$, $j = 1, 2$, be defined in* (4.3), (4.4) *and let $\psi_{j,\mu,t}$, $j = 3, 4$, be defined in* (4.5), (4.6), *respectively. Then we have the following*:

1.
$$\|T_{\psi_{1,\lambda,t}}(f)\|_{H^p} \leq C_p(t)\|f\|_{H^p} \iff \left|\frac{1}{p}-\frac{1}{2}\right| \leq \frac{\lambda}{n-1}.$$

In this case, $C_p(t) = C_p$ is a constant independent of the t-variable.

2.
$$\|T_{\psi_{2,\lambda,t}}(f)\|_{H^p} \leq C_p(t)\|f\|_{H^p} \iff \left|\frac{1}{p}-\frac{1}{2}\right| \leq \frac{\lambda+1}{n-1}.$$

In this case, $C_p(t) = C_p t$.

3.
$$\|T_{\psi_{3,\mu,t}}(f)\|_{H^p} \leq C_p(t)\|(\mathbf{I}-\Delta)^{\frac{\mu}{2}} f\|_{H^p} \iff \left|\frac{1}{p}-\frac{1}{2}\right| \leq \frac{\mu}{n-1}.$$

In this case, $C_p(t) = C_p(1+t)^{(n-1)|\frac{1}{p}-\frac{1}{2}|}$.

4.
$$\|T_{\psi_{4,\mu,t}}(f)\|_{H^p} \leq C_p(t)\|(\mathbf{I}-\Delta)^{\frac{\mu}{2}} f\|_{H^p} \iff \left|\frac{1}{p}-\frac{1}{2}\right| \leq \frac{\mu+1}{n-1}.$$

In this case, $C_p(t)$ satisfies the following: *when $\mu \geq 0$, $C_p(t) = C_p(1+t)^{\gamma(n,p)}$ with $\gamma(n,p) = \max\{(n-1)\,|(1/p)-(1/2)|-1, 0\}$; when $-1 \leq \mu < 0$,*

$$C_p(t) = \begin{cases} C_p t, & t \geq 1, \\ C_p t^{1+\mu}, & 0 < t < 1. \end{cases}$$

If we look at the proof of Theorem 4.2 carefully, the crucial ingredient we used there was the surface $\{\xi \in \mathbb{R}^n : |\xi| = 1\}$ has nonvanishing Gaussian curvature everywhere. It is easy to generalize Theorem 4.2 to the following theorem (see Miyachi [27]):

Theorem 4.4. *Let ϕ be a positive homogeneous function of degree* 1. *Assume that ϕ is smooth and positive on $\mathbb{R}^n\backslash\{0\}$ and that the Gaussian curvature of the surface*

$$\Sigma = \{\xi \in \mathbb{R}^n : \phi(\xi) = 1\}$$

never vanishes on Σ. Let a and b be positively homogeneous functions of degree $-\lambda$, $\lambda \geq 0$. Suppose that a and b are smooth on $\mathbb{R}^n \setminus \{0\}$. Then the function

$$\psi_t(\xi) = \Phi(\phi(\xi))(a(\xi)e^{it\phi(\xi)} + b(\xi)e^{-it\phi(\xi)})$$

is a Fourier multiplier on $H^p(\mathbb{R}^n)$ if and only if $|(1/p) - (1/2)| \leq \lambda/(n-1)$. Moreover, there exist positive constants C_1 and C_2 such that for $t \geq 1$,

$$C_1 t^{(n-1)\left|\frac{1}{p}-\frac{1}{2}\right|} \leq \|\psi_t\|_{\mathcal{M}(H^p)} \leq C_2 t^{(n-1)\left|\frac{1}{p}-\frac{1}{2}\right|}.$$

Notice that the constants $C_p(t)$'s given in Corollary 4.3 are sharp. This can be seen by Theorem 4.4 and the inequality

$$\|\psi\|_{L^\infty} \leq C_p \|\psi\|_{\mathcal{M}(H^p)}, \quad 0 < p < \infty.$$

A similar argument also allows us to obtain the following result:

Corollary 4.5 (Marcinkiewicz multiplier theorem). *Let ϕ be as mentioned in Theorem 4.4, $\lambda > 0$, $n \geq 2$, and*

$$N = \max\left\{\left[\frac{n\lambda}{n-1}\right] + 1, \left[\frac{n}{2}\right] + 1\right\}.$$

Suppose that $\psi \in C^N(\mathbb{R}^n \setminus \{0\})$ and

$$R^{2(\lambda+|\mathbf{k}|)-n} \int_{R \leq |\xi| \leq 2R} |D^{\mathbf{k}}\psi(\xi)|^2 d\xi \leq A^2 \quad \forall\ R > 0$$

for all $0 < |\mathbf{k}| \leq N$. Then we have

$$\left\|{}^{-1}\left[\Phi(\phi(\xi))\psi(\xi)e^{\pm i\phi(\xi)}\hat{f}(\xi)\right]\right\|_{H^p_{s+\lambda}} \leq CA\|f\|_{H^p_s}$$

whenever $|(1/p) - (1/2)| \leq \lambda/(n-1)$.

In fact, we may generalize the above results to the following operator: for $f \in \mathcal{S}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$T(f)(x) = \int_{\mathbb{R}^n} e^{i\Phi(x,\xi)}\psi(x,\xi)\hat{f}(\xi)d\xi,$$

where the phase function $\Phi(x,\xi)$ and the amplitude function $\psi(x,\xi)$ satisfy the following properties:

1. $\psi \in C^\infty(\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}))$;

2. for each $\xi \in \mathbb{R}^n$, $\operatorname{supp}(\psi(\cdot, \xi)) \subset K$ with K a compact subset in $\mathbb{R}^n$;
3. $\Phi : \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}) \to \mathbb{R}$ is smooth;
4. there exist $C > 0$ and $\lambda \geq 0$ such that
$$\left|\partial_x^\alpha \partial_\xi^{\mathbf{k}} \psi(x, \xi)\right| \leq C(1 + |\xi|)^{-\lambda - |\mathbf{k}|} \quad \text{for all} \quad x \in K;$$
5. Φ is homogeneous of degree 1 in the ξ-variable, i.e., $\Phi(x, \delta\xi) = \delta\Phi(x, \xi)$, for all $\delta > 0$, $x \in K$, and $\xi \in \mathbb{R}^n \setminus \{0\}$;
6. the rotational curvature of Φ never vanishes, i,e.,
$$\det\left(\frac{\partial^2 \Phi(x, \xi)}{\partial x_j \partial \xi_k}\right) \neq 0$$
for all $x \in K$, $\xi \in \mathbb{R}^n \setminus \{0\}$.

Under the above assumptions, Seeger, Sogge and Stein proved the following celebrated result (see [30]):

Assume that $n \geq 2$ and $0 < \lambda < (n-1)/2$. The operator T is a bounded operator from $L^p(\mathbb{R}^n)$ into itself, $1 < p < \infty$, if and only if
$$\left|\frac{1}{p} - \frac{1}{2}\right| \leq \frac{\lambda}{n-1}.$$

In this talk, we want to apply the above theorem to H^p estimates for solutions of hyperbolic equations. In order to make this article self-contained, we will prove H^p result for Seeger-Sogge-Stein theorem when T is a multiplier operator, i.e., $\psi(x, \xi) = \psi(\xi)$. Let us first consider the L^2 estimates for the operator T.

Theorem 4.6. *Assume that the phase function $\Phi(x, \xi)$ and the amplitude function $\psi(x, \xi)$ satisfy the above conditions* (1) *to* (6) *with $\lambda = 0$. Then T defines a bounded operator from $L^2(\mathbb{R}^n)$ into itself.*

Proof. Since $\|f\|_{L^2} = \|\hat{f}\|_{L^2}$, we just need to consider the L^2-boundedness of the following operator:
$$T'(f)(x) = \int_{\mathbb{R}^n} e^{i\Phi(x,\xi)} \psi(x, \xi) f(\xi) d\xi.$$
Let S be a measurable subset of $\{\xi \in \mathbb{R}^n : |\xi| = 1\}$ such that $\operatorname{diam}(S) \leq \delta$, where δ is a very small positive number. We may assume that for $\psi(x, \xi) \neq 0$ with $\xi \neq 0$, we have
$$\xi \in \Gamma = \left\{\xi \in \mathbb{R}^n : \xi \neq 0, \frac{\xi}{|\xi|} \in S\right\}.$$

Therefore,

$$\begin{aligned}\|T'(f)\|^2_{L^2} &= \int_{\mathbb{R}^n} (T'f)(x)\overline{(T'f)(x)}dx \\ &= \int_{\mathbb{R}^n} e^{i(\Phi(x,\xi)-\Phi(x,\zeta))}\psi(x,\xi)\overline{\psi(x,\zeta)}f(\xi)\overline{f(\zeta)}d\xi d\zeta dx \\ &= \int_{\mathbb{R}^n} K(\xi,\zeta)f(\xi)\overline{f(\zeta)}dx,\end{aligned}$$

where

$$K(\xi,\zeta) = \int_{\mathbb{R}^n} e^{i(\Phi(x,\xi)-\Phi(x,\zeta))}\psi(x,\xi)\overline{\psi(x,\zeta)}dx.$$

It reduces to showing that

$$\sup_{\zeta\in\mathbb{R}^n}\int_{\mathbb{R}^n}|K(\xi,\zeta)|d\xi \le C, \qquad \sup_{\xi\in\mathbb{R}^n}\int_{\mathbb{R}^n}|K(\xi,\zeta)|d\zeta \le C. \tag{4.7}$$

Once we achieve this goal, then by Schur's lemma, we have

$$\|T'(f)\|^2_{L^2} \le C\|f\|^2_{L^2}.$$

In order to prove (4.7), we just need to show that

$$|K(\xi,\zeta)| \le C(1+|\xi-\zeta|)^{-(n+1)}$$

with $x \in K$ and ξ, $\zeta \in \Gamma$. First we want to prove the following two inequalities

$$\left|D^{\mathbf{k}}_x(\Phi(x,\xi)-\Phi(x,\zeta))\right| \le C_{\mathbf{k}}|\xi-\zeta| \tag{4.8}$$

and

$$|\bigtriangledown_x(\Phi(x,\xi)-\Phi(x,\zeta))| \ge C_1|\xi-\zeta| \tag{4.9}$$

for all multi-indices $\mathbf{k} \in (\mathbb{Z}_+)^n$. Since both $|\xi-\zeta|$ and $|\Phi(x,\xi)-\Phi(x,\zeta)|$ are homogeneous of degree 1, it suffices to assume that

$$|\xi| = 1 \ge |\zeta|.$$

When $|\zeta| > 1/2$, by the mean-value theorem, it is easy to obtain (4.8). When $|\zeta| \le 1/2$, we have

$$\left|D^{\mathbf{k}}_x(\Phi(x,\xi)-\Phi(x,\zeta))\right| \le \left|D^{\mathbf{k}}_x\Phi(x,\xi)\right|+\left|D^{\mathbf{k}}_x\Phi(x,\zeta)\right| \le C_{\mathbf{k}}|\xi|+C_{\mathbf{k}}|\zeta| \le C'_{\mathbf{k}}|\xi-\zeta|,$$

which is (4.8). In order to prove (4.9), we need to look at the phase function $\Phi(x,\xi)$ more carefully. By assumption (6) of the phase function $\Phi(x,\xi)$, we know that $\frac{\partial^2\Phi(x,\xi)}{\partial x_j \partial \xi_k}$ is homogeneous of degree zero and nonsingular away from the origin. Therefore, there exists a constant C_1 such that

$$\left| \left(\sum_{k=1}^{n} \frac{\partial^2\Phi(x,\xi)}{\partial x_j \partial \xi_k} \eta_k \right)_j \right| \geq C_1|\eta| \tag{4.10}$$

for all $x \in K$, $\xi \in \mathbb{R}^n \setminus \{0\}$ and all $\eta \in \mathbb{R}^n$. Denote $\bigtriangledown_x \Phi(x,\xi) = \Phi_x(x,\xi)$. Then $\Phi_x(x,\xi)$ is homogeneous of degree 1 in the ξ-variable. By Euler's identity, we have

$$\Phi_x(x,\xi) = \sum_{k=1}^{n} \frac{\partial \Phi_x(x,\xi)}{\partial \xi_k} \xi_k.$$

Hence,

$$|\Phi_x(x,\xi)| \geq C_1|\xi| \quad \text{for all} \quad x \in K, \quad \xi \in \mathbb{R}^n \setminus \{0\}. \tag{4.11}$$

Next, consider the Taylor's expansion

$$\Phi_x(x,\xi) - \Phi_x(x,\zeta) = \sum_{k=1}^{n} \frac{\partial \Phi_x(x,\xi)}{\partial \xi_k}(\xi_k - \zeta_k) + \mathcal{O}(|\xi - \zeta|^2).$$

For $x \in K$, $|\xi| = 1$ and $|\xi - \zeta| \leq 1/2$, the term $\sum_{k=1}^{n} \frac{\partial \Phi_x(x,\xi)}{\partial \xi_k}(\xi_k - \zeta_k)$ can be estimated by using (4.10) and the estimate for the term $\mathcal{O}(|\xi - \zeta|^2)$ is trivial. Therefore, we conclude that there exists a very small positive number δ_1 such that

$$\begin{aligned} &|\Phi_x(x,\xi) - \Phi_x(x,\zeta)| \\ &\geq \frac{C_1}{2}|\xi - \zeta| \quad \text{for all} \quad x \in K, \quad |\xi| = 1 \quad \text{and} \quad |\xi - \zeta| \leq \delta_1. \end{aligned} \tag{4.12}$$

Now, we may use (4.11) and (4.12) to obtain (4.9). Since both sides of (4.9) are homogeneous of degree 1 and $|\xi| = 1 \geq |\eta|$, it is easy to see that (4.9) holds for $|\xi - \zeta| \leq \delta_1$. Now we may assume that $|\xi - \zeta| > \delta_1$. Since $\xi, \zeta \in \Gamma$, we have

$$1 - |\zeta| = \left| \frac{\zeta}{|\zeta|} - \zeta \right| \geq |\xi - \zeta| - \left| \xi - \frac{\zeta}{|\zeta|} \right| \geq \delta_1 - \delta \geq \frac{\delta_1}{2}$$

with δ a very small positive number. Since $\Phi_x(x,\xi)$ is homogeneous of degree 1, we have

$$\begin{aligned} \Phi_x(x,\xi) - \Phi_x(x,\zeta) &= \Phi_x(x,\xi) - \Phi_x(x,|\zeta|\xi) + \Phi_x(x,|\zeta|\xi) - \Phi_x(x,\zeta) \\ &= (1 - |\zeta|)\Phi_x(x,\xi) + |\zeta| \left(\Phi_x(x,\xi) - \Phi_x\left(x, \frac{\zeta}{|\zeta|}\right) \right) = I + II. \end{aligned}$$

Since $1-|\zeta| \geq \delta_1/2$, by (4.11), we have

$$|I| \geq C_1(1-|\zeta|) \geq C_1\frac{\delta_1}{2}.$$

By (4.8), we have

$$|II| \leq |\zeta| \cdot C_{\mathbf{0}} \cdot \left|\xi - \frac{\zeta}{|\zeta|}\right| \leq C_{\mathbf{0}}\delta$$

because $|\zeta| \leq 1$ and $\xi, \zeta \in \Gamma$. Here δ is a very small positive number. Therefore, there exists a constant $C > 0$ such that

$$|\Phi_x(x,\xi) - \Phi_x(x,\zeta)| \geq C_1\frac{\delta_1}{2} - C_{\mathbf{0}}\delta \geq C_1\frac{\delta_1}{3} \geq C|\xi - \zeta|.$$

Denote $\Phi(x,\xi) - \Phi(x,\zeta) = \Theta(x;\xi,\zeta)$ and $\psi(x,\xi)\overline{\psi(x,\zeta)} = \varphi(x;\xi,\zeta)$. Then we have

$$\frac{1}{i}\sum_{j=1}^{n}\frac{1}{|\bigtriangledown_x \Theta|^2}\frac{\partial\Theta}{\partial x_j}\cdot\frac{\partial e^{i\Theta}}{\partial x_j} = \frac{1}{i}\sum_{j=1}^{n}\theta_j\frac{\partial e^{i\Theta}}{\partial x_j} = e^{i\Theta},$$

where $\theta_j = \frac{1}{|\bigtriangledown_x\Theta|^2}\frac{\partial\Theta}{\partial x_j}$. Therefore,

$$\begin{aligned} K(\xi,\zeta) &= \int_{\mathbb{R}^n} e^{i\Theta(x;\xi,\zeta)}\varphi(x;\xi,\zeta)dx = \frac{1}{i}\int_{\mathbb{R}^n}\left(\sum_{j=1}^{n}\theta_j\frac{\partial e^{i\Theta}}{\partial x_j}\right)\varphi dx \\ &= i\sum_{j=1}^{n}\int_{\mathbb{R}^n} e^{i\Theta}\left(\frac{\partial\theta_j}{\partial x_j}\varphi + \theta_j\frac{\partial\varphi}{\partial x_j}\right)dx. \end{aligned}$$

Applying integration by parts $n+1$ times to the above identity, we obtain

$$K(\xi,\zeta) = i^{n+1}\sum_{|\mathbf{k}|+\ell=n+1} C\int_{\mathbb{R}^n} e^{i\Theta}(\partial_x^{k_1}\theta_{j_1})\cdots(\partial_x^{k_s}\theta_{j_s})(\partial_x^{\ell}\varphi)dx. \tag{4.13}$$

By (4.8) and (4.9), we know that the integrand in (4.13) is nonvanishing for $x \in K$ and ξ, $\zeta \in \Gamma$. Moreover, for $0 < |\mathbf{k}| \leq n+1$, we have

$$\begin{aligned} |\partial_x^{\mathbf{k}}\theta_j| &\leq C \cdot \sum_{r=1,\ldots,|\mathbf{k}|,\, k_j\neq 0} |\bigtriangledown_x \Theta|^{-1-r}|\partial_x^{k_1}\partial_{j_1}\Theta|\cdots|\partial_x^{k_s}\partial_{j_s}\Theta| \\ &\leq C\cdot|\xi-\zeta|^{-1}. \end{aligned} \tag{4.14}$$

For $|\mathbf{k}| = 0$, we have

$$|\theta_j| = \frac{1}{|\bigtriangledown\Theta|^2}\cdot\left|\frac{\partial\Theta}{\partial x_j}\right| \leq C\cdot|\xi-\zeta|^{-1}. \tag{4.15}$$

Plugging (4.14) and (4.15) into (4.13), we have

$$|K(\xi,\zeta)| \le C \cdot \sum_{\ell \le n+1} \int_K |\xi-\zeta|^{-(n+1)} \left|\partial_x^\ell(\psi(x,\xi)\overline{\psi(x,\zeta)}\right| dx \le C \cdot |\xi-\zeta|^{-(n+1)}.$$

On the other hand,

$$|K(\xi,\zeta)| \le \int_K \left|\partial_x^\ell(\psi(x,\xi)\overline{\psi(x,\zeta)}\right| dx \le C,$$

since K is compact and $\lambda = 0$. Summarizing the above discussion, we have

$$|K(\xi,\zeta)| \le C \cdot (1+|\xi-\zeta|)^{-(n+1)}.$$

This completes the proof of the theorem. ∎

Let us now consider the following Cauchy problem

$$\begin{cases} P(D_t, D_x)u(t,x) = 0, & x \in \mathbb{R}^n,\ t \in \mathbb{R}^+, \\ D_t^j u(0,x) = g_j(x), \quad j = 0,1,\ldots,m-1, & x \in \mathbb{R}^n. \end{cases}$$

Here the operator $P(D_t, D_x)$ is a homogeneous constant coefficient partial operator of degree m in D_t, $D_{x_1}, \ldots, D_{x_n}$ which is strictly hyperbolic, i.e., the symbol $\mathcal{P}(\tau,\xi)$ can be factorized as

$$\mathcal{P}(\tau,\xi) = (\tau-\phi_1(\xi))\cdots(\tau-\phi_m(\xi)),$$

where the characteristic roots $\{\phi_j(\xi)\}_{j=1}^m$ are homogeneous of degree 1 and are ordered as

$$\phi_1(\xi) > \phi_2(\xi) > \cdots > \phi_m(\xi) \quad \text{for} \quad \xi \ne 0.$$

We further assume that each characteristic root $\phi_j(\xi)$, $j = 1,\ldots,m$, is either identically positive for $\xi \ne 0$ or identically negative. Then the solution of the above Cauchy problem can be represented as

$$u(t,x) = \sum_{j=1}^{m} \sum_{\lambda=0}^{m-1} \Psi_t^{j,\lambda}(D) g_j.$$

Here the symbol of the operator $\Psi_t^{j,\lambda}(D)$ is of the form

$$\psi_t^{j,\lambda}(\xi) = e^{it\phi_j(\xi)} a_{j,\lambda}(\xi)$$

with $a_{j,\lambda}(\xi) \in S^{-\lambda}$ and $0 \notin \operatorname{supp}(a_{j,\lambda})$. Let $t > 0$. We have

$$[\Psi_t^{j,\lambda}(D) g_j](x) = t^j [\Psi_1^{j,\lambda}(D)(g_j(t\cdot))] \left(\frac{x}{t}\right).$$

Hence the problem reduces to studying the following multiplier operator:

$$T_\psi(D) = {}^{-1}\left[\psi(\xi)\right] = {}^{-1}\left[e^{i\phi(\xi)}a_\lambda(\xi)\right],$$

where $\phi \in C^\omega(\mathbb{R}^n \setminus \{0\})$ is homogeneous of degree 1 and $a_\lambda(\xi) \in C^\infty(\mathbb{R}^n)$ is homogeneous of degree $-\lambda$ for large $|\xi|$ and vanishes near the origin. We shall assume $\phi(\xi) > 0$ for $\xi \neq 0$ since it is one of the characteristic roots $\{\phi_j\}_{j=1}^m$. Estimates for the case $\phi(\xi) < 0$ can be easily derived from those for the case $\phi(\xi) > 0$. Furthermore, we assume that the hypersurface

$$\Sigma = \{\xi \in \mathbb{R}^n : \phi(\xi) = 1\}$$

is strictly convex, i.e., every tangent plane of Σ never lies on Σ except for the tangent point. In particular, in case $m = 2$, the Gaussian curvature of Σ never vanishes. Let us investigate the properties of the kernel

$$K(x) = \mathcal{F}_\xi^{-1}[\psi(\xi)](x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{\{ix\cdot\xi+\phi(\xi)\}} a_\lambda(\xi) d\xi. \tag{4.16}$$

Let the spherical map ν of the surface Σ be

$$\nu : p \in \Sigma \ \mapsto \ \frac{\bigtriangledown\phi(p)}{|\bigtriangledown \phi(p)|} \in S^{n-1},$$

and let $\kappa(p)$ be the Gaussian curvature at the point $p \in \Sigma$ with respect to the spherical map ν. Since Σ is strictly convex, the spherical map ν is a homomorphism. From calculation below, it can be seen that the kernel $K(x)$ has a singularity on the hypersurface

$$\Sigma^* = \{-\bigtriangledown \phi(\xi) : \xi \in \Sigma\} = \{x \in \mathbb{R}^n : H(x) = 0\},$$

where

$$H(x) = |x| - \left|\bigtriangledown\phi\left(\nu^{-1}\left(-\frac{x}{|x|}\right)\right)\right|.$$

The expression (4.16) of the kernel is an oscillatory integral. Therefore, we may rewrite it as

$$K(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{\{ix\cdot\xi+\phi(\xi)\}} (L^*)^M a_\lambda(\xi) d\xi$$

for all positive integers M. Here

$$L = \frac{(x + \bigtriangledown\phi)\cdot\bigtriangledown_\xi}{i|x + \bigtriangledown\phi|^2}$$

and L^* is the transpose of L. From this we can easily obtain the following:

Proposition 4.7. *The kernel $K(x)$ is smooth in $\mathbb{R}^n \setminus \Sigma^*$ and we have*

$$\left(\frac{\partial}{\partial x}\right)^\beta K(x) = \mathcal{O}(|x|^{-M}) \quad as \quad |x| \to \infty$$

for every β and for every $M > 0$.

By the compactness of the sphere S^{n-1} and the rotation invariance of the geometric preoperties, we may assume that $a_\lambda(\xi)$ in (4.16) is supported in a sufficiently small open conic neighborhood Γ of the "north pole" $e_n = (0, \ldots, 0, 1) \in S^{n-1}$. Then by (4.16) again, we can just pay attention to x near the point $- \bigtriangledown \phi(e_n) \in \Sigma^*$.

Since Euler's identity

$$\phi(\xi) = \xi \cdot \bigtriangledown\phi(\xi)$$

yields $\phi'_{x_n}(e_n) = \phi(e_n) > 0$, the hypersurface Σ can be expressed locally as

$$\Sigma \cap \Gamma = \{(y, h(y)) : \, y \in U\}$$

by the implicit function theorem. Here $U \subset \mathbb{R}^{n-1}$ is a sufficiently small open neighborhood of the origin and $h : U \to \mathbb{R}$ is a real analytic function.

The strict convexity of the hypersurface Σ implies that the function h is concave and the map $h' : U \to h'(U) \subset \mathbb{R}^{n-1}$ is a homeomorphism.

For the point x near $- \bigtriangledown \phi(e_n) \in \Sigma^*$, we define the point $z \in U$ by

$$(z, h(z)) = \nu^{-1}(-x/|x|) \in \Sigma^*.$$

If we write $x = (x', x_n)$ with $x' = (x_1, \ldots, x_{n-1})$, this is equivalent to the following:

$$h'(z) = -\frac{x'}{x_n}$$

because of the trivial equality

$$-\frac{x}{|x|} = \frac{\bigtriangledown\phi}{|\bigtriangledown \phi|}(z, h(z))$$

and of the fact that the vector $(-h'(z), 1)$ is normal to the hypersurface Σ at the point $(z, h(z))$.

Then the Gaussian curvature κ is represented as

$$\kappa\left(\nu^{-1}\left(-\frac{x}{|x|}\right)\right) = \frac{(-1)^{n-1} \det h''(z)}{\{1 + |\bigtriangledown h(z)|^2\}^{\frac{n+1}{2}}}.$$

On the other hand, by Euler's identity

$$(z, h(z)) \cdot \bigtriangledown\phi(z, h(z)) = 1,$$

we have

$$H(x) = -x_n \left| \bigtriangledown\phi\left(\kappa\left(\nu^{-1}\left(-\frac{x}{|x|}\right)\right)\right)\right| (x_n^{-1} + h(z) - h'(z)\cdot z).$$

Besides, we may decompose the kernel $K(x)$ as following:

$$K(x) = \sum_{j=1}^{\infty} K_j(x),$$

where K_j is defined as follows:

$$\begin{aligned} K_j(x) &= \mathcal{F}_{\xi}^{-1}\left[\psi(\xi)\Phi_j(x_n\phi(\xi))\right](x) \\ &= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{i(x\cdot\xi+\phi(\xi))} a_\lambda(\xi)\Phi_j(x_n\phi(\xi))d\xi. \end{aligned}$$

Here $\{\Phi_j(t)\}_{j=0}^{\infty}$ is a partition of unity of Littlewood-Paley, that is,

$$\Phi(t) \in C_0^\infty(\{t:\, t>0\}), \quad \Phi_j(t) = \Phi\left(\frac{|t|}{2^j}\right), \quad j \geq 1 \quad \text{and} \quad \sum_{j=1}^{\infty}\Phi_j(t) = 1.$$

Then we have the following result (see Sugimoto [32] and Chang-Sugimoto [12]):

Proposition 4.8. *For sufficiently small $\varepsilon, \eta \geq 0$, every term $K_j(x)$ has the estimate*

$$\left\| \kappa\left(\nu^{-1}\left(-\frac{x}{|x|}\right)\right)^{-\eta} H(x)^{\varepsilon} \left(\frac{\partial}{\partial x}\right)^{\mathbf{k}} K_j(x)\right\|_{L^1} \leq C_{\mathbf{k},\varepsilon,\eta} 2^{j\left(\frac{(n-1)}{2}-\lambda+|\mathbf{k}|-\varepsilon\right)}$$

Here the constant $C_{\mathbf{k},\varepsilon,\eta}$ is independent of the number j. In particular, there exists a constant $C_{\mathbf{k}}$ such that

$$\left\| \left(\frac{\partial}{\partial x}\right)^{\mathbf{k}} K_j(x)\right\|_{L^1(\{x\in\mathbb{R}^n:\, |x|\geq\rho\})} \leq C_{\mathbf{k}} 2^{j\left(\frac{(n-1)}{2}-\lambda+|\mathbf{k}|\right)},$$

for all $\rho \geq 2$

Immediately, we may apply the above proposition to the kernel $K_j(x)$ with $\lambda = (n-1)\left((1/p) - (1/2)\right)$, i.e., we need to check the following:

$$\begin{aligned}
&\int_{|x|\geq \alpha|y|} |K_j(x-y) - K_j(x)|dx \\
&\leq \left(\sum_{2^j \leq |y|^{-1}} + \sum_{2^j > |y|^{-1}} \right) \int_{|x|\geq\alpha|y|} |K_j(x-y) - K_j(x)|dx \\
&= \Sigma_1 + \Sigma_2.
\end{aligned}$$

• Estimate of Σ_1:

$$\begin{aligned}
\int_{|x|\geq\alpha|y|} |K_j(x-y) - K_j(x)|dx &\leq C \int_{|x|\geq\alpha|y|} |y| \cdot \left| \left(\frac{\partial}{\partial x}\right)^{\mathbf{k}} K_j(x-\omega y) \right| dx \\
&\leq C \int_{|x+\omega y|\geq\alpha|y|} |y| \cdot \left| \left(\frac{\partial}{\partial x}\right)^{\mathbf{k}} K_j(x) \right| dx \\
&\leq C|y| \cdot \int_{|x|\geq\frac{\alpha}{2}|y|} \left| \left(\frac{\partial}{\partial x}\right)^{\mathbf{k}} K_j(x) \right| dx.
\end{aligned}$$

Here $|\mathbf{k}| = 1$ and $0 < \omega < 1$. The third inequality above holds because the following reason:

$$|x| \geq (\alpha - \omega)|y| \geq (\alpha - 1)|y| \geq \left(\alpha - \frac{\alpha}{2}\right)|y| \geq \frac{\alpha}{2}|y|.$$

• Estimate of Σ_2:

$$\begin{aligned}
\int_{|x|\geq\alpha|y|} |K_j(x-y) - K_j(x)|dx &\leq \int_{|x|\geq\alpha|y|} |K_j(x-y)|dx + \int_{|x|\geq\alpha|y|} |K_j(x)|dx \\
&\leq \int_{|x+y|\geq\alpha|y|} |K_j(x)|dx + \int_{|x|\geq\alpha|y|} |K_j(x)|dx \\
&\leq 2\int_{|x|\geq\frac{\alpha}{2}|y|} |K_j(x)|dx.
\end{aligned}$$

Once again, the third inequality above holds because of the following reason:

$$|x| \geq (\alpha - 1)|y| \geq \left(\alpha - \frac{\alpha}{2}\right)|y| \geq \frac{\alpha}{2}|y|$$

since $\alpha \geq 2$.

Now we may apply Proposition 4.8 to obtain the following result: if $\lambda \leq (n-1)/2$ then $((n-1)/2) - \lambda + 1 > 0$,

$$|\Sigma_1| \leq C \sum_{2^j \leq |y|^{-1}} 2^{j\left(\frac{n-1}{2}-\lambda+1\right)}|y|$$
$$\leq C|y|^{-\left(\frac{n-1}{2}-\lambda+1\right)}|y| \leq C|y|^{-\left(\frac{n-1}{2}-\lambda\right)},$$

and

$$|\Sigma_2| \leq C \sum_{2^j \geq |y|^{-1}} 2^{j\left(\frac{n-1}{2}-\lambda\right)} \leq C|y|^{-\left(\frac{n-1}{2}-\lambda\right)}.$$

Hence

$$|\Sigma_1 + \Sigma_2| \leq C|y|^{-\left(\frac{n-1}{2}-\lambda\right)}.$$

It follows that

$$\int_{|x|\geq\alpha|y|} |K_j(x-y) - K_j(x)|dx < C.$$

This tells us that the multiplier operator T_ψ is weak type (1,1). It follows that T_ψ defines a bounded operator from $H^1(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$.

Let $\mathcal{R}_J$, $J = (j_1, \ldots, j_\ell) \in (\mathbb{Z}_+)^\ell$, be the Riesz transform of order ℓ, i.e., the Fourier multiplier transform T_{ψ_J} with

$$\psi_J(\xi) = \left(-i\frac{\xi_{j_1}}{|\xi|}\right) \cdots \left(-i\frac{\xi_{j_1}}{|\xi|}\right), \quad \xi \in \mathbb{R}^n.$$

Here the factor $(-i\xi_j/|\xi|)$ shall be replaced by 1 if $j = 0$. Then we have the following theorem (see Fefferman and Stein [18]):

Theorem 4.9. *Let $p > (n-1)/(n-1+\ell)$. Then $f \in L^2(\mathbb{R}^n) \cap H^p(\mathbb{R}^n)$ if and only if $\mathcal{R}_J(f) \in L^2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ for all $J \in (\mathbb{Z}_+)^\ell$; and there exist constants C_1 and C_2 depending only on p, n, and ℓ such that*

$$C_1 \sum_J \|\mathcal{R}_J(f)\|_{L^p} \leq \|f\|_{H^p} \leq C_2 \sum_J \|\mathcal{R}_J(f)\|_{L^p}$$

for all $f \in L^2(\mathbb{R}^n) \cap H^p(\mathbb{R}^n)$.

By Theorem 4.6, we know that T_ψ is strong type (2,2). Now using Theorem 4.9 and the fact that T_ψ is of weak-type (1,1), we conclude that T_ψ is a bounded operator from $H^1(\mathbb{R}^n)$ into itself since it is a multiplier operator (see Stein [34, Chapter 3]). By interpolation theorem between $L^2(\mathbb{R}^n)$ and $H^1(\mathbb{R}^n)$ (see Folland and Stein [19]), we know that $T_\psi(D)$ is bounded from $L^p(\mathbb{R}^n)$ into itself, $1 < p < 2$. Since T_ψ is a convolution operator, we also know that

T_ψ is bounded from $L^p(\mathbb{R}^n)$ into itself, $2 < p < \infty$, by duality argument. Moreover, since $T_\psi(D)$ is basically a convolution operator, we also conclude that $T_\psi(D)$ defines a bounded operator from $BMO(\mathbb{R}^n)$ into itself. A locally integrable function f is said to be in $BMO(\mathbb{R}^n)$ (the function of bounded mean oscillation) if

$$\|f\|_\# \equiv \sup_B \left\{ \frac{1}{|B|} \int_B |f(x) - f_B|^p dx \right\}^{\frac{1}{p}} < \infty, \quad \text{some} \quad 1 \le p < \infty.$$

Here B ranges over all balls in $\mathbb{R}^n$ and f_B denotes the average of f over the ball. Recently, applying Theorems 2.4, 3.8 and Proposition 4.8, we obtain H^p estimates for the operator $T_\psi(D)$ with $p_\theta < p < 1$. We will not go through the detail here and readers can consult a forthcoming paper (Chang and Sugimoto [12]).

Theorem 4.10. *If the hypersurface* Σ *is strictly convex and* $|(1/p) - (1/2)| \le \lambda/(n-1)$, *then*

1. *the operator* $T_\psi(D)$ *is bounded on the Hardy-Sobolev spaces* $H^p_s(\mathbb{R}^n)$ *with* $s \in \mathbb{R}$ *and* $p_\theta < p < \infty$, *where* $p_\theta < 1$ *is an index depending on the vanishing order of the Gaussian curvature*;
2. *the operator* $T_\psi(D)$ *is bounded on the Besov spaces* $B^s_{p,q}(\mathbb{R}^n)$ *with* $1 \le p, q \le \infty$ *and* $s \in \mathbb{R}$.

Here the Besov spaces are a generalization of classes of Hölder continuous functions. For instance, $B^s_{\infty,\infty}$, $s > 0$, is "almost" the same as the class of functions which are $[s]$-times differentiable and whose derivatives are Hölder continuous of order $s - [s]$.

Acknowledgement

In appreciation, the author would like to thank the Mathematical Society of the Republic of China and the organizing committee of 1999 annual meeting, especially the Mathematics Department at Fu Jen Catholic University and the Mathematics Institute of the Academia Sinica, for their invitation to participate at the conference. Much gratitude is extended to them for their warm hospitalities during the author's visit to Taiwan.

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Department of Mathematics, Georgetown University
Washington D.C., 20057, USA
E-mail: chang@archimedes.math.georgetown.edu