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EXACT BOUNDARY CONTROLLABILITY FOR HEAT EQUATION WITH TIME DEPENDENT COEFFICIENTS

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Abstract. We consider the exact boundary controllability for one-dimensional linear heat equation with coefficients depending on the space variable and the time variable. We show that the functions of Gevrey class 2 are reachable when the initial functions are continuous.

1. Introduction

The aim of this work is to study the exact boundary controllability problem for one-dimensional linear heat equation with coefficients depending on the space variable and the time variable. We consider the following initial boundary value problem for a linear heat equation with Dirichlet boundary conditions

$$w_t - w_{xx} = a(x,t)w_x + b(x,t)w + c(x,t) \text{ on } (0,1)\times(0,\infty), \tag{1.1}$$

$$w(0,t) = 0 \text{ for } t \geq 0, \tag{1.2}$$

$$w(x,0) = w_0(x) \text{ for } x \in [0,1], \tag{1.3}$$

$$w(1,t) = h(t) \text{ for } t \geq 0, \tag{1.4}$$

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where $c(x,t)$ is an infinitely differentiable function and $a(x,t)$ and $b(x,t)$ are Gevrey class 2 functions in $[0,1]\times[0,\infty)$. What this means is that positive constants C_i, H_i, $i=1,2$, exist such that

$$|\partial_x^{\alpha_1}\partial_t^{\alpha_2}a(x,t)|\le C_1H_1^{\alpha_1+\alpha_2}(2(\alpha_1+\alpha_2))!,$$

$$|\partial_x^{\alpha_1}\partial_t^{\alpha_2}b(x,t)|\le C_2H_2^{\alpha_1+\alpha_2}(2(\alpha_1+\alpha_2))!$$

for $(x,t)\in[0,1]\times[0,\infty)$, $\alpha_1\ge 0$, $\alpha_2\ge 0$. The problem of exact boundary controllability for (1.1)–(1.4) can be stated as follows. Given $T>0$ and a function $w_f(x)$ in an apppropriate space, is it possible to find a corresponding controller $h(t)$ so that the solution of the resulting problem (1.1)-(1.4) satisfies $w(x,T)=w_f(x)$ for $x\in[0,1]$ for every initial data $w_0(x)$ in another appropriate space?

The method we use here is based on the work of Y.-J. L. Guo and W. Littman [6] in which the control problem is transferred to two well-posed problems. For our case, the method proceeds roughly as follows:

(1) Extend the domain of the initial data w_0 to be $[0,2]$ so that the extended w_0 is still continuous and $w_0(x)\equiv 0$ in a neighborhood of 2. We also extend the domain of $a(x,t)$, $b(x,t)$ and $c(x,t)$ to be $\{(x,t):0\le x\le 2, 0<t<\infty\}$ so that all properties of $a(x,t)$, $b(x,t)$ and $c(x,t)$ are maintained.

(2) With the new modified initial data $w_0(x)$ and the functions $a(x,t)$, $b(x,t)$ and $c(x,t)$, solve the initial-boundary value problem:

$$v_t-v_{xx}=a(x,t)v_x+b(x,t)v+c(x,t)\quad\text{on } (0,2)\times(0,\infty), \tag{1.5}$$

$$v(0,t)=0\quad\text{for } t\ge 0, \tag{1.6}$$

$$v(2,t)=0\quad\text{for } t\ge 0, \tag{1.7}$$

$$v(x,0)=w_0(x)\quad\text{for } x\in(0,2). \tag{1.8}$$

(3) Let ψ be a cut-off function satisfying $\psi(t)=1$ for $t\le T/2$ and $\psi(t)=0$ for $t\ge T$. Let

$$g(t)=v_x(0,t)\psi(t),$$

where v is the solution in (2).

(4) Solve the Cauchy problem

$$u_{xx} = u_t - a(x,t)u_x - b(x,t)u - c(x,t) \quad \text{for } t \geq T_0, x > 0, \tag{1.9}$$

$$u(0,t) = 0, u_x(0,t) = g(t) \quad \text{for } t \geq T_0 \tag{1.10}$$

in the x-direction to get a solution which vanishes for $t \geq T$ and equals the solution v for $t \leq T/2$, where T_0 is a positive constant.

(5) The boundary function is obtained by setting $h(t) = u(1,t)$.

The initial-boundary value problem (1.5)-(1.8) can be solved by the standard method. To solve the second problem (1.9)-(1.10), we use the nonlinear Cauchy-Kowalevski Theorem. If the solution $u(x,t)$ of (1.9)-(1.10) exists beyond $x = 1$, we obtain a controller by reading the values of $v(x,t)$ and $u(x,t)$ at $x = 1$ where $v(x,t)$ and $u(x,t)$ are solutions of (1.5)-(1.8) and (1.9)-(1.10) respectively. To estimate the length of the x-interval of existence for the solution $u(x,t)$, we shall recheck the constants in the proof of the nonlinear Cauchy-Kowalevski Theorem. In [6], the authors consider the control problem for semilinear heat equations and the result of the null boundary controllability for semilinear heat equations is obtained for continuously differentiable and sufficiently small initial data. The small condition on initial data is imposed to ensure that the interval of existence for the problem similar to problem (1.9)-(1.10) is greater than 1. In this work, we only assume that the initial data are continuous without imposing the smallness condition. The linearity of differential equation and the Gevrey class 2 properties for the coefficients $a(x,t)$ and $b(x,t)$ will help us to show that the x-interval of existence for problem (1.9)-(1.10) is greater than 1.

From the solutions of the problems (1.5)-(1.8) and (1.9)-(1.10), we derive the null boundary controllability for linear heat equations and thus the exact boundary controllability for (1.1).

A great many developments in the controllability theory of the linear heat equation were initiated by Fattorini and Russell. These have been presented in numerous articles (see, e.g., [3, 4]). We mention the work of Fattorini and Russell [4] about the exact controllability for linear heat equations of the form

$$u_t = (p(x)u_x)_x + q(x).$$

In this work, we also consider linear heat equation but with coefficients depending on the space variable and the time variable.

The paper is organized as follows. In Section 2, we use the nonlinear Cauchy-Kowalevski Theorem to solve the Cauchy problem (1.9)-(1.10). Since we need to estimate the interval of existence, we restate the nonlinear Cauchy-Kowalevski Theorem in detail in this section. In Section 3, we discuss the null boundary controllability problem. Then we use this result to obtain the exact boundary controllability for (1.11) in Section 4.

2. Solutions of the Cauchy Problem in the x-Direction

In this section, we shall use the nonlinear Cauchy-Kowalevski Theorem to solve the following problem:

$$u_{xx} = u_t - a(x,t)u_x - b(x,t)u - c(x,t) \quad \text{for } x > 0, t \geq T_0, \tag{2.1}$$

$$u(0,t) = 0, \quad u_x(0,t) = g(t) \quad \text{for } t \geq T_0, \tag{2.2}$$

where $g(t)$ is a Gevrey class 2 function and T_0 is a positive constant. We will prove that the solution exists and the x-inteval of existence is greater than 1.

The nonlinear Cauchy-Kowalevski Theorem was originally due to Ovcyannikov and is exploited (see, e.g., [2, 10]) in a number of ways to obtain results in the study of the nonlinear abstract Cauchy problem

$$\begin{aligned} &\frac{du}{dx} = F(u,x), \quad |x| < \eta, \quad \eta > 0, \\ &u(0) = u_0. \end{aligned}$$

Here the solutions are sought, as functions of the variable x, in a scale of Banach spaces $\{X_s\}$. The nonlinear Cauchy-Kowalevski Theorem is a generalization of the well-known Cauchy-Kowalevski Theorem and is reduced to the Cauchy-Kowalevski Theorem when all data are real analytic.

We shall use the same method as used in [6] to solve problem (2.1)-(2.2). Since we shall estimate the parameters in the nonlinear Cauchy-Kowalevski Theorem to obtain the interval of existence, we shall restate the Theorem here. We begin by considering a 1-parameter family of Banach spaces $\{X_s\}$ where the parameter s is allowed to vary in $[0,1]$.

Definition 2.1 $\{X_s\}_{0\leq s\leq 1}$ is a scale of Banach spaces if for any $s \in [0,1]$, X_s is a linear subspace of X_0 and if $s' \leq s$ then $X_s \subset X_{s'}$ and the natural injection of X_s into $X_{s'}$ has norm less than or equal to 1.

We denote by $\|\cdot\|_s$ the norm of X_s.

For each i, $i = 1, \cdots, m$, let $\{X_s^i\}_{0\leq s\leq 1}$ be a scale of Banach spaces with norm $\|\cdot\|_s^i$. Consider the system of differential equations

$$\frac{du_i}{dx} = F_i(u_1, u_2, \cdots, u_m, x), \quad |x| < \eta,\ \eta > 0,\ i = 1, \cdots, m, \tag{2.3}$$

$$u_i(0) = u_{i,0}, \quad i = 1, \cdots, m, \tag{2.4}$$

where the u_i, as functions of the variable x, are in X_s^i, $i = 1, \cdots, m$.

We need the following assumptions.

(H1) $u_{i,0} \in X_s^i$ for every $s \in [0, 1]$ and satisfies

$$\|u_{i,0}\|_s \leq R_{i,0}$$

for some $R_{i,0} < \infty$ for $i = 1, \cdots, m$.

(H2) There are $R_i > R_{i,0} > 0$, $i = 1, \cdots, m$, $\eta > 0$, such that for every pair of numbers s, s' with $0 \leq s' < s \leq 1$ the mapping $F_i(u_1, \cdots, u_m, x)$, $i = 1, \cdots, m$, is continuous from the set

$$\{u_1 \in X_s^1 \mid \|u_1\|_s < R_1\} \times \cdots \times \{u_m \in X_s^m \mid \|u_m\|_s < R_m\} \times \{x \mid |x| < \eta\}$$

into $X_{s'}^i$.

(H3) There are constants C_i, $i = 1, \cdots, m$, such that for every pair of numbers s, s' with $0 \leq s' < s \leq 1$, for all $\|u_j\|_s < R_j$, $\|v_j\|_s < R_j$, $j = 1, \cdots, m$, and for all x, $|x| < \eta$, we have

$$\begin{aligned}&\|F_i(u_1, u_2, \cdots, u_m, x) - F_i(v_1, v_2, \cdots, v_m, x)\|_{s'} \\ &\leq \frac{C_i}{(s - s')^{\alpha_i}} [\vartheta_i^1 \|u_1 - v_1\|_s + \cdots + \vartheta_i^m \|u_m - v_m\|_s],\end{aligned}$$

$$i = 1, \cdots, m,$$

where the number ϑ_i^j is set to be zero if F_i is independent of u_j and to be one otherwise, for some parameters $\alpha_i \geq 0$, $i = 1, \cdots, m$, such that for any collection of m^2 numbers c_i^j, the degree of $P(\lambda, \mu)$ with respect to λ, μ is at most m, where the expression $P(\lambda, \mu)$ of two variables λ, μ is defined by

$$P(\lambda, \mu) = \det(\lambda I - [\mu^{\alpha_i} \vartheta_i^j c_i^j]),$$

with I the $m \times m$ identity matrix and the degree is defined to be the highest degree among all monomials in $P(\lambda, \mu)$.

(H4) $F_i(0,\cdots,0,x)$ is a continuous function of x, $|x|<\eta$, with values in X_s^i for every $s<1$ and satisfies

$$\|F_i(0,\cdots,0,x)\|_s \le \frac{K_i}{(1-s)^{\alpha_i}}, \quad 0\le s<1,$$

for some constants K_i, $i=1,\cdots,m$, with α_i defined in (H3).

Then we have the following existence and uniqueness theorem for solutions of (2.3)-(2.4).

Theorem 2.1 [6]. *Under the preceding hypotheses* (H1)–(H4), *there is a positive constant ρ such that the Cauchy problem* (2.3)-(2.4) *has a unique solution $\{u_i(x),\ i=1,\cdots,m\}$, which are continuously differentiable functions of x, $|x|<\rho(1-s)$, with values in X_s^i such that $\|u_i(x)\|_s<R_i$ for every $s<1/2$.*

Remark 2.1. The proof of Theorem 2.1 [6] gives the estimate of the interval of existence. For $m=3$, which is the case we will consider, the constant ρ in Theorem 2.1 is any positive constant less than

$$\begin{aligned}\frac{1}{2}\min\bigg\{&\frac{1}{108M},\frac{R_1-R_{1,0}-24A}{24S},\frac{R_2-R_{2,0}-24A}{24S},\\ &\frac{R_3-R_{3,0}-192A}{192S}\bigg\},\end{aligned} \tag{2.5}$$

where $M=\max\{C_1,C_2,2C_3\}$, $A=2[R_{2,0}+C_2(R_{1,0}+R_{2.0})]$, $S=4R_{2,0}$, with constants $C_i,R_i,R_{i,0}$ in the assumptions $(H1)$-$(H3)$, $i=1,2,3$.

To apply Theorem 2.1 to solve the Cauchy problem (2.1)-(2.2), we choose the following scale of Banach spaces.

Definition 2.2. Let K be a compact interval and let θ_0 and θ_1 be two positive constants such that $\theta_0<\theta_1<\infty$. Given $s\in[0,1]$, we define the space $B_s(K)$ to be the set of all $C^\infty(K)$ functions ϕ satisfying

$$\|\phi\|_s \equiv \sup_{n\ge 0}\max_{t\in K}\frac{\tilde{n}^4\theta(s)^n}{\lambda(2n)!}|\phi^{(n)}(t)|<\infty,$$

where $1/\theta(s)=(1-s)/\theta_0+s/\theta_1$, $\tilde{n}=\max(n,1)$, and λ is any positive constant satisfying

$$\lambda\le 1\Big/\left[2+2^4\sum_{k=1}^{\infty}(1/k)^4\right].$$

It is easy to check that $\{B_s(K)\}_{0\le s\le 1}$ is a scale of Banach spaces.

The Gevrey class 2 functions which play an important role in this paper are defined as follows.

Definition 2.3. Let Ω be a subset of R^n and $\delta > 0$. A C^∞ function f in Ω is said to be of Gevrey class δ in Ω (in short, $f \in \gamma^\delta(\Omega)$) if there exist positive constants C and H such that

$$|D_x^\alpha f(x)| \le CH^{|\alpha|}(\delta|\alpha|)!$$

for all multi-indices α and for all $x \in \Omega$, where $\alpha! = \Gamma(\alpha+1)$ and Γ is the usual gamma function.

It is clear that any function which is of Gevrey class δ in Ω is bounded.

The following relationship between the spaces $B_s(K)$ and the Gevrey class 2 functions can be found in [6, Proposition 4.4].

(a) The space $B_s(K)$ is contained in γ^2 for all $s \in [0,1]$.

(b) Suppose $\phi : R \to R$ is an infinitely differentiable function defined in K and there are positive constants C and H such that

$$|\phi^{(j)}(t)| \le CH^j(2j)!$$

for all t and for all $j = 1, 2, \cdots$. If the constant θ_1 in defining $B_s(K)$ satisfies $\theta_1 < 1/H$, then $\phi \in B_s(K)$ for all $s \in [0,1]$.

Similarly, the following two lemmas can be easily deduced.

Lemma 2.1. *Let $D = [a,b] \times [c,d]$, $f(x,t) \in \gamma^2(D)$ and C, H be the constants for the Gevrey class 2 functions for $f(x,t)$. If the constant θ_1 in Definition 2.2 satisfies $\theta_1 < 1/H$, then*

(1) *$f(x,\cdot) \in B_s([c,d])$ for all $x \in [a,b]$ and $s \in [0,1]$, and*

(2) *$\sup_{x\in[a,b]} \|f(x,\cdot)\|_1 < \infty$, where $\|\cdot\|_1$ is the norm of $B_1([c,d])$.*

Lemma 2.2. *Let $\{B_s^i(K)\}_{0\le s\le 1}$ be two scales of Banach spaces as defined in Definition 2.2 corresponding to constants θ_0^i, θ_1^i, $i = 1, 2$. If $\theta_1^2 \le \theta_0^1$ and $f(x,t) \in \gamma^2(D)$, $D = [a,b] \times [c,d]$, then $B_0^1(K) \subset B_1^2(K)$, and*

$$\sup_{x\in[a,b]} \|f(x,\cdot)\|_1^1 \ge \sup_{x\in[a,b]} \|f(x,\cdot)\|_1^2,$$

where $\|\cdot\|_1^i$ is the norm of $B_1^i([c,d])$, $i = 1, 2$.

Furthermore, by [6, Proposition 4.2], the partial differentiation $\partial/\partial t$ defines a bounded linear operator from $B_s(K)$ into $B_{s'}(K)$ for $0 \le s' < s \le 1$ with norm less than or equal to $C/(s-s')^2$, where C is a positive constant which can be taken as $(4/e)^2\theta_0/(\theta_1-\theta_0)^2$. We note that the constant C can be made as small as we wish by taking the constant θ_0 sufficiently small while keeping the constant θ_1 fixed in the definition of $B_s(K)$.

Now, we are ready to prove the main result of this section as follows.

Theorem 2.2. *Suppose that $g \in \gamma^2([T_0,\infty))$ with support $[T_0,T]$, $T > T_0$, and $a(x,t)$, $b(x,t) \in \gamma^2(\Omega)$, $c(x,t) \in C^\infty(\Omega)$, where*

$$\Omega = \{(x,t) \mid 0 \le x \le 2, T_0 \le t < \infty\}.$$

Then a classical solution $u(x,t)$ of (2.1), (2.2) exists and the x-interval of existence is greater than 1.

Proof. To apply Theorem 2.1, we convert the problem (2.1)–(2.2) to a first-order system of differential equations by introducing the variables $u_1 = u$, $u_2 = u_x$, and $u_3 = u_t$. Then (2.1)–(2.2) can be rewritten as

$$\begin{aligned}
&\frac{du_1}{dx}(x,\cdot) = u_2(x,\cdot),\\
&\frac{du_2}{dx}(x,\cdot) = u_3(x,\cdot) - a(x,\cdot)u_2(x,\cdot) - b(x,\cdot)u_1(x,\cdot) - c(x,\cdot),\\
&\frac{du_3}{dx}(x,\cdot) = \frac{\partial}{\partial t}u_2(x,\cdot),
\end{aligned}$$

with the Cauchy data

$$u_1(0,\cdot) = 0,\ u_2(0,\cdot) = g(\cdot),\ u_3(0,\cdot) = 0.$$

Let $K = [T_0, T+\epsilon]$ and $D = [0,2]\times K$, where ϵ is any finite number. Since $a(x,t), b(x,t) \in \gamma^2(D)$ and $g(t) \in \gamma^2([T_0,\infty))$, there exist positive constants M_i, H_i, $i = 1,2,3$, such that

$$|\partial_x^{\alpha_1}\partial_t^{\alpha_2} a(x,t)| \le M_1 H_1^{\alpha_1+\alpha_2}(2(\alpha_1+\alpha_2))!,$$

$$|\partial_x^{\alpha_1}\partial_t^{\alpha_2} b(x,t)| \le M_2 H_2^{\alpha_1+\alpha_2}(2(\alpha_1+\alpha_2))!$$

for all $(x,t) \in D$ and α_1, α_2 are any nonnegative integers and

$$|\partial_t^j g(t)| \le M_3 H_3^j (2j)!$$

for all $t \in K$ and any nonnegative integer j. Let θ_0^1, θ_1^1 be two constants satisfying $0 < \theta_0^1 < \theta_1^1 < \min(1/H_1, 1/H_2, 1/H_3)$ and $(4/e)^2\theta_0^1/(\theta_1^1-\theta_0^1)^2 <$

1/2. Let $\{B^1_s\}_{0\le s\le 1}$ be the scale of Banach spaces as defined in Definition 2.2 with constants $\theta_0 = \theta^1_0$ and $\theta_1 = \theta^1_1$. Then it is easy to check that all hypotheses (H1)-(H4) of Theorem 2.1 are satisfied with $C_1 = 1$, $C_2 = \max(1, \sup_{x\in[0,2]} \|a(x,\cdot)\|_1, \sup_{x\in[0,2]} \|b(x,\cdot)\|_1) < \infty$ and $C_3 = (4/e)^2\theta^1_0/(\theta^1_1 - \theta^1_0)^2 < 1/2$. By Theorem 2.1, there exists a constant $\rho_1 > 0$ such that (2.1)-(2.2) has a solution $u(x,\cdot) \in B^1_0$ for $|x| < \rho_1$.

If $\rho_1 > 1$, then the theorem is proved. If $\rho_1 < 1$, then we will proceed as above with new Cauchy data at $x = \rho/2$ and new scale of Banach spaces. That is, we consider

$$\begin{aligned} w_{xx} =& w_t - a(x,t)w_x - b(x,t)w - c(x,t) \\ & \text{for } x \in (\rho/2, \infty), t \ge T_0, \end{aligned} \tag{2.6}$$

$$w(\rho/2,\cdot) = u(\rho/2,\cdot) \quad w_x(\rho/2,\cdot) = u_x(\rho/2,\cdot) \quad \text{for } t \ge T_0, \tag{2.7}$$

where $u(x,t)$ is the solution obtained above.

Since $u(x,\cdot) \in B^1_0$, we have $u(x,\cdot) \in \gamma^2(K)$. We define a new scale of Banach spaces$\{B^2_s\}_{0\le s\le 1}$ corresponding to constants θ^2_0, θ^2_1, where θ^2_0, θ^2_1 are two constants satisfying $0 < \theta^2_0 < \theta^2_1 < \theta^1_0$ such that the constant $(4/e)^2\theta^2_0/(\theta^2_1 - \theta^2_0)^2 < 1/2$ and θ^2_1 is small enough so that $u(\rho/2,\cdot)$, $u_x(\rho/2,\cdot)$, $u_t(\rho/2,\cdot)$, $a(\rho/2,\cdot)$, $b(\rho/2,\cdot) \in B^2_s$ for $s \in [0,1]$. By Lemma 2.2, we have $B^1_0 \subset B^2_1$ and thus the Cauchy data of problem (2.6)-(2.7) and $u_t(\rho/2,\cdot)$ belong to B^2_1. Again, by Theorem 2.1, there exists a constant $\rho_2 > 0$ such that (2.6)-(2.7) has a solution $w(x,\cdot) \in B^2_0$ for $|x - \rho_1/2| < \rho_2$. The procedure can be proceeded if it is needed. For the x-interval of existence, we need to estimate ρ_i, $i = 1, 2, \cdots$.

According to the proof of Theorem 2.1 in [6], for $i = 1, 2, \cdots$, ρ_i is any positive constant less than

$$\begin{aligned} \frac{1}{2}\min\Bigg\{ & \frac{1}{108M_i}, \frac{R^i_1 - R^i_{1,0} - 24A_i}{24S_i}, \frac{R^i_2 - R^i_{2,0} - 24A_i}{24S_i}, \\ & \frac{R^i_3 - R^i_{3,0} - 192A_i}{192S_i}\Bigg\}, \end{aligned} \tag{2.8}$$

where $M_i = \max\{C^i_1, C^i_2, 2C^i_3\}$, $A_i = 2[R^i_{2,0} + C^i_2(R^i_{1,0} + R^i_{2.0})]$, $S_i = 4R^i_{2,0}$ with constants C^i_j in the assumption (H3) of Theorem 2.1, $R^i_{j,0}$ is the $\|\cdot\|^i_1$-norm of the jth Cauchy data in the ith procedure and R^i_j is any constant greater than $R^i_{j,0}$ for $j = 1, 2, 3$.

Since $C^i_1 = 1$, $C^i_2 = \max\{1, \sup_{x\in[0,2]} \|a(x,\cdot)\|^i_1, \sup_{x\in[0,2]} \|b(x,\cdot)\|^i_1\} \ge 1$, $C^i_3 < 1/2$ for $i = 1, 2, \cdots$, we have

$$M_i = C^i_2$$

and thus the first term on the right-hand side of (2.8) is

$$\frac{1}{108M_i} = \frac{1}{108C_2^i}.$$

Since R_j^i is any constant greater than $R_{j,0}^i$, $j = 1, 2, 3$, $i = 1, 2, \cdots$, R_j^i can be chosen so that the minimum on the right-hand side of (2.8) is $\frac{1}{108C_2^i}$ and thus from (2.8) the interval of existence ρ_i is any positive constant satisfying

$$\rho_i < \frac{1}{216C_2^i} \quad \text{for } i = 1, 2, \cdots$$

From Lemma 2.2 and the choice of θ_0^i, θ_1^i, we have

$$C_2^i \geq C_2^{i+1} \quad \text{for all } i,$$

and so

$$C_2^1 \geq C_2^i \quad \text{for all } i.$$

Thus the length of the interval of existence of the solution in each step is at least $1/217C_2^1$ and hence by iterating the above procedure in finitely many times, the total interval of existence can be greater than 1.

This proves the Theorem. ■

3. Existence of Null Boundary Controller

In this section, we shall prove the existence of the boundary controller $h(t)$ that steers a prescribed initial data w_0 to zero for the problem (1.1)–(1.4). The controller $h(t)$ will be continuously differentiable on a finite time duration $0 \leq t \leq T$ with $T > 0$.

First, we define a terminology.

Definition 3.1. Let Ω be a subset of R^n and $0 < \alpha < 1$. A function f defined in Ω is uniformly Hölder continuous of order α in Ω if there exists a positive constant M such that

$$|f(x) - f(y)| \leq M|x - y|^\alpha$$

for all $x, y \in \Omega$.

Now, we state the principal result of this section.

Theorem 3.1. *Suppose $a(x,t)$, $b(x,t)$ and $c(x,t)$ are functions defined in $\Omega = \{(x,t) : 0 \leq x \leq 1, 0 < t < \infty\}$, are uniformly Hölder continuous of order*

α in Ω, $0 < \alpha < 1$, are analytic in both arguments in a neighborhood of the origin, $a(x,t)$, $b(x,t)$ belong to Gevrey class 2 in x, t in Ω, and $c(x,t) \in C^\infty(\Omega)$. Let the initial data $w_0(x)$ be a continuous function in $[0,1]$ and vanish at 0. Then for any finite time $T > 0$, there exists a controller $h(t) \in C^\infty((0,\infty)) \cap C([0,\infty))$ such that the solution $w(x,t)$ of (1.1)–(1.4) satisfies $w(x,T) \equiv 0$ for $x \in [0,1]$.

Proof. We organize the proof in a series of steps.

Step 1. Extend the domain of the initial data $w_0(x)$ to be $[0,2]$ so that $w_0(x)$ is continuous and $w_0(x) \equiv 0$ in a neighborhood of 2. We also extend the domain of $a(x,t)$, $b(x,t)$ and $c(x,t)$ to be $\{(x,t) : 0 \le x \le 2, 0 < t < \infty\}$ so that all properties of $a(x,t)$, $b(x,t)$ and $c(x,t)$ are maintained.

Step 2. We solve the initial-boundary value problem with the new modified initial condition:

$$w_t - w_{xx} = a(x,t)w_x + b(x,t)w + c(x,t) \quad \text{on } (0,2)\times(0,\infty), \tag{3.1}$$

$$w(0,t) = 0 \quad \text{for } t \ge 0, \tag{3.2}$$

$$w(2,t) = 0 \quad \text{for } t \ge 0, \tag{3.3}$$

$$w(x,0) = w_0(x) \quad \text{for } x \in (0,2). \tag{3.4}$$

It is well-known that the solution $w(x,t)$ exists [8]. Let $T > 0$ be any given finite time and $\epsilon < T$ be any small positive number. Then it is clear that the solution $w(x,t)$ is a C^∞ function for $0 \le x \le 2\epsilon$ and $\epsilon \le t \le T$.

Step 3. We claim that the solution $w(x,t)$ obtained in Step 2 belongs to Gevrey class 2 in t for $t \le T$. Let $u_0(x) = w(x,\epsilon)$, where $\epsilon < T$ is any small positive number as in Step 2. Since $w(x,t)$ is a $C^\infty([0,2\epsilon]\times[\epsilon,T])$ solution of the problem

$$\begin{aligned}
&w_t - w_{xx} = a(x,t)w + b(x,t)w_x + c(x,t) \quad \text{on } (0,2\epsilon)\times(\epsilon,T],\\
&w(0,t) = 0 \quad \text{for } \epsilon \le t \le T,\\
&w(x,\epsilon) = u_0(x) \quad \text{for } x \in (0,2\epsilon),
\end{aligned}$$

it follows from a theorem of D. Kinderlehrer and L. Nirenberg [7] that $w(x,t)$ is real analytic in x and is of Gevrey class 2 in t for $0 \le x \le \epsilon$ and $\epsilon \le t \le T$. Thus $w_x(0,t)$ belongs to the Gevrey class 2 in t for $\epsilon \le t \le T$.

Step 4. Next, we modify $w_x(0,t)$ to be a function $w_x(0,t)\psi(t)$ with support in $[0,T]$. Here $\psi(t) \in \gamma^2[0,\infty)$ satisfies

$$\begin{aligned} &0 \le \psi(t) \le 1, \\ &\psi(t) = 0 \quad \text{for } t \ge T, \\ &\psi(t) = 1 \quad \text{for } 0 \le t \le (T+\epsilon)/2. \end{aligned}$$

Let

$$g(t) = \begin{cases} w_x(0,t)\psi(t) & \text{for } \epsilon \le t \le T, \\ 0 & \text{for } t \ge T. \end{cases}$$

Since the Gevrey class of functions forms an algebra which is closed under multiplication, $g(t) \in \gamma^2$ in t for $t \ge \epsilon$ and vanishes for $t \ge T$.

Step 5. In this step, we solve the Cauchy problem:

$$u_{xx} = u_t - a(x,t)u_x - b(x,t)u - c(x,t) \quad \text{on } (0,2) \times (\epsilon, \infty), \tag{3.5}$$

$$u(0,t) = 0, \quad u_x(0,t) = g(t) \quad \text{for } t \ge \epsilon. \tag{3.6}$$

It follows from Theorem 2.2 that there exist a constant $\rho > 1$ and a classical solution $u(x,t)$ of (3.5)–(3.6) which is twice continuously differentiable in x, t, bounded for $0 < x < \rho$, $t \ge \epsilon$ and vanishes for $t \ge T$.

Step 6. By L. Nirenberg's Theorem [9], it is easy to derive that $w(x,t)$ and $u(x,t)$ are identical on $[0,1] \times [\epsilon, (T+\epsilon)/2]$. Now, we read off the required boundary controller $h(t)$ through $w(x,t)$ and $u(x,t)$ by defining $h(t) = w(1,t)$ for $0 \le t \le \epsilon$ and $h(t) = u(1,t)$ for $t \ge \epsilon$.

This proves the theorem. ∎

4. Exact Boundary Controllability

In this section, we use the result of the null boundary controllability of problem (1.1)-(1.4) in Section 3 to obtain the exact boundary controllability of the same problem, i.e., given a finite time $T > 0$ and any final data $w_f(x)$ in an appropriate space, to find a controller $h(t)$ so that the solution of problem (1.1)-(1.4) satisfies

$$w(x,T) = w_f(x)$$

for all $x \in [0,1]$ for any initial data $w_0(x)$ in another appropriate space.

By defining $v(x,t) = w(x,t) - w_f(x)$, the exact boundary controllability of problem (1.1)-(1.4) is transferred to the null boundary controllability of the following problem

$$v_t - v_{xx} = a(x,t)v_x + b(x,t)v(x,t) + \tilde{c}(x,t) \quad \text{on } (0,1)\times(0,\infty), \tag{4.1}$$

$$v(0,t) = 0 \quad \text{for } t \geq 0, \tag{4.2}$$

$$v(x,0) = w_0(x) - w_f(x) \quad \text{for } x \in (0,1), \tag{4.3}$$

$$v(1,t) = \tilde{h}(t) \quad \text{for } t \geq 0, \tag{4.4}$$

where $\tilde{c}(x,t) = c(x,t) + b(x,t)w_f(x) + a(x,t){w_f}'(x) + w_f''(x)$ and $\tilde{h}(t) = h(t) - w_f(1)$.

From Theorem 3.1, we can easily derive the following theorem.

Theorem 4.1. *Suppose $a(x,t)$, $b(x,t)$, $c(x,t)$ and $w_0(x)$ satisfy the same conditions in Theorem* 3.1. *Given any finite time $T > 0$, if $w_f(x) \in C^\infty([0,1])$, analytic in a neighborhood of the origin and $w_f(0) = 0$, then there exists a boundary controller $h(t) \in C^\infty((0,\infty)) \cap C([0,\infty))$ such that the solution of* (1.1)-(1.4) *satisfies*

$$w(x,T) = w_f(x)$$

for all $x \in [0,1]$.

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