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ON SUMMABILITY IN L^p - NORM ON GENERAL VILENKIN GROUPS

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Abstract. Sufficient conditions are given in order that a sequence of linear operators $L_n(\Lambda, \cdot)$ defined by

$$L_n(\Lambda, f) := \sum_{k=0}^{n} \lambda_{nk} \hat{f}(k) \chi_k \quad (n \in N_0), \quad \hat{f}(k) := \int_G f \overline{\chi}_k \ (k \in N_0),$$

converges in L^q- norm to identity, where $f \in L^q(G)$, $q \in [1, \infty]$, $\lambda_{n0} = 1$ $(\forall n \in N_0)$, $\lambda_{nk} = 0$ $(\forall k > n, \forall n \in N_0)$ and G is a general Vilenkin group. In case of bounded Vilenkin groups, our result coincides with an earlier result of Blyumin.

1. Introduction

A Vilenkin group G is an infinite compact totally disconnected Abelian group whose topology satisfies the second axiom of countability. Vilenkin [18] has shown that topology in G can be given by a basic chain of neighbourhoods of zero

$$G = G_0 \supset G_1 \supset G_2 \cdots \supset G_n \supset \cdots \supset \{0\}, \quad \bigcap_{n=0}^{\infty} G_n = \{0\}, \tag{1.1}$$

consisting of open subgroups of the group G, such that the factor group G_n/G_{n+1} is a cyclic group of a prime order p_{n+1}, for every $n \in N_0$. We shall call the group G *bounded* if and only if the sequence (p_n) is bounded.

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It is possible to supply G with the normalized Haar measure μ such that $\mu(G_n) = m_n^{-1}$, where $m_n := p_1 \cdot p_2 \cdots p_n$ $(m_0 := 1)$.

For every $p \in [1, \infty)$, let $L^p(G)$ denote the L^p space on G with respect to the measure μ. The class of continuous complex functions on G will be denoted by $C(G)$.

Remark 1. If $1 \leq p_1 < p_2 < \infty$, then $L^{p_2}(G) \subset L^{p_1}(G)$.

Let Γ denote the group of characters of the group G, and let $\Gamma_n = G_n^{\perp}$ denote the annihilator of G_n in Γ. The dual group Γ is a discrete countable Abelian group with torsion [6, (24.15) and (24.26)]. Vilenkin [18] has proved that there exists Paley-type ordering of elements in Γ: Let us choose a $\chi \in \Gamma_{k+1} \backslash \Gamma_k$ and denote it by χ_{m_k}. Every nonnegative integer n has a unique representation as

$$(1.2) \qquad \begin{aligned} &n = \sum_{i=0}^{N} a_i m_i, \ a_i \in \{0, 1, 2, \ldots, p_{i+1} - 1\} \quad (i = 0, 1, 2, \ldots, N), \\ &a_N \neq 0, \ N = N(n). \end{aligned}$$

Let χ_n be the character defined by

$$(1.3) \qquad \chi_n = \prod_{k=0}^{N} \chi_{m_k}^{a_k}.$$

It is straightforward that

$$(1.4) \qquad \Gamma_n = \{\chi_j : 0 \leq j < m_n\} \ (\forall n \in N_0).$$

Sequence $(\chi_n)_{n \in N_0}$ is called a *Vilenkin system*. We shall say that this system is *bounded* if the group G is bounded. For every $n \in N_0$, there exists $g_n \in G_n \backslash G_{n+1}$ such that

$$(1.5) \qquad \chi_{m_n}(g_n) = e^{2\pi i / p_{n+1}}.$$

Every $g \in G$ can be represented in a unique way as

$$(1.6) \qquad g = \sum_{n=0}^{\infty} a_n g_n,$$

where $a_n \in \{0, 1, \ldots, p_{n+1} - 1\}$. Then

$$(1.7) \qquad G_n = \left\{ g \in G : g = \sum_{i=0}^{\infty} a_i g_i, \ a_i = 0, 0 \leq i < n \right\}.$$

A Vilenkin series $\sum_{n=0}^{\infty} c_n\chi_n$ is a Fourier series if there is a function $f \in L^1(G)$ such that

$$c_n = \hat{f}(n) := \int_G f\overline{\chi}_n \quad (\forall n \in N_0), \tag{1.8}$$

where $\overline{z}$ denotes the complex conjugate of z. In that case, the nth partial sum of the series is given by

$$S_n(f) = \sum_{k=0}^{n-1} \hat{f}(k)\chi_k = f * D_n, \tag{1.9}$$

where D_n, defined by

$$D_n := \sum_{k=0}^{n-1} \chi_k, \tag{1.10}$$

is the Dirichlet kernel of index n on G, and

$$f * \varphi(x) = \int_G f(x-t)\varphi(t)d\mu(t) \tag{1.11}$$

is the convolution of functions f and φ on G. Let us state here some properties of the kernel $(D_n)_{n\in N_0}$ that will be used in the sequel ([18] and [9]).

(1.12) For every $n \in N_0$ and $x \in G$, $|D_n(x)| \leq n$.

(1.13) $D_{m_n}(x) = m_n \cdot I_{G_n}(x)$, where I_A denotes the characteristic function of

a set A.

(1.14) If $n \in N_0$ is given by (1.2), then:

a) $$D_n = \sum_{i=0}^{N} D_{m_i} \frac{1-\chi_{m_i}^{a_i}}{1-\chi_{m_i}} \prod_{s=i+1}^{N} \chi_{m_s}^{a_s},$$

where $\dfrac{1-\chi_{m_t}^{a_t}(x)}{1-\chi_{m_t}(x)} = \sum_{j=0}^{a_t-1} \chi_{m_t}^{j}(x) = 0$ whenever $a_t = 0$ (even if $\chi_{m_t}(x)$ $=1$), and

$$\prod_{s=i+1}^{N} \chi_{m_s}^{a_s} = \begin{cases} \chi_{m_{i+1}}^{a_{i+1}} \cdots \chi_{m_N}^{a_N} & \text{for every } i \in \{0,1,\ldots,N-1\} \\ \chi_0, & \text{for } i = N \ (\chi_0(x) = 1, \ \forall x \in G). \end{cases}$$

b) If $x \in G\backslash G_s$ and $k = \sum_{i=0}^{s-1} a_i m_i \, (1 \le s \le N)$, then $D_n(x) = D_k(x) \cdot \chi_{m_s}^{a_s}(x) \cdots \chi_{m_N}^{a_N}(x)$.

c) $$D_n(x) = \chi_n \left(\sum_{i=0}^{N} \frac{D_{m_i}}{\chi_{m_i}^{a_i}} \cdot \frac{1 - \chi_{m_i}^{a_i}}{1 - \chi_{m_i}} \right).$$

Combining (1.13), (1.14) and Lemma 2 from [10], one obtains

$$\int_{G_s \backslash G_{s+1}} |D_n| = O(\log\ n) \quad \text{(uniformly in } n \in N_0). \tag{1.15}$$

Let

$$\Lambda = [\lambda_{nk}] \quad (n, k \in N_0) \tag{1.16}$$

be a matrix of numbers with the following properties: $\lambda_{n0} = 1$ $(\forall n \in N_0)$ and $\lambda_{nk} = 0$ $(\forall k > n,\ \forall n \in N_0)$. The matrix (1.16) defines in a natural way a sequence of linear operators $L_n(\Lambda, \cdot)$ on $L^1(G)$ by:

$$L_n(\Lambda, f) := \sum_{k=0}^{n} \lambda_{nk} \hat{f}(k) \chi_k \ (n \in N_0), \text{ where } \hat{f}(k) := \int_G f \overline{\chi}_k \ (k \in N_0). \tag{1.17}$$

For every $q \in [1, \infty]$ and every function $f \in L^q(G)$, let us consider the value

$$\| f - L_n(\Lambda, f) \|_q, \tag{1.18}$$

that represents the distance between $L_n(\Lambda, f)$ and the function f in the corresponding metric. We are mainly interested in the following problem: *What conditions on matrix* (1.16) *are sufficient to ensure that*

$$\| f - L_n(\Lambda, f) \|_q \to 0 \ \textit{as} \ n \to \infty, \quad (\forall f \in L^q(G), \ \ \forall q \in [1, \infty])? \tag{1.19}$$

In [1, pp. 132 – 134], it has been proved (for all Vilenkin systems) that conditions

$$\lambda_{nk} \to 1 \ (n \to \infty) \ \text{ for every } k \in N_0, \ \text{ and} \tag{1.20}$$

$$\left\| \sum_{k=0}^{n} \lambda_{nk} \chi_k \right\|_1 \le C < \infty \ \text{ for every } \ n \in N_0 \tag{1.21}$$

are sufficient in order that (1.19) holds.

A natural question to be raised is of a condition involving only matrix entries in place of (1.21). For bounded Vilenkin systems, an answer to that question is given by the following theorem.

Theorem A [1, p. 134, Theorem 4.21]. *If for some $p \in (1,2]$ and every $n \in N_0$, matrix* (1.16) *satisfies condition*

$$(1.22)\qquad \begin{aligned} & n^{1/p'}\left(\sum_{k=0}^{n}|\Delta\lambda_{nk}|^p\right)^{1/p} \le C < \infty, \quad \text{where} \\ & \Delta\lambda_{nk} = \begin{cases} \lambda_{nk}-\lambda_{nk+1}, & 0 \le k < n \\ \lambda_{nn}, & k = n \\ 0, & k > n \end{cases} \qquad (n \in N_0), \end{aligned}$$

$\frac{1}{p}+\frac{1}{p'}=1$, *then for bounded Vilenkin systems conditions* (1.20) *and* (1.21) *are fulfilled, so that* (1.19) *holds.*

Our main result is the following theorem.

Theorem. *Let G be a Vilenkin group and $\Gamma = (\chi_n)_{n\in N_0}$ the dual group of the group G. If for some $p \in (1,2]$ and every $n \in N_0$ the matrix* (1.16) *satisfies the condition*

$$(1.23)\qquad \sum_{i=1}^{N} m_i^{1/p'} \log p_i \left(\sum_{k=1}^{n}|\Delta\lambda_{nk}|^p\right)^{1/p} + \log p_{N+1}\sum_{k=1}^{n}|\Delta\lambda_{nk}| = O(1),$$

where

$$\Delta\lambda_{nk} = \begin{cases} \lambda_{nk}-\lambda_{nk+1}, & 0 \le k < n \\ \lambda_{nn}, & k = n, \\ 0, & k > n, \end{cases} \qquad \frac{1}{p}+\frac{1}{p'}=1 \ (n \text{ and } N \text{ are related by } (1.2)),$$

then $\|f - L_n(\Lambda, f)\|_q \to 0$ as $n \to \infty$ ($\forall f \in L^q(G), \forall q \in [1,\infty]$).

Remark 1. For a bounded Vilenkin system, conditions (1.22) and (1.23) are equivalent. Indeed, if the sequence (p_n) is bounded by some constant M, then from $m_N \le n < p_{N+1} \cdot m_N \le M \cdot m_N$ immediately follows that the condition (1.23) implies condition (1.22).

Conversely, if (p_n) is bounded and (1.22) is satisfied, then from

$$\log\, p_{N+1} \sum_{k=1}^{n} |\Delta\lambda_{nk}| \leq (\log\, M) n^{1/p'} \left(\sum_{k=1}^{n} |\Delta\lambda_{nk}|^p\right)^{1/p}$$

and

$$\begin{aligned}\sum_{i=1}^{N} m_i^{1/p'} \log\, p_i &\leq (\log\, M) m_N^{1/p'} \left[1 + \left(\frac{1}{p_N}\right)^{1/p'} + \ldots + \left(\frac{1}{p_N \ldots p_2}\right)^{1/p'}\right] \\ &\leq C m_N^{1/p'} \leq C n^{1/p'},\end{aligned}$$

where C is an absolute constant, one obtains that (1.22) implies (1.23).

Remark 2. Behavior of the value (1.18), depending upon constructive or structural properties of function f, has been studied (for bounded Vilenkin systems) by S. L. Blyumin in [2] and [3]. In the trigonometric case, appropriate analogues had earlier been given by S. B. Stechkin [12], G. A. Fomin [5] and M. F. Timan [16]. $(C,1)$-summability of series over multiplicative systems of functions has been studied by N. Ya. Vilenkin [18], H. E. Chrestenson [4] and R. Zh. Nurpeisov [8]. Summability over arbitrary systems of characters of 0-dimensional groups satisfying condition $\overline{\lim}\, p_n < \infty$ has been studied by A. M. Zubakin and G. S. Survilo. Zubakin [19] has proved that $\lim\limits_{n\to\infty} \sigma_n^{(\alpha)}(f,x) = \lim\limits_{t\to\infty} (f(x+t) + f(x-t)/2$ for $f \in L^1([0,1])$, where $\sigma_n^{(\alpha)}(f,x)$ are (C,α)-means. Moreover, Zubakin has given sufficient conditions for uniform summability of series of continuous functions by some triangular summability methods [20]. For systems satisfying condition $\sup\limits_n p_n = p < \infty$, methods of summing series using triangular matrices have been studied by G. S. Survilo ([14] and [15]). He [13] has transferred theorems of D. E. Men'shov [7] about (C,α)-summability $(0 < \alpha < 1)$ to this setting.

It is well-known that $(C,1)$-summability of Vilenkin-Fourier series depends a lot upon the nature of the sequence (p_n) which defines the structure of G. For example, if Vilenkin system (χ_n) is bounded, then the Vilenkin-Fourier series

$$\sum_{n=0}^{\infty} \hat{f}(n) \chi_n(x) \tag{1.24}$$

of every function $f \in C(G)$ is uniformly $(C,1)$-summable towards f. However, J. J. Price [10] has proved that in the case of an unbounded Vilenkin system there exists a function $f \in C(G)$ such that $\left|\overline{\lim\limits_{n\to\infty}} \sigma_n(f,0)\right| = \infty$. P. Simon

[11] has proved even more: If (p_n) grows sufficiently fast, then it is possible to construct a function $f \in L^1(G)$ that satisfies the smoothness condition

$$\omega^{(1)}\left(f, \frac{1}{m_k}\right) = O((\log m_k)^{-1}) \; (k \to \infty)$$

$$\left(\omega_n^{(p)}(f) := \sup_{h \in G_n} \|T_h f - f\|^p, \, p \in [1, \infty], T_h f(x) := f(x+h)\right)$$

and such that $(S_n(f, x))$ is divergent a.e.

N. I. Tsutserova [17] has established the following relation between the modulus of continuity $\omega_n(f)$ and the $(C, 1)$-summability of its Fourier series: If $f \in C(G)$ and $\omega_{n-1}(f) \log p_n = o(1) \, (n \to \infty)$, then $\sigma_n(f) \to f$ uniformly on G. On the other hand, if (χ_n) is an unbounded Vilenkin system, then there exists a $f \in C(G)$ that satisfies condition $\omega_{n-1} \log p_n = O(1) \, (n \to \infty)$ and whose Vilenkin-Fourier series is not $(C, 1)$-summable anywhere on G. R. Zh. Nurpeisov [8] has proved that this situation cannot be improved even if we pass to a subsequence of the sequence $(\sigma_n(f))$. More precisely, he has proved that if (χ_n) is an unbounded Vilenkin system, there exists $f \in C(G)$ that satisfies the condition $\omega_{n-1} \log p_n = O(1)(n \to \infty)$ such that $\sigma_{m_n}(f, x)$ does not converge uniformly on G. In the same paper, he has given the following characterization of uniform convergence of $(C, 1)$-means of index m_n for the class $H^\omega(G) := \{f : \omega_n(f) \le C\omega_n\}$, where $\omega = (\omega_n)$ is an arbitrary nonincreasing zero sequence: If $f \in H^\omega(G)$, then $\sigma_{m_n}(f, x)$ converges uniformly on G towards f if and only if

$$\omega_{n-1} \log p_n = o(1) \, (n \to \infty).$$

Nurpeisov has also proved [8, Theorem 4] that the Vilenkin-Fourier series of a function $f \in C(G)$ that satisfies condition $\omega_{n-2}(f) \log \left(\max_{1 \le j \le n+1} \{p_j\}\right) = o(1) \, (n \to \infty)$ is uniformly $(C, 1)$-summable towards f on G.

2. Proof of the Theorem

It is sufficient to prove that under assumptions of our theorem relations (1.20) and (1.21) hold. From results of G. A. Fomin [5, (13), (14) and (15)] immediately follows that (1.23) implies (1.20). What we need to prove is that (1.23) implies (1.21). For $n = m_{N+1} - 1$, one obtains:

$$\left\| \sum_{k=0}^{n} \lambda_{nk} \chi_k \right\|_1 = \left\| \sum_{k=0}^{n} \Delta\lambda_{nk} D_{k+1} \right\|_1 \le |\lambda_{nn}| + \left\| \sum_{k=1}^{n} \Delta\lambda_{nk-1} D_k \right\|_1 .$$

For $m_N \le n < M_{N+1} - 1$, one obtains

$$\left\|\sum_{k=0}^{n} \lambda_{nk}\chi_k\right\|_1 = \left\|\sum_{k=0}^{n} \Delta\lambda_{nk} D_{k+1}\right\|_1 = \left\|\sum_{k=1}^{n'} \Delta\lambda_{nk-1} D_k\right\|_1 ,$$

where $n' = n + 1 \le m_{N+1} - 1$.

Therefore, it is sufficient to prove that under assumptions of the theorem, the relation

$$\left\|\sum_{k=1}^{n} c_k D_k\right\|_1 = O(1) \quad \text{holds,} \tag{1.25}$$

where we put c_k instead of $\Delta\lambda_{nk-1}$ for every $k \in \{1, 2, \dots, n\}$.

In general, we have

$$\left\|\sum_{k=1}^{n} c_k D_k\right\|_1 = \left(\int\limits_{G_{N+1}} + \int\limits_{G_N \setminus G_{N+1}} + \int\limits_{G \setminus G_N}\right) \left|\sum_{k=1}^{n} c_k D_k\right| = I_1 + I_2 + I_3. \tag{1.26}$$

We will estimate integrals I_j $(j = 1, 2, 3)$. Now

$$I_1 = \int\limits_{G_{N+1}} \left|\sum_{k=1}^{n} c_k D_k\right| = \left|\sum_{k=1}^{n} k c_k\right| \mu(G_{N+1}) \le \sum_{k=1}^{n} |c_k|, \tag{1.27}$$

because $D_k(x) = k$ for every $k \le n$, every $x \in G_{N+1}$ and $m_N \le n < m_{N+1}$.

$$\begin{aligned} I_2 = &\int\limits_{G_N \setminus G_{N+1}} \left|\sum_{k=1}^{n} c_k D_k\right| \le \sum_{k=1}^{n} |c_k| \int\limits_{G_N \setminus G_{N+1}} |D_k| \\ &\le C_1 \log p_{N+1} \sum_{k=1}^{n} |c_k| \quad \text{(by (1.15)).} \end{aligned} \tag{1.28}$$

$$\begin{aligned} I_3 &= \int\limits_{G \setminus G_N} \left|\sum_{k=1}^{n} c_k D_k\right| = \int\limits_{G \setminus G_N} \left|\sum_{k=1}^{n} c_k \left(\sum_{i=0}^{N} D_{m_i} \frac{1 - \chi_{m_i}^{a_i}}{1 - \chi_{m_i}} \overline{\chi_{m_i}^{a_i}}\right) \chi_k\right| \\ &= \sum_{s=0}^{N-1} \int\limits_{G_s \setminus G_{s+1}} \left|\sum_{k=1}^{n} c_k \left(\sum_{i=0}^{s} m_i \frac{1 - \chi_{m_i}^{a_i}}{1 - \chi_{m_i}} \overline{\chi_{m_i}^{a_i}}\right) \chi_k\right| \end{aligned}$$

(by (1.13) and (1.14) c)). Therefore

$$\begin{aligned} I_3 \le &\sum_{s=0}^{N-1} \Bigg(\int\limits_{G_s \setminus G_{s+1}} \left|\sum_{k=1}^{n} c_k m_s \frac{1 - \chi_{m_s}^{a_s}}{1 - \chi_{m_s}} \overline{\chi_{m_s}^{a_s}} \cdot \chi_k\right| \\ &+ \int\limits_{G_s \setminus G_{s+1}} \left|\sum_{k=1}^{n} c_k \left(\sum_{i=0}^{s-1} m_i \frac{1 - \chi_{m_i}^{a_i}}{1 - \chi_{m_i}} \overline{\chi_{m_i}^{a_i}}\right) \chi_k\right| \Bigg) = \sum_{s=0}^{N-1} (I_3^{(1)} + I_3^{(2)}). \end{aligned} \tag{1.29}$$

Recall that $g_s \in G_s \backslash G_{s+1}$ was chosen such that $\chi_{m_s}(g_s) = e^{2\pi i/p_{s+1}}$. Let us put

$$B_{k,\nu,s} := c_k m_s \frac{1-\chi_{m_s}^{a_s}(\nu \cdot g_s)}{1-\chi_{m_s}(\nu \cdot g_s)} \overline{\chi_{m_s}^{a_s}}(\nu \cdot g_s)\chi_k(\nu \cdot g_s).$$

Obviously,

$$|B_{k,\nu,s}| \le |c_k| m_s \left| \frac{1-e^{2\pi\nu a_s i/p_{s+1}}}{1-e^{2\pi\nu i/p_{s+1}}} \right|.$$

Now applying the Hölder and then F. Riesz inequality, one obtains

$$\begin{aligned}
I_3^{(1)} &= \int\limits_{G_s\backslash G_{s+1}} \left| \sum_{k=1}^n c_k m_s \frac{1-\chi_{m_s}^{a_s}}{1-\chi_{m_s}} \overline{\chi_{m_s}^{a_s}} \chi_k \right| = \sum_{\nu=1}^{p_{s+1}-1} \int\limits_{\nu g_s + G_{s+1}} \left| \sum_{k=1}^n c_k m_s \frac{1-\chi_{m_s}^{a_s}}{1-\chi_{m_s}} \overline{\chi_{m_s}^{a_s}} \chi_k \right| \\
&= \sum_{\nu=1}^{p_{s+1}-1} \int\limits_{G_{s+1}} \left| \sum_{k=1}^n c_k m_s \frac{1-\chi_{m_s}^{a_s}(\nu g_s)}{1-\chi_{m_s}(\nu g_s)} \overline{\chi_{m_s}^{a_s}}(\nu g_s)\chi_k(\nu g_s)\chi_k(x) \right| \\
&\le \sum_{\nu=1}^{p_{s+1}-1} \int\limits_{G_{s+1}} \left| \sum_{k=1}^n B_{k,\nu,s}\chi_k(x) \right| \le \sum_{\nu=1}^{p_{s+1}-1} m_{s+1}^{-1/p} \left(\int\limits_{G_{s+1}} \left| \sum_{k=1}^n B_{k,\nu,s}\chi_k(x) \right|^{p'} \right)^{1/p'} \\
&\le \sum_{\nu=1}^{p_{s+1}-1} m_{s+1}^{-1/p} \left(\int\limits_{G} \left| \sum_{k=1}^n B_{k,\nu,s}\chi_k(x) \right|^{p'} \right)^{1/p'} \le \sum_{\nu=1}^{p_{s+1}-1} m_{s+1}^{-1/p} \left(\sum_{k=1}^n |B_{k,\nu,s}|^p \right)^{1/p} \\
&\le \sum_{\nu=1}^{p_{s+1}-1} m_s m_{s+1}^{-1/p} \left| \frac{1-e^{2\pi\nu a_s i/p_{s+1}}}{1-e^{2\pi\nu i/p_{s+1}}} \right| \left(\sum_{k=1}^n |c_k|^p \right)^{1/p} \\
&= \sum_{\nu=1}^{p_{s+1}-1} m_s m_{s+1}^{-1/p} \left| \frac{\sin\left(\frac{\pi\nu a_s}{p_{s+1}}\right)}{\sin\left(\frac{\pi\nu}{p_{s+1}}\right)} \right| \left(\sum_{k=1}^n |c_k|^p \right)^{1/p} \\
&\le \frac{m_s p_{s+1}}{2} m_{s+1}^{-1/p} \left(\sum_{\nu=1}^{p_{s+1}-1} \frac{1}{\nu} \right) \left(\sum_{k=1}^n |c_k|^p \right)^{1/p} \\
&\le C_2 m_{s+1}^{1/p'} \log\, p_{s+1} \left(\sum_{k=1}^n |c_k|^p \right)^{1/p},
\end{aligned}$$

where we used inequality $\left| \frac{\sin\left(\frac{\pi\nu a_s}{p_{s+1}}\right)}{\sin\left(\frac{\pi\nu}{p_{s+1}}\right)} \right| \le \frac{p_{s+1}}{2\nu}$ [18, (2.4)]. Hence,

$$I_3^{(1)} \le C_2 m_{s+1}^{1/p'} \log\, p_{s+1} \left(\sum_{k=1}^n |c_k|^p \right)^{1/p}. \tag{1.30}$$

$$I_3^{(2)} = \int\limits_{G_s\setminus G_{s+1}} \left|\sum_{k=1}^{n} c_k \left(\sum_{i=0}^{s-1} m_i \frac{1-\chi_{m_i}^{a_i}}{1-\chi_{m_i}} \overline{\chi_{m_i}^{a_i}}\right)\chi_k\right|$$
$$= \int\limits_{G_s\setminus G_{s+1}} \left|\sum_{k=1}^{n} c_k \left(\sum_{i=0}^{s-1} a_i m_i\right)\chi_k(x)\right|,$$

because $\chi_{m_i} \in G_s^{\perp}$ $(\forall i \le s-1)$. Set $B_{k,s} := c_k \sum_{i=0}^{s-1} a_i m_i$ and notice that $|B_{k,s}| \le m_s |c_k|$. Applying the Hölder inequality for integrals and then F. Riesz inequality, one obtains

$$I_3^{(2)} = \int\limits_{G_s\setminus G_{s+1}} \left|\sum_{k=1}^{n} B_{k,s}\chi_k(x)\right|$$
$$\le \left(\frac{1}{m_s} - \frac{1}{m_{s+1}}\right)^{1/p} \left(\int\limits_{G_s\setminus G_{s+1}} \left|\sum_{k=1}^{n} B_{k,s}\chi_k(x)\right|^{p'}\right)^{1/p'}$$
$$\le \left(\frac{1}{m_s} - \frac{1}{m_{s+1}}\right)^{1/p} \left(\int\limits_{G} \left|\sum_{k=1}^{n} B_{k,s}\chi_k(x)\right|^{p'}\right)^{1/p'} \tag{1.31}$$
$$\le \left(\frac{1}{m_s} - \frac{1}{m_{s+1}}\right)^{1/p} \left(\sum_{k=1}^{n} |B_{k,s}|^p\right)^{1/p}$$
$$\le m_s^{1/p'} \left(\sum_{k=1}^{n} |c_k|^p\right)^{1/p}.$$

Relations (1.31), (1.30) and (1.29) yield

$$I_3 \le 2C_2 \sum_{s=1}^{N} m_s^{1/p'} \log\, p_s \left(\sum_{k=1}^{n} |c_k|^p\right)^{1/p}. \tag{1.32}$$

From (1.32), (1.28), (1.27) and (1.26) follows

$$\left\|\sum_{k=1}^{n} c_k D_k\right\|_1 \le C\left[\log\, p_{N+1} \sum_{k=1}^{n} |c_k| + \sum_{s=1}^{N} m_s^{1/p'} \log\, p_s \left(\sum_{k=1}^{n} |c_k|^p\right)^{1/p}\right], \tag{1.33}$$

where C is an absolute constant (instead of C one can take $1+3/\log 2$, which can be proved by a simple calculation). From (1.33) and assumption (1.23) of the theorem, (1.25) follows.

This proves the theorem.

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