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SUPSETS ON PARTIALLY ORDERED TOPOLOGICAL LINEAR SPACES

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Abstract. We introduce supsets and infsets for subsets of a partially ordered topological linear space. These notions generalize the usual notions of supremum and infimum in Riesz spaces. We shall investigate properties of supsets and infsets in this paper.

1. Partially Ordered Linear Space and the Supset of a Subset

Let E be a linear space over the real field. Let us consider a convex cone P in E which is generating and proper. Namely, the following two conditions are satisfied:

(a) $E = P - P$,

(b) $P \cap (-P) = \{0\}$.

We say that $x \geq y$ (or, equivalently, $y \leq x$) if $x - y \in P$. It is well-known that conditions (a) and (b) are equivalent to the following five conditions for a given subset P of E:

(1) $x \geq y$ and $y \geq x$ imply $x = y$.

(2) $x \geq y$ and $y \geq z$ imply $x \geq z$.

(3) $x \geq y$ implies $x + z \geq y + z$ for all z in E.

(4) $x \geq 0$ implies $\alpha x \geq 0$ for all positive scalars α.

(5) For every x in E, there exist $x_1 \geq 0$ and $x_2 \geq 0$ such that $x = x_1 + x_2$.

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E is called a *partially ordered linear space* and P is called an *order* of E provided conditions (a) and (b) (or, equivalently, (1), (2), (3), (4) and (5)) are satisfied. Elements of P are said to be *positive* in E.

Let $A = \{a_\lambda : \lambda \in \Lambda\}$ be a subset of a partially ordered linear space E with order P. We define the *supset* of A (or sup A) to be $\bigvee A = \bigvee_{\lambda\in\Lambda} a_\lambda = \{z \in E : z \geq a_\lambda$ for all λ in Λ and $z = w$ whenever $z \geq w$ and $w \geq a_\lambda$ for all λ in $\Lambda\}$. Sometimes, we use the notation $\sup A$ instead of $\bigvee A$. Hence, $\bigvee A$ is the set of all minimal elements of $U\{A\} = \{z \in E : z \geq a\ \forall a \in A\}$. Elements of $U\{A\}$ are called *upper bounds* of A. Similarly, we define the *infset* $\bigwedge A$ (or inf A) of A to be $\bigwedge A = \{z \in E : z$ is a maximal element of $L\{A\}\}$, where $L\{A\} = \{z \in E : z \leq a\ \forall a \in A\}$ is the set of *lower bounds* of A. A is said to be *upper bounded* (resp. *lower bounded*) if $U\{A\} \neq \emptyset$ (resp. $L\{A\} \neq \emptyset$). We shall discuss when a upper bounded (resp. lower bounded) set has a nonvoid supset (resp. infset).

A partially ordered linear space E is said to satisfy *Condition* (A) if for every upper bound a of a subset A of E there exists a minimal element x in $U\{A\}$ such that $x \leq a$. E satisfies Condition (A) if and only if $\sup A + P = U\{A\}$ for all upper bounded subsets A of E. Later, we shall show that every finite-dimensional partially ordered linear space with a closed order always satisfies Condition (A). From here to the end of this section, we state some elementary observations, in which $A = \{a_\lambda : \lambda \in \Lambda\}$ is a subset of a partially ordered linear space E satisfying Condition (A).

Proposition 1.1. *If* $\sup A$ *is a singleton* $\{u\}$, *then* u *is the least upper bound of* A. *If* $\inf A$ *is a singleton* $\{l\}$, *then* l *is the greatest lower bound of* A.

The least upper bound of A is called the *supremum* of A. The greatest lower bound of A is called the *infimum* of A.

Proposition 1.2.

1. $-\bigvee_{\lambda\in\Lambda} a_\lambda = \bigwedge_{\lambda\in\Lambda}(-a_\lambda)$; *or, equivalently,* $-\sup A = \inf(-A)$.
2. *For every positive number* α, $\alpha \bigvee_{\lambda\in\Lambda} a_\lambda = \bigvee_{\lambda\in\Lambda} \alpha a_\lambda$ *or, equivalently,* $\alpha \sup A = \sup \alpha A$.
3. *For every positive number* α, $\alpha \bigwedge_{\lambda\in\Lambda} a_\lambda = \bigwedge_{\lambda\in\Lambda} \alpha a_\lambda$; *or, equivalently,* $\alpha \inf A = \inf \alpha A$.

Proposition 1.3. *For every* b *in* E, *we have*:

1. $\bigvee_{\lambda\in\Lambda} a_\lambda + b = \bigvee_{\lambda\in\Lambda}(a_\lambda + b)$ *or, equivalently,* $\sup A + b = \sup\{A + b\}$.
2. $\bigwedge_{\lambda\in\Lambda} a_\lambda + b = \bigwedge_{\lambda\in\Lambda}(a_\lambda + b)$ *or, equivalently,* $\inf A + b = \inf\{A + b\}$.

Proposition 1.4. $\sup A = \sup(\mathrm{co}A)$, *where* $\mathrm{co}A$ *is the convex hull of* A.

Proposition 1.5. *If* x *is a positive element of* E (*i.e.*, $x \in P$), *then* $x \vee 0 = \{x\}$ *and* $x \wedge 0 = \{0\}$.

Proposition 1.6. *If* $a \vee b \neq \emptyset$ *for some* a *and* b *in* E, *then* $a \wedge b \neq \emptyset$ *and*

$$a + b - (a \wedge b) = a \vee b.$$

Proof. Since $a \wedge b = -\{(-a) \vee (-b)\}$, we have by Propositions 1.2 and 1.3 that

$$a + b - (a \wedge b) = a + b + \{(-a) \vee (-b)\} = b \vee a = a \vee b. \qquad \blacksquare$$

The following example shows that $\sup\{\sup A\} \neq \sup A$ in general.

Example 1.7. Let E be the 3-dimensional Euclidean space R^3 with order P generated by four points $(1,0,0)$, $(1,1,0)$, $(1,0,1)$ and $(1,1,1)$. Let $z = (0,0,1)$ and $0 = (0,0,0)$. Let $A = \{z, 0\}$. Then $\sup A = \{a \in R^3 : a = (1, \alpha, 0), 0 \leq \alpha \leq 1\}$ and $\sup\{\sup A\} = \{a \in R^3 : a = (2,2,\beta), 1 \leq \beta \leq 2\}$. Hence, $\sup A \neq \sup\{\sup A\}$ in this case.

It may also happen that $\inf\{\inf A\} \neq \inf A$. However, we have the following

Theorem 1.8.

1. $\sup\inf\sup A = \sup A$.
2. $\inf\sup\inf A = \inf A$.

2. Monotone Complete Order and Dual Order

Let E be a partially ordered linear space. If E satisfies Condition (A), it is easy to see that the supset $\sup A$ of an upper bounded set A is not empty. Similarly, the infset $\inf A$ of a lower bounded subset A of E is not empty as well. In this section, we shall consider a sufficient condition to ensure that E satisfies Condition (A). To this end, we recall the notion of a monotone complete order.

A subset A of E is a *linear set* if every two elements x and y of A is comparable, *i.e.*, $x \leq y$ or $y \leq x$. We say that E is *monotone complete* (in an order P) if every upper bounded linear subset A of E has the least upper bound, *i.e.*, $\sup A \neq \emptyset$ consisting of a single element.

Let E be a topological linear space with a linear topology τ. An order of E determined by a convex cone P is called a *topologically continuous order* (or, equivalently, τ is called an *order continuous topology*) if every directed linear subset $\{a_\lambda\}$ with $\inf a_\lambda = 0$ converges to 0 in τ.

Theorem 2.1. *Let E be a partially ordered linear space with order continuous topology and ordered by a closed convex cone P. If E is monotone complete, then the supset* $\sup A$ *of every nonempty upper bounded subset A of E is not empty. Moreover, E has Condition (A). (However,* $\sup A$ *does not necessarily consist of a single element in this case.)*

Proof. Let A be a nonempty upper bounded subset of E. Hence $U\{A\}$ is not empty. Let $a \in U\{A\}$. Then we can find a maximal linear subset B of E which contains a and is contained in $U\{A\}$ by Zorn's maximal theorem. By monotone completeness of the order P, $\inf B = \{b\}$ is a singleton. Since the linear topology τ of E is order continuous and P is closed, we have $b \in U\{A\}$ and $b \leq a$. Hence b is a minimal element of $U\{A\}$ and thus $\sup A$ is not empty. ∎

In the following, E^* denotes the dual of a partially ordered normed space E. Let P^* be the positive cone dual to P in E^*, *i.e.*, $P^* = \{f \in E^* : f(x) \geq 0\ \forall x \in P\}$. By definition, $P^* \cap \{-P^*\} = \{0\}$. But P^* might not necessarily be an order in E^* in general. In fact, $P^* - P^*$ is not necessarily the whole of E^*. When E is a Banach space with closed order P, T. Ando [2] gave several equivalent conditions to ensure that P^* is an order in E^*, *i.e.*, $P^* - P^* = E^*$.

Theorem 2.2. *Let E be a Banach space with closed order P. If $P^* - P^* = E^*$, then E^* is monotone complete in the order determined by P^*. Moreover, the weak* topology of E^* is order continuous with respect to P^*. Hence,* $\sup A^*$ *is nonempty for every nonempty upper bounded subset A^* of E^*. In this case, E^* satisfies Condition (A) in the order P^*.*

Proof. By the definition of P^* and the theorem of Banach-Steinhaus, we see that E^* is monotone complete. Since the weak* topology of E^* is order continuous, the assertion follows from Theorem 2.1. ∎

It is shown in [2] that for a closed order P in a Banach space E, $E^* = P^* - P^*$ if and only if every order interval $[x, y] = \{z \in E : x \leq z \leq y\}$ in E is norm-bounded.

Corollary 2.3. *Every finite-dimensional Euclidean space E with a closed order P always satisfies Condition (A).*

3. Norm and Order

Let E be a partially ordered normed space. A norm $\|\cdot\|$ of E is called an *ordered norm* if $0 \leq x \leq y$ implies $\|x\| \leq \|y\|$. We shall investigate when a norm is equivalent to an ordered norm.

For a symmetric absorbing convex subset V of E, we shall define the *P-envelop* of V by

$$E_P(V) = (V - P) \cap (V + P).$$

Lemma 3.1.

$$E_P(E_P(V)) = E_P(V).$$

Proof. Let $U = (V+P)\cap(V-P)$. Then, $V \subset U$. Since $U+P \subset V+P+P = V + P$ and $U - P \subset V - P - P = V - P$, we have $(U + P) \cap (U - P) \subset (V + P) \cap (V - P) = U$. But, we always have $U \subset E_P(U)$. The assertion follows. ∎

Theorem 3.2. *If $U = E_P(V)$ is the unit ball of E in a norm $\|\cdot\|_U$, then this norm is an ordered norm.*

Proof. We shall show that if $x_1 \in U$ and $x_1 \geq x_2 \geq 0$ then $\|x_1\|_U \geq \|x_2\|_U$. In fact, the norm $\|\cdot\|_U$ is the Minkowski functional of E defined by U. It thus suffices to show that $\alpha x_1 \in U$ for some $\alpha \geq 1$ implies $\alpha x_2 \in U$. Since $x_1 = x_2 + p$ for some p in P, $\alpha x_2 = \alpha x_1 - \alpha p \in U - P$. On the other hand, $\alpha x_2 = x_2 + (\alpha - 1)x_2 \in U + P$. This means that $\alpha x_2 \in E_P(U) = U$ by Lemma 3.1. Therefore, $\|x_1\|_U \geq \|x_2\|_U$ as asserted. ∎

Theorem 3.3. *Let E be a partially ordered normed linear space with an order P. The norm of E is equivalent to an ordered norm if and only if $(V + P) \cap (V - P) \subset \alpha V$ for some $\alpha > 0$, where V is the unit ball of E.*

Proof. Suppose, without loss of generality, that the norm $\|\cdot\|$ of E with unit ball V is an ordered norm. Let $U = (V + P) \cap (V - P)$. Since $U = \bigcup\{z \in E : x_1 \leq z \leq x_2$ for some x_1, x_2 in V with $x_1 \leq x_2\}$, by the order interval relation $[x_1, x_2] = x_1 + [0, x_2 - x_1]$, we have

$$\|z\| \leq \|x_1\| + \|x_2 - x_1\| \leq 3 \quad \forall z \in U.$$

Hence $U \subset 3V$.

Conversely, if $U = (V + P) \cap (V - P) \subset \alpha V$ for some $\alpha > 0$, then the norm $\|\cdot\|_U$ of E with unit ball $U = (V + P) \cap (V - P)$ is equivalent to the original norm of E with unit ball V. Moreover, $\|\cdot\|_U$ is an ordered norm by Theorem 3.2. ∎

4. Riesz Spaces and Distributive Law

In this section, we shall consider the distributive law in a partially ordered linear space E. If E is a Riesz space, it is known that the distributive law holds in E. We shall consider when a partially ordered linear space E becomes a Riesz space. The following provide some criteria.

Proposition 4.1. *Let E be an n-dimensional Euclidean space with a closed order P. If P is generated by a set of n linearly independent elements of E, then E is a Riesz space.*

Corollary 4.2. *Let E be a 2-dimensional Euclidean space with a closed order P. Then E is a Riesz space.*

Theorem 4.3. *Let E be a Hausdorff topological linear space. Let P be an order in E such that $P \setminus \{0\}$ is open in E. If E has dimension greater than 2, then E cannot be a Riesz space in the order P.*

We shall make a remark here that if E is one-dimensional then E is a Riesz space in any order P and $P \setminus \{0\}$ is open in this case.

Proof. Suppose on the contrary that E is a Riesz space in P. At first we shall notice that the topological boundary of P relative to $P \setminus \{0\}$ is equal to $P^- \setminus P$. Since E has dimension greater than 2 and so $E \setminus \{0\}$ is connected, we can conclude that $P^- \setminus P \neq \emptyset$. So, there exists $0 \neq x \in P^- \setminus P$.

Let $y = x \vee 0 \in P$. Then $x < y$ and $0 < y$. Since $P \setminus \{0\}$ is open and $y \in P \setminus \{0\}$, there is a positive number α with $0 < \alpha < 1$ such that

$$z = \alpha x + (1 - \alpha)y \in P.$$

It is easy to see that $0 < z, x < z$ and $z = \alpha x + (1 - \alpha)y < y$. But this is a contradiction to the fact that $y =$ least upper bound for x and 0. This establishes our assertion. ■

We shall present an example of a closed order P in which a 3-dimensional Euclidean space is not a Riesz space.

Example 4.4. Let E be a 3-dimensional Euclidean space R^3. Let P be a generating proper convex cone in E generated by the 4 elements $(1, 0, 0)$, $(1, 0, 1)$, $(1, 1, 0)$ and $(1, 1, 1)$. Then, there is no least upper bound for the two elements $0 = (0, 0, 0)$ and $z = (0, 0, 1)$ of E. For example, both $a = (1, 0, 1)$ and $b = (1, 1, 1)$ are greater than 0 and z in the order P. But a and b are not comparable. This says that E is not a Riesz space in the order P.

We now consider the distributive law in partially ordered linear spaces.

Proposition 4.5. *Let E be a Riesz space. Then for all x_1, x_2 and y in E, we have*

$$(x_1 \vee x_2) \wedge y = (x_1 \wedge y) \vee (x_2 \wedge y).$$

Similarly, we also have

$$(x_1 \wedge x_2) \vee y = (x_1 \vee y) \wedge (x_2 \vee y).$$

However, the distributive law does not hold, in general, in a partially ordered linear space E. For every pair of subsets A and B of E, we define

$$A \vee B = \sup(A \cup B).$$

There are many possible ways to define $A \vee B$ other than the one stated above. In this paper, though, we consider only the above condition.

Proposition 4.6.

1. *$A \cup B = C \cup D$ implies $A \vee B = C \vee D$.*
2. *$A \vee B = B \vee A$.*
3. *$(A \vee B) \vee C = A \vee (B \vee C)$.*

We can also define $A \wedge B = \inf(A \cup B)$. With these definitions, we shall provide an example of a partially ordered linear space in which the distributive law holds for some elements x_1, x_2 and y.

Proposition 4.7. *Let E be a partially ordered space and $z \in E$. Then, $y \in z \vee (-z)$ implies $y \geq 0$.*

Proof. Let $y \in z \vee (-z) = (2z \vee 0) - z$. Then, we can find an a from $2z \vee 0$ such that $y = a - z = (1/2)a - z + (1/2)a$. Since $(1/2)a \geq z$ and $a \geq 0$, we conclude that $y \geq 0$. ∎

Example 4.8. Let E be a 3-dimensional Euclidean space with an order as in Examples 1.7 and 4.4. We shall show that the distributive law is true for some elements and false for others in E. Let $z = (0, 0, 1)$. Then

$$(z \wedge 0) \vee (-z \wedge 0) = (z \vee -z) \wedge 0 = 0.$$

But, if we take z, $(1/2)z = (0, 0, 1/2)$ and 0, then the distributive law fails to hold for these three elements.

In the following, we shall verify the converse of Proposition 4.5.

Theorem 4.9. *If the distributive law*

$$(x_1 \vee x_2) \wedge y = (x_1 \wedge y) \vee (x_2 \wedge y)$$

holds for all elements x_1, x_2 *and* y *in a partially ordered linear space* E, *then* E *is a Riesz space.*

Proof. Suppose that $x \wedge 0$ is a subset consisting of more than two elements and $y \leq x$ and $y \neq x$. We shall show that

$$((x \vee y) \wedge 0) \cap ((x \wedge 0) \vee (y \wedge 0)) = \emptyset.$$

In particular,

$$(x \vee y) \wedge 0 \neq (x \wedge 0) \vee (y \wedge 0).$$

Let $z \in ((x \vee y) \wedge 0) \cap ((x \wedge 0) \vee (y \wedge 0))$. Since $y \leq x$, we have $(x \vee y) \wedge 0 = x \wedge 0$. On the other hand, $z \in (x \wedge 0) \vee (y \wedge 0)$. It follows that $z \geq w$ for all w in $x \wedge 0$. Hence, z is the maximum of the subset $x \wedge 0$. It says that $x \wedge 0 = \{z\}$. This conflicts with the assumption that $x \wedge 0$ contains more than two elements. ■

5. Supsets for Two Non-comparable Elements

In this section, we shall consider $a \vee b$ for any non-comparable pair a and b of elements of a partially ordered Hausdorff topological linear space E. It is not easy to determine the exact form for $a \vee b$. In some cases, we can present $a \vee b$ in terms of boundary sets. To this end, we need to introduce some definitions.

Throughout this section, E is always an Euclidean space in a closed order P. A subset F of the order P is called a *face* if there exists a supporting hyperplane H of P such that $F = P \cap H$. We shall use the notation $\dim F$ as the dimension of the affine hull of F for a convex subset F of P.

Theorem 5.1. *Let* a *and* b *be any non-comparable pair of elements of a partially ordered Hausdorff topological linear space* E *in the order* P. *If* $\dim F \leq 1$ *for all faces* F *of* P, *then we have*

$$a \vee b = \partial(a + P) \cap \partial(b + P),$$

where ∂ *means boundary*, i.e., ∂C *is the topological boundary of a subset* C *of* E.

We prepare the proof with the following

Lemma 5.2. *If* $x \in \partial P$ *and* $0 \leq y \leq x$, *then* $y \in \partial P$.

Proof. If y belongs to the interior P° of P, then $x+(x-y) \in P$. It follows that $[y, x+x-y) \subset P^\circ$ since P is convex. Hence $x = (1/2)(y+x+x-y) \in P^\circ$, a contradiction. ∎

Proof of Theorem 5.1. Assume that $x_0 \in a \vee b$ and $x_0 \in (a+P)^\circ$, the interior of $a+P$. Since the affine hull of $a+P$ equals E, there exists a number $\lambda > 0$ such that $z = (1-\lambda)b + \lambda x_0 \in a+P$. Hence, $z < x_0$ and $z \in (a+P) \cap (b+P)$. As a result, x_0 is not a minimal element. This contradiction establishes that $x_0 \in \partial(a+P) \cap \partial(b+P)$.

Conversely, let $x_0 \in \partial(a+P) \cap \partial(b+P)$ and suppose $y_0 \leq x_0$ for some $y_0 \in U\{a, b\}$, the set of upper bounds of $\{a, b\}$. By virtue of the fact that $a \leq y_0 \leq x_0$ and Lemma 5.2, we have $y_0 \in [a, x_0] \subset \partial(a+P)$. Similarly, we have $y_0 \in [b, x_0] \subset \partial(b+P)$. Hence, we have $[a, x_0] \cap (a+P)^\circ = \emptyset = [b, x_0] \cap (a+P)^\circ$. By the separation theorem, there exist a closed hyperplane H_1 such that $[y_0, x_0] \subset H_1$ separating $a+P$, and another closed hyperplane H_2 such that $[y_0, x_0] \subset H_2$ separating $b+P$. By assumption, $H_1 \cap (a+P)$ is a half line which contains a and x_0. Also, $H_2 \cap (b+P)$ is a half line which contains b and x_0. Since a and b are not comparable in the order P, these two half lines have different directions. This means that $y_0 = x_0$ and so $x_0 \in a \vee b$. ∎

To illustrate Theorem 5.1, we consider the following

Example 5.3. Let E be the set of all Hermitian operators on a 2-dimensional Euclidean space. Let P be the set of all positive semi-definite operators. More precisely, E is considered as a 3-dimensional space in the order $P = \{(a, b, c) : a \geq 0, b \geq 0, ab \geq c^2\}$. It is easy to see that the dimension of every face of P is less than or equal to 1 and so the assumption of Theorem 5.1 is satisfied. For any $p = (a, b, c)$ in E with real coordinates a, b and c, we have

$$\begin{aligned} p \vee 0 = \{(x, y, z) : x \geq 0, y \geq 0, x-a \geq 0, y-b \geq 0, xy \\ = z^2, (x-a)(y-b) = (z-c)^2\}. \end{aligned}$$

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