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NONLINEAR ERGODIC THEOREMS FOR SEMIGROUPS OF NON-LIPSCHITZIAN MAPPINGS IN HILBERT SPACES

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Abstract. Let C be a nonempty subset (not necessarily closed and convex) of a Hilbert space, and $S = \{T(t); t \geq 0\}$ be a semigroup of non-Lipschitzian mappings on C. In this paper we study almost-convergence of almost-orbits of S.

1. Introduction and Preliminaries

Throughout this paper, let H be a real Hilbert space with inner product $(\cdot, \cdot)$ and norm $\|\cdot\|$, and let C be a nonempty subset of H. We do not assume that C is closed and convex. A family $S = \{T(t); t \geq 0)$ of mappings $T(t)$ is said to be a *semigroup* on C, if

(a_1) $T(t)$ is a mapping from C into itself for $t \geq 0$,

(a_2) $T(0)x = x$ and $T(t+s)x = T(t)T(s)x$ for $x \in C$ and $t,\ s \geq 0$

and

(a_3) for each $x \in C$, $T(\cdot)x$ is strongly measurable and bounded on every bounded subinterval of $[0, \infty)$.

For a semigroup S on C, we set $F = \{x \in C; T(t)x = x \text{ for all } t \geq 0\}$ and an element in F is called a *fixed point* of S.

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Lemma 2.2. *Let $u(\cdot)$ be a function satisfying* (2.1)*. Then we have the following* (I) *and* (II).

(I) *The following statements* (i), (ii) *and* (iii) *are mutually equivalent*:

(i) $\overline{\lim}_{s\to\infty}\overline{\lim}_{t\to\infty}\sup_{r\geq 0}[(u(t+r),u(t))-(u(s+r),u(s))]\leq 0$;

(ii) $\overline{\lim}_{s\to\infty}\overline{\lim}_{t\to\infty}\sup_{r\geq 0}[\|u(t+r)+u(t)\|^2-\|u(s+r)+u(s)\|^2]\leq 0$;

(iii) $\overline{\lim}_{s\to\infty}\overline{\lim}_{t\to\infty}\sup_{r\geq 0}[\|u(s+r)-u(s)\|^2-\|u(t+r)-u(t)\|^2]\leq 0$ *and* $\|u(t)\|$ *is convergent as* $t\to\infty$.

(II) *If* $u(\cdot)$ *satisfies the equivalent conditions in* (I), *then* $u(\cdot)$ *is strongly almost-convergent to its asymptotic center* y, *i.e.*,

$$\lim_{t\to\infty}(1/t)\int_0^t u(r+h)dr = y \text{ uniformly in } h\geq 0. \tag{2.10}$$

Proof. (I) is a direct consequence of the identity $\|u(s+r)\pm u(s)\|^2 = \|u(s+r)\|^2\pm 2(u(s+r),u(s))+\|u(s)\|^2$ for $s,r\geq 0$.

(II) Suppose that $u(\cdot)$ satisfies condition (i) in (I). It is easy to see that $\|u(t)\|$ is convergnt as $t\to\infty$. Since (i) implies (i$'$) in Remark 2.2, we see from Lemma 2.1$'$ that $u(\cdot)$ is weakly almost-convergent to its asymptotic center y and

$$\lim_{t\to\infty}(u(t),y)=\|y\|^2. \tag{2.11}$$

Set $y(t,h)=(1/t)\int_0^t u(r+h)dr$ for $t>0$ and $h\geq 0$. (2.10) holds if and only if $\lim_{n\to\infty}y(t_n,h_n)=y$ for every sequence $\{t_n\}$ with $t_n\to\infty, t_n>0$ and every sequence $\{h_n\}$ with $h_n\geq 0$.

Now, let $\{t_n\}$ and $\{h_n\}$ be sequences such that $t_n\to\infty, t_n>0$ and $h_n\geq 0$. We want to show

$$\lim_{n\to\infty}y(t_n,h_n)=y. \tag{2.12}$$

Since w-$\lim_{t\to\infty}y(t,h)=y$ uniformly in $h\geq 0$, we have w-$\lim_{n\to\infty}y(t_n,h_n)=y$ and therefore $\|y\|\leq\underline{\lim}_{n\to\infty}\|y(t_n,h_n)\|$. Therefore, to prove (2.12) it suffices to show the following

$$\overline{\lim_{n\to\infty}}\|y(t_n,h_n)\|\leq\|y\|. \tag{2.13}$$

Let $\varepsilon>0$ be arbitrarily given. By $\overline{\lim}_{s\to\infty}\overline{\lim}_{\tau\to\infty}\sup_{\eta\geq 0}[(u(\tau+\eta),u(\tau))-(u(s+\eta),u(s))]\leq 0$ (condition (i)) and (2.11), we can choose $s>0$ and $T(=T(s))>0$ such that $(u(s),y)<\|y\|^2+\varepsilon$ and

$$(u(\tau+\eta),u(\tau))-(u(s+\eta),u(s))<\varepsilon \text{ for } \tau\geq T \text{ and } \eta\geq 0. \tag{2.14}$$