TAIWANESE JOURNAL OF MATHEMATICS
Vol. 4, No. 1, pp. 105-117, March 2000

KINETIC CONDITION AND THE GIBBS FUNCTION

Fumioki Asakura

Dedicated to Professor Fon-Che Liu on his sixtieth birthday

Abstract. We study the Cauchy problem for a 3×3-system of conservation laws describing the phase transition: $u_t - v_x = 0$, $v_t - \sigma(u)_x = 0$, $(e + \frac{1}{2}v^2)_t - (\sigma v)_x = 0$. A phase boundary is said to be admissible if it satisfies the Abeyaratne-Knowles kinetic condition. We give a physical account of the kinetic condition by means of the *Gibbs function*. We also obtain a useful description of the entropy function using the Gibbs function.

1. Introduction

We study the model equations of thermo-elasticity in Lagrangian form:

$$u_t - v_x = 0, \quad v_t - \sigma_x = 0, \tag{1}$$

$$\left(e + \frac{1}{2}v^2\right)_t - (\sigma v)_x = 0, \tag{2}$$

where u: strain, σ: stress, v: velocity, e: internal energy which is a function of u and the specific entropy η. These quantities satisfy the fundamental thermodynamic identity:

$$de = Td\eta + \sigma du, \tag{3}$$

where T is the temperature. If η is a constant, the system of equations is reduced to the simpler system (1), which is called the *isentropic equations*.

Received December 9, 1999.
Communicated by P. Y. Wu.
2000 *Mathematics Subject Classification*: 35L65, 35L67, 35L45.
Key words and phrases: Hyperbolic system, conservation law, phase boundary, entropy.

When T is a constant, it follows from the basic identity (3) that the energy function is linear in η and σ depends only on u:

$$e = T\eta + \psi(u), \quad \psi(u) = \int \sigma(u)\, du. \tag{4}$$

Hence, the two equations (1) are equivalent to the isentropic equations involving only u and v; the third equation (2) is in the form:

$$T\eta_t = -\left(\psi(u) + \frac{1}{2}v^2\right)_t + (\sigma v)_x \tag{5}$$

whose right-hand side depends only on u and v.

We are mainly concerned with these *isothermal* elastic materials. If $\sigma' > 0$, then the system of equations constitutes a hyperbolic system. Chen-Dafermos [7] showed that a global-in-time solution exists for given initial data, provided $\sigma'' > 0$ for $x > x_0$ and $\sigma'' < 0$ for $x < x_0$. They used the method of compensated compactness, hence the entropy function η obtained is a measure. Bereux-Bonnetier-LeFloch [6] gave certain conditions for η to be a function. Here, we study the material which undergoes a crystal-elastic phase transition. We assume that the stress $\sigma = \sigma(u)$ is a C^2-function and there exist $\alpha,\ \beta$ ($\alpha < \beta$) such that

$$\sigma'(u) = \begin{cases} > 0 & \text{for} \quad u < \alpha, \\ < 0 & \text{for} \quad \alpha < u < \beta, \\ > 0 & \text{for} \quad u > \beta. \end{cases} \tag{6}$$

Abeyaratne-Knowles [1, 2, 3] studied models consisting of piecewise linear stress functions. We note that the system of equations (1), (2) is *hyperbolic* for $u < \alpha,\ u > \beta$ and *elliptic* for $\alpha < u < \beta$; the region $\Omega_\alpha = \{(u, v) :\ u < \alpha\}$ is called the *α-phase* and $\Omega_\beta = \{(u, v) :\ u > \beta\}$ the *β-phase*. A discontinuous solution connecting states in Ω_α and Ω_β is called a *phase boundary* (see Menikoff-Plohr [17] for general accounts). Since the discontinuous solution is not unique for given initial data, we have to impose a physically relevant condition for the discontinuity. For example, we have a lot of stationary discontinuous solutions connecting two states, but the *Maxwell states*: only ones satisfying the equal-area principle are important. For shock waves in the hyperbolic region, we adopt the Lax entropy condition (see Lax [13, 14]). Hattori [11, 12] adopted the entropy rate criterion of Dafermos [9] and showed that the Maxwell states are admissible in this sense. In [1, 2], Abeyaratne-Knowles proposed the *kinetic condition* and singled out a unique admissible solution. This condition is very convenient in the hyperbolic theory: LeFloch [15] formulated a weak form of the condition and obtained global solutions for given initial data to isentropic equations with piecewise linear stress function; Asakura [5] (and

a brief note [4]), obtained global solutions whose initial data are perturbations of the Maxwell states to equations with nonlinear stress function.

In this paper, we first give a form of the kinetic condition using the Gibbs function. Using this form we generalize the existence theorem in [5] for non-isentropic equations and derive a description of the entropy function.

2. Model Equations

In this section we shall derive the basic equations (1), (2) of 1-dimensional motion of an elastic bar having constant cross-sectional area A. Assume that the bar is laid along the x-axis in a reference configuration and set in longitudinal motion. During such motion, a particle initially located at x moves to $X = x + \widehat{u}(x,t)$ at time t; $\widehat{u}(x,t)$ is called the *displacement*. Amount of relative deformation is computed as

$$\frac{\Delta X - \Delta x}{\Delta x} = \widehat{u}_x + O(1)\Delta x.$$

We set $u = \widehat{u}_x$: the *strain* and $v = \widehat{u}_t$: the *particle velocity*. Since the mapping $x \to X$ is one-to-one, we require $u > -1$. Let ρ_0 denote the *mass density* (constant) in the reference configuration, σ the *stress* and e the *internal energ*. Then the balance of momentum and the conservation of the total energy are expressed by

$$\frac{d}{dt}\int_{x_1}^{x_2} \rho_0 v\, dx = \sigma(x_2,t) - \sigma(x_1,t), \tag{7}$$

$$\frac{d}{dt}\int_{x_1}^{x_2} \rho_0 \left(e + \frac{1}{2}v^2\right) v\, dx = \sigma(x_2,t)v(x_2,t) - \sigma(x_1,t)v(x_1,t). \tag{8}$$

We may assume that the coordinates are *Lagrangian* by setting $x \to \rho_0 x + x_0$ and hence $\rho_0 = 1$. Dividing by $x_2 - x_1$ and letting $x_2 \to x_1$, we obtain the second equation of (1) and the equation (2); the first one is the compatibility condition.

Let us consider a small deformation $X = x + \Delta\widehat{u}(x)$. The work done in the portion $[x_1, x_2]$ of this elastic material is expressed as $\Delta W = -K\Delta L$, where $K = \sigma A$: the tension force and $\Delta L = X_2 - X_1 - (x_1 - x_2)$: the elongation. We denote by E and S, respectively, the total internal energy and the total entropy. The first and second laws of thermodynamics (see Fermi [10]) show that

$$\begin{aligned} \Delta E &= T\Delta S + K\Delta L \\ &= T\Delta S + \sigma A\{\Delta\widehat{u}(x_2) - \Delta\widehat{u}(x_1)\}. \end{aligned}$$

Then dividing the above expression by $A(x_2 - x_1)$ and letting $x_2 \to x_1$, we have

$$\Delta e = T\Delta\eta + \sigma\Delta\widehat{u}_x.$$

Thus, using differentials, we can express the first and second laws of thermodynamics as

$$de = Td\eta + \sigma du. \tag{9}$$

First, let u and η be independent variables. Expression (9) shows that

$$\sigma = \left(\frac{\partial e}{\partial u}\right)_\eta, \quad T = \left(\frac{\partial e}{\partial \eta}\right)_u. \tag{10}$$

Introducing the *free energy* $\psi = e - T\eta$, we choose u and T as independent variables and obtain

$$d\psi = -\eta dT + \sigma du. \tag{11}$$

Hence

$$\sigma = \left(\frac{\partial \psi}{\partial u}\right)_T, \quad \eta = -\left(\frac{\partial \psi}{\partial T}\right)_u. \tag{12}$$

When the elastic material is in contact with a heat reservoir which is at the constant temperature T, such motion is called *isothermal*. It follows from (10) that e is linear in η:

$$e = T\eta + e_0(u),$$

where e_0 coincides with the free energy ψ. We have also by (10) that $e_0' = \sigma$. Thus we have

Proposition 2.1. *If the motion is isothermal, the stress σ and the free energy ψ depend only on the strain u; the internal energy e and ψ have the form*

$$e = T\eta + \psi(u), \quad \psi(u) = \int \sigma(u)\, du. \tag{13}$$

3. Phase Boundaries and the Kinetic Condition

The *Riemann problem* is the initial value problem with the initial data:

$$(u, v, \eta)|_{t=0} = \begin{cases} (u_L, v_L, \eta_L) & x < 0, \\ (u_R, v_R, \eta_R) & x > 0. \end{cases} \tag{14}$$

where $u_L, v_L, \eta_L, u_R, v_R, \eta_R$ are constants. The jump discontinuity of the form:

$$(u, v, \eta) = \begin{cases} (u_-, v_-, \eta_-) & x < st, \\ (u_+, v_+, \eta_+) & x > st, \end{cases} \tag{15}$$

is a weak solution ($u_\pm, v_\pm, \eta_\pm$: constants) if and only if it satisfies the *Rankine-Hugoniot condition* :

$$\begin{aligned} s(u_+ - u_-) &= -(v_+ - v_-), \\ s(v_+ - v_-) &= -(\sigma_+ - \sigma_-), \\ s(e_+ + \tfrac{1}{2}v_+^2 - e_- - \tfrac{1}{2}v_-^2) &= -(\sigma_+ v_+ - \sigma_- v_-). \end{aligned} \tag{16}$$

In particular, thermodynamic quantities $e_\pm, \sigma_\pm, u_\pm$ must satisfy the *Hugoniot equation* :

$$e_+ - e_- = \frac{1}{2}(\sigma_+ + \sigma_-)(u_+ - u_-). \tag{17}$$

The jump discontinuity of the form (15) is said to be a *phase boundary* if the states belong to different phases and satisfy the Rankine-Hugoniot condition (16). Phase boundaries represent crystal-elastic states. These states are in thermo-elastic equilibrium if and only if the strains satisfy

$$\sigma(u_+) = \sigma(u_-), \quad \int_{u_-}^{u_+} \sigma(u)du = \sigma_\pm(u_+ - u_-). \tag{18}$$

Unique strains determined by this principle are called the *Maxwell strains*, denoted by $u_- = u_m$, $u_+ = u_m^*$. By setting $v_+ = v_-$ (denoted by v_m), $U_m = (u_m, v_m)$, $U_m^* = (u_m^*, v_m)$ constitute stationary solutions that are called *Maxwell states*.

Suppose that the material is in thermal contact with a heat reservoir which is at a constant temperature T and at a constant stress σ. We define the *Gibbs free energy* G by

$$G = E - TS - KL.$$

Then the change of entropy is

$$\Delta S = -\frac{\Delta G}{T}. \tag{19}$$

Hence we can deduce that this material is at thermal equilibrium if and only if the Gibbs free energy attains its minimum (see Reif [19]).

Now we assume that the material is composed of two phases: mass M_α of α-phase and M_β of β-phase. Then G can be written as

$$G = M_\alpha g_\alpha + M_\beta g_\beta,$$

where g_α and g_β are the *specific Gibbs potentials* of each phase. Since $M_\alpha + M_\beta = M$: constant, M_α is taken to be the independent variable. Then it follows that

$$\Delta G = (g_\alpha - g_\beta)\Delta M_\alpha. \tag{20}$$

Hence, denoting the mass density of the α-phase by m_α, we have

Proposition 3.1.

$$de = Td\eta + \sigma du + (g_\alpha - g_\beta)dm_\alpha. \tag{21}$$

This is the fundamental law of thermodynamics by taking the phase transition into account.

For simplicity, we consider the case where $du = 0$. We suppose that the material has a transitional region having a cross-section A and a small thickness L_0. Moreover, the material is supposed to be in α-phase on the left side of the transition region and β-phase on the right side. Recall that our coordinate is Lagrangian: $\Delta M_\alpha = A\Delta x$; assume that $|\Delta x| \ll L_0$ and the phase transition occurs in quasi-static manner. Then it follows from (19) and (20) that

$$\begin{aligned} T\Delta\eta &= -(g_\alpha - g_\beta)\frac{\Delta M_\alpha}{AL_0} \\ &= -(g_\alpha - g_\beta)\frac{\Delta x}{L_0}. \end{aligned}$$

Hence the chemical work ΔW exerted for the transition from β-phase into α-phase is

$$\Delta W = (AL_0)T\Delta\eta = -A(g_\alpha - g_\beta)\Delta x.$$

Dividing both sides by Δx and letting $\Delta x \to 0$, we have

$$-\frac{dW}{dx} = A(g_\alpha - g_\beta). \tag{22}$$

Thus, letting $L_0 \to 0$, we find that $A(g_\alpha - g_\beta)$ is the chemical force acted on the phase boundary.

Now we postulate that the rate of the change of the mass M_α is a function of the chemical force:

$$\frac{dM_\alpha}{dt} = \Phi(A(g_\alpha - g_\beta)),$$

where $\Phi(\theta)$ satisfies $\Phi'(\theta) \geq 0,\ \Phi(0) = 0$. Hence we have

$$\frac{dx}{dt} = \frac{1}{A}\Phi(A(g_\alpha - g_\beta)), \tag{23}$$

which is the *kinetic condition* on the phase boundary.

Let us consider the jump discontinuity (15). The *mechanical Gibbs function* $\widehat{g}$ is defined by

$$\widehat{g} = e + \frac{1}{2}\widehat{s}^2 - T\eta - \sigma u, \tag{24}$$

where $\widehat{s} = su$ is the Eulerian propagation speed. Clearly, g depends only on u.

Proposition 3.2. *If the motion is isothermal, we have*

$$\widehat{g}_+ - \widehat{g}_- = \int_{u_-}^{u_+} \sigma(u)du - \frac{1}{2}(\sigma_+ + \sigma_-)(u_+ - u_-). \tag{25}$$

Proof. It follows from Proposition 2.1 that

$$\begin{aligned}
\widehat{g}_+ - \widehat{g}_- &= \psi_+ - \psi_- + \frac{1}{2}s^2(u_+^2 - u_-^2) - \sigma_+u_+ + \sigma_-u_- \\
&= \int_{u_-}^{u_+} \sigma(u)\,du + \frac{1}{2}(\sigma_+ - \sigma_-)(u_+ + u_-) - \sigma_+u_+ + \sigma_-u_- \\
&= \int_{u_-}^{u_+} \sigma(u)du - \frac{1}{2}(\sigma_+ + \sigma_-)(u_+ - u_-).
\end{aligned}$$

Thus we have the proposition. ∎

If $s = 0$, then $\widehat{g} = g$. We find by (18) that the Maxwell strains satisfy

$$g_+ = g_-. \tag{26}$$

From now on, we simply denote $\widehat{g} = g$ and call this quantity the *Gibbs function*. We say that the phase boundary (14) is admissible if and only if it satisfies the Abeyaratne-Knowles *kinetic equation*

$$s_\phi = \Phi(g_+ - g_-), \tag{27}$$

where $\Phi(\theta)$ is a nondecreasing function satisfying $\Phi(0) = 0$. This condition is a generalization of (26) to propagating phase boundaries and $A^{-1}\Phi(A\theta)$ is denoted simply by $\Phi(\theta)$. The quantity $g_+ - g_-$ is called the *driving traction* denoted by $f(u_+, u_-)$ in Abeyaratne-Knowles [1, 2, 3].

Finally, we note that the Hugoniot equation (17) is expressed as

$$\eta_+ - \eta_- = -\frac{1}{T}(g_+ - g_-), \tag{28}$$

which will be useful in the following sections.

4. Isentropic Equations

Let us first study the isentropic equations (1). For technical reasons, we assume that

$$\sigma''(u) \neq 0, \quad \text{for} \quad u < \alpha,\ u > \beta. \tag{29}$$

Existence and large time stability of weak solutions are obtained in Asakura [5], which are briefly reviewed in this section.

We can solve the Riemann problem in this case with the initial data

$$(u, v)|_{t=0} = \begin{cases} (u_L, v_L) & x < 0, \\ (u_R, v_R) & x > 0. \end{cases} \tag{30}$$

Hereafter, we always assume that Φ is a C^2-function and

$$\Phi'(0) > 0. \tag{31}$$

Following Abeyaratne-Knowles, we also assume that the new phase does not occur from any point in the interior of the hyperbolic regions, which is called the *nucleation condition*. Suppose that $U_L = (u_L, v_L)$ and $U_R = (u_R, v_R)$ are contained in small neighborhoods of the Maxwell states $U_m = (u_m, v_m)$ and $U_m^* = (u_m, v_m^*)$, respectively. Then we can construct a self-similar solution which consists of 4 constant regions U_L, $U_- \in \Omega_\alpha$ and U_+, $U_R \in \Omega_\beta$, where U_L, U_- are connected by a 1-rarefaction wave or shock wave in Ω_α, U_-, U_+ by a single phase boundary and U_+, U_R by a 2-rarefaction wave or shock wave in Ω_β. However the solution of this form is not unique and these solutions constitute a one-parameter family.

Now we consider the kinetic equation (27) with (31). This condition, together with the Rankine-Hugoniot condition, determines u_+ as a function of u_-. We proved in [5] the following.

Lemma 4.1. *If $\Phi(0) = 0$ and $\Phi'(0) > 0$, then u_+ and s_ϕ are functions of u_- in a neighborhood of u_m and*

$$\left.\frac{du_+}{du_-}\right|_{u_-=u_m} = \frac{\sigma'(u_m)}{\sigma'(u_m^*)}, \quad \left.\frac{ds_\phi}{du_-}\right|_{u_-=u_m} = -\Phi'(0)\sigma'(u_m)(u_m^* - u_m). \tag{32}$$

We note that the derivative of u_+ does not depend on the particular value of $\Phi'(0)$ and coincides with Hattori's description coming from the maximum entropy dissipation rate admissibility condition. By using these relations, the following existence theorem is obtained.

Theorem 4.1 [5, Theorem 2.1]. *If* $|U_L - U_m|,\ |U_R - U_m^*|$ *are sufficiently small, then there exists a unique admissible solution which consists of* 4 *constant regions connected by rarefaction waves, shock waves and a phase boundary. Moreover these constant states are differentiable with respect to the initial data* $U_L,\ U_R$.

These solutions are used to construct approximate solutions with general initial data $U_0(x) = U_R(x)$ for $x > 0,\ U_L(x)$ for $x < 0$. We have the following existence theorem of global solutions by the wave-front tracking alternative of the Glimm method. We use important ideas of Chern [8] in estimating approximate solutions.

Theorem 4.2 [5, Theorems 2.2, 2.3]. *Suppose that the initial perturbation is sufficiently small in total variation, i.e., the quantity*

$$T.V.(U_L(x) - U_m)|_{x<0} + T.V.(U_R(x) - U_m^*)|_{x>0} \tag{33}$$

is sufficiently small. Then there exists a weak global solution with a single phase boundary which is Lipschitz continuous curve in (x,t)*-space. Moreover, the limit*

$$\lim_{\delta \to \pm 0} U(\chi(t) + \delta, t) = U_{\pm}(t) \tag{34}$$

exists except for countably many t *at the phase boundary; the Rankine-Hugoniot conditions* (16) *and the kinetic equation*

$$\dot{\chi}(t) = \Phi(g_+(t) - g_-(t)) \tag{35}$$

hold at these points.

5. Non-Isentropic Equations

Using the solution to the isentropic equations, we have a unique admissible solution to the Riemann problem for the general isothermal system (1), (2).

Theorem 5.1. *If the Riemann initial data* $(u_L,\ v_L,\ \eta_L)$ *and* $(u_R,\ v_R,\ \eta_L)$ *are close to the Maxwell states, then there exists a unique admissible solution which consists of* 5 *constant regions connected by rarefaction waves, shock waves and a phase boundary. Moreover these constant states are Lipschitz continuous with respect to the initial data.*

Proof. Theorem 4.1 shows that for given $U_L = (u_L, v_L) \in \Omega_\alpha$ and $U_R = (u_R, v_R) \in \Omega_\beta$ such that $|U_L - U_m|,\ |U_R - U_m^*|$ are sufficiently small, there

exists a unique admissible solution to the Riemann problem with initial data (30); this solution consists of 4 constant regions: U_L, U_R, U_+ and U_- connected by rarefaction waves, shock waves and a phase boundary. By the Hugoniot equation (28), $\eta_\pm$ are defined by

$$\begin{aligned} \eta_- - \eta_L &= -\frac{1}{T}(g_- - g_L), \\ \eta_+ - \eta_R &= -\frac{1}{T}(g_+ - g_R). \end{aligned} \tag{36}$$

Moreover, we define $\overline{\eta}_\pm$ as the following:

$$\begin{aligned} \overline{\eta}_- &= \begin{cases} \eta_L - \frac{1}{T}(g_- - g_L) - \frac{1}{T}(g_+ - g_-) & g_+ < g_-, \\ \eta_L - \frac{1}{T}(g_- - g_L) & g_+ \geq g_-, \end{cases} \\ \overline{\eta}_+ &= \begin{cases} \eta_R - \frac{1}{T}(g_+ - g_R) & g_+ < g_-, \\ \eta_R - \frac{1}{T}(g_+ - g_R) - \frac{1}{T}(g_- - g_+) & g_+ \geq g_-. \end{cases} \end{aligned} \tag{37}$$

Then $\overline{\eta}_- - \eta_-$ $(g_+ < g_-)$ and $\overline{\eta}_+ - \eta_+$ $(g_+ > g_-)$ are the changes of entropy across the phase boundary and η defined by

$$\eta = \begin{cases} \overline{\eta}_- & x < 0, \\ \overline{\eta}_+ & x > 0 \end{cases} \tag{38}$$

together with $\overline{\sigma}_\pm = \sigma_-$ (or σ_+) and $\overline{v}_\pm = v_-$ (or v_+) constitutes a contact discontinuity which completes the solution. It is easy to see that the above constant states are Lipschitz functions of the initial data.

These Riemann solutions constitute approximate solutions with general initial data

$$(u, v, \eta) = \begin{cases} (u_L(x), v_L(x), \eta_L(x)) & x < 0, \\ (u_R(x), v_R(x), \eta_R(x)) & x > 0. \end{cases} \tag{39}$$

Since the equations (1) are equal to the isentropic equations, we find by Theorem 4.2 that there exist weak global solutions u and v with a single phase boundary provided the initial data are close to the Maxwell states and the perturbation is sufficiently small in total variation. Hence we have only to construct the entropy η.

Theorem 5.2. *For given initial data with bounded total variation, the specific entropy exists and has the form* :

$$\eta(x,t) = \eta_0(x) - \frac{1}{T}\int_0^t d_\tau g(u(x,\tau)) \tag{40}$$

for almost all x.

Proof. Estimates for $T.V.u(*,t)$ and $T.V.v(*,t)$ are contained in the proof of Theorem 4.2. Using the expressions (36), (37), we have

$$\begin{aligned} & |\eta_+ - \eta_R| + |\eta_- - \eta_L| + |\overline{\eta}_+ - \eta_+| + |\overline{\eta}_- - \eta_-| + |\overline{\eta}_+ - \overline{\eta}_-| \\ = \; & \frac{1}{T}(|\eta_R - \eta_L| + 2|g_+ - g_-| + 2|g_+ - g_R| + 2|g_- - g_L|), \end{aligned} \tag{41}$$

that is estimated by $T.V.u(*,t) + T.V.v(*,t)$. Hence we obtain the estimates of $T.V.\eta(*,t)$.

The constructions (36), (37) of η indicate that the front-tracking approximation η^h will be defined by

$$\eta^h(x,t) = \eta_0^h(x) - \frac{1}{T}\sum_l \left[g^h\right]_{x=x_l} \tag{42}$$

for all x for which the vertical line through $(x,0)$, denoted by l_x, does not lie on stationary ($s=0$) phase boundaries and the summation runs over all discontinuities $x = x_l(t)$ that meet the line l_x. On the jump discontinuities, η^h is defined to be the mean value of both sides.

Let $t_1, t_2, \ldots, t_n, \ldots$ denote the collision times and $x = x_l(t)$ (a line with gradient s_l) the wave front at J_m. We have, for C^1-functions ϕ with bounded support,

$$\begin{aligned} & \int\int_{R_+\times[0,T]} \{\phi_t(T\eta^h + \frac{1}{2}(v^h)^2) + \psi^h) - \phi_x(\sigma^h v^h)\}\, dxdt + \int_R \phi(x,0)\eta_0^h(x)\, dx \\ = & \sum_l \int_{t_m}^{t_{m+1}} \left\{ s_l[T\eta^h + \psi^h + \tfrac{1}{2}(v^h)^2]_{x=x_l} + [\sigma^h v^h]_{x=x_l} \right\} \phi(x_l,t)\, dt \\ = & \sum_l \int_{t_m}^{t_{m+1}} s_l \left\{ T\left[\eta^h\right]_{x=x_l} + \left[g^h\right]_{x=x_l} \right\} \phi(x_l,t)\, dt = 0. \end{aligned}$$

Thus we find that the limit function satisfies the equation in the weak sense and has the form (40).

Remark 5.1. The description (40) of the entropy function η is also valid for the large amplitude solutions obtained by Nishida [18] for model equations of isothermal ideal gas.

Remark 5.2. We find by (40) that the limit

$$\eta_\infty(x) = \lim_{t\to\infty} \eta(x,t)$$

exists for almost all x. Moreover, Liu [16] shows that $\eta(*,t)$ converges to η_∞ in $L^1(R)$ at the rate $t^{-1/2}$.

References

1. R. Abeyaratne and J. K. Knowles, On the dissipative response due to discontinuous strains in bars of unstable elastic material, *Internat. J. Solids Structures* **24** (1988), 1021-1044.
2. R. Abeyaratne and J. K. Knowles, Kinetic relations and the propagation of phase boundaries in solids, *Arch. Rational Mech. Anal.* **114** (1991), 119-154.
3. R. Abeyaratne and J. K. Knowles, Dynamics of propagating phase boundaries: adiabatic theory for thermoelastic solids, *Phys. D* **79** (1994), 269-288.
4. F. Asakura, Large time stability of propagating phase boundaries, in: *Hyperbolic Problems*: *Theory*, *Numerics*, *Applications*, M. Fey, R. Jeltsch, eds., 7th International Conference in Zürich 1998, Vol. 1, Birkhäuser, 1999, pp. 21-30.
5. F. Asakura, Large time stability of Maxwell phase boundaries, *Methods Appl. Anal.* **6** (1999).
6. F. Bereux, E. Bonnetier and P. LeFloch, Gas dynamics system: two special cases, *SIAM J. Math. Anal.* **28** (1997), 499-515.
7. G.-Q. Chen and C. Dafermos, The vanishing viscosity method in one-dimensional thermoelasticity, *Trans. Amer. Math. Soc.* **147** (1995), 531-541
8. I.-L. Chern, Stability theorem and truncation error analysis for the Glimm scheme and for a front tracking method for flows with strong discontinuities, *Comm. Pure Appl. Math.* **42** (1989), 815-844.
9. C. M. Dafermos, The entropy rate admissibility criterion for solutions of hyperbolic conservation laws, *J. Differential Equations* **14** (1973), 202-212.
10. E. Fermi, *Thermodynamics*, Dover Pub., New York, 1956.
11. H. Hattori, The Riemann problem for a van der Waals fluid with entropy rate admissible criterion – isothermal case, *Arch. Rational Mech. Anal.* **92** (1986), 246-263
12. H. Hattori, The Riemann problem and the existence of weak solutions to a system of mixed-type in dynamic phase transition, *J. Differential Equations* **146** (1998), 287-319.
13. P. D. Lax, Shock waves and entropy, in: *Contributions to Nonlinear Functional Analysis*, E. Zarantonello, ed., Academic Press, New York, 1971, pp. 603-634.
14. P. D. Lax, *Hyperbolic Systems of Conservation Laws and the Mathematical Theory of Shock Waves*, CBMS-NSF Regional Conference Series in Applied Mathematics 11, SIAM, 1973.

15. P. LeFloch, Propagating Phase boundaries: formulation of the problem and existence via the Glimm method, *Arch. Rational Mech. Anal.* **123** (1993), 153-197.

16. T.-P. Liu, Decay to N-waves of solutions of general systems of nonlinear hyperbolic conservation laws, *Comm. Pure Appl. Math.* **30** (1977), 585-610.

17. R. Menikoff and B. J. Plohr, The Riemann problem for fluid flow of real materials, *Rev. Modern Phys.* **61** (1989), 75-130.

18. T. Nishida, *Nonlinear Hyperbolic Equations and Related Topics in Fluid Dynamics*, Publ. Math. No. 79-02, Univ. Paris-Sud, 1978.

19. F. Reif, *Statistical Physics*, Berkeley Physics Course, Vol. 5, McGraw-Hill, 1965.

Faculty of Engineering, Osaka Electro-Communication University
18-8 Hatsucho, Neyagawa, Osaka 572-8530, Japan
E-mail: asakura@isc.osakac.ac.jp