

## SOME FRACTAL PROPERTIES OF BROWNIAN PATHS

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Dedicated to Professor Fon-Che Liu on his sixtieth birthday

**Abstract.** In this paper, we survey some recent results concerning the fractal structure of Brownian sample paths. The following aspects are discussed: (1) average densities of Brownian trails and intersections; (2) dimension spectra of Brownian zeroes; (3) multifractal properties of Brownian substitutions.

### 1. INTRODUCTION

Brownian motion is referred to as a highly irregular motion proceeded by a small particle in some medium, which was first observed by the British botanist Robert Brown. The mathematical formulation is due to Norbert Wiener, which we describe as follows. Let  $B_d = \{B_d(t)\}$  be a stochastic process defined on some probability space  $(\Omega, P)$  and taking values in  $R^d$ . We say that  $B_d$  is the *standard  $d$ -dimensional Brownian motion* if it satisfies the following:

- (i) the sample path  $t \rightarrow B_d(t, \omega), \omega \in \Omega$ , is continuous;
- (ii) the increments  $B_d(t_j) - B_d(t_{j-1}), 1 \leq j \leq k, t_0 = 0$ , are stationary and (stochastically) independent for all  $k \geq 2$ ;
- (iii) the distribution of  $(B_d(t) - B_d(0))/\sqrt{t}$  is standard normal in  $R^d$ ;
- (iv)  $B_d(0) = 0$ .

The Brownian motion process (also named *Wiener process*) described above is (*stochastically self-similar of index 1/2*) by which it means that, for any  $c > 0$ ,

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the time-scaled process  $\{B_d(ct)\}$  and the space-scaled process  $\{\sqrt{c}B_d(t)\}$  are equivalent in the sense of finite-dimensionally distributional equivalence. This self-similar property is central in our study, from which various dimension formulae concerning Brownian paths can be figured out.

Brownian sample paths exhibit highly erratic patterns, despite the continuity; we can appreciate such pictures from many books on stochastic processes. Thus, this should be a rich source of fractal analysis/geometry and be one prevailing topic in nonlinearity (even though the transition density of the process is just heat kernel). We are concerned with the following fractals:

$$\begin{aligned} [B_d] &= \{x : x = B_d(t) \text{ some } t\}, \\ Z &= \{t : B_1(t) = 0\}, \\ I_d &= [B_d] \cap [B'_d]. \end{aligned}$$

In the above,  $B'_d$  denotes an independent copy of  $B_d$ . Thus, the three sets are simply the trail (range), the zero set, and the set of intersections of Brownian paths. Note that the zero set is meaningful only for the 1-dimensional case, while the trail and the intersection are meaningful only for the multi-dimensional case. These are due to the fact that the 1-dimensional Brownian motion is point-recurrent while it is not so for the multi-dimensional case. These sets are random, since they depend on a particular sample path realization  $B_d(t, \omega)$ , and so we must interpret any statement about these sets and their associated measures (random, too) as being true “with probability one”. Let  $\dim K$  denote the Hausdorff dimension of a Borel  $K$ . The following results are well-known:

$$\begin{aligned} d = 1, & \quad \dim Z = 1/2, \\ d \geq 2, & \quad \dim [B_d] = 2, \\ d = 2, 3, & \quad \dim I_d = d - 2(d - 2). \end{aligned}$$

We refer to Taylor [16] for a convenient reference on the theory of random fractals arising from the sample paths of stochastic processes, in which the detailed definitions and properties of Hausdorff and other dimension indices are described.

Nowadays the fractal analysis of measures rather than sets has been focused. In our concern, there are natural measures associated with the above fractals  $Z$ ,  $[B_d]$  and  $I_d$ ; they are respectively Brownian *local time measure*, *occupation measure* and *intersection measure*. These measures are regarded as *fractal measures*, since each of them is singularly continuous (non-atomic and supported by a set of Lebesgue measure zero) and exhibits a certain self-similarity which is inherited from the self-similarity of the process. The main difference (and difficulty) from pure analysis is that the self-similarity is now always in the stochastic sense rather than the strict (analytic) sense. Therefore, even though the assertions of the theorems are formulated in almost sure

statements, we cannot proceed the reasoning just in an analytic (pathwise) way; instead we have to deliberate our arguments by combining both analysis and probability.

The rest of this paper is divided into three sections; each one is devoted to one aspect of Brownian fractal structures. The paper is concise; we merely state the main results together with some explanations on the main idea behind the proofs.

## 2. AVERAGE DENSITIES

Let  $\mu$  be a locally finite regular Borel measure in  $R^d$ , which is fractal in the sense described in the previous section. Let  $\phi(r)$  be a suitable gauge function. Bedford-Fisher [1] introduced the concept of *average density* of  $\mu$  as an analogue of Lebesgue density. The latter one fails to exist for many  $\mu$ , in view of a famous result of J. M. Marstrand. The average density of *order two* and *order three* of  $\mu$  at  $x$ , w.r.t.  $\phi$ , are defined respectively as

$$AD_2(\mu, x) = \lim_{\epsilon \downarrow 0} \frac{1}{\log(1/\epsilon)} \int_{\epsilon}^1 \frac{\mu(B(x, r))}{\phi(r)} \frac{dr}{r},$$

$$AD_3(\mu, x) = \lim_{\epsilon \downarrow 0} \frac{1}{\log \log(1/\epsilon)} \int_{\epsilon}^{1/\epsilon} \frac{\mu(B(x, r))}{\phi(r)} \frac{dr}{r \log(1/r)},$$

where  $B(x, r)$  denotes the closed ball with center  $x$  and radius  $r$ . Bedford-Fisher introduced and discussed the above definition via the classical summation techniques of Hardy and Riesz. They also discussed some specific cases of the measures associated with the middle-third Cantor set, cookie-cutter Cantor set, and Brownian zeroes. Falconer-Xiao [3] adapted the arguments to prove, among other things, the  $AD_2$  for the occupation measure of  $B_d$ ,  $d \geq 3$ , which is the measure  $\mu = \mu(\omega)$  defined by

$$\mu(A) = \text{Leb} \{t : B_d(t) \in A\}, \quad A \subset R^d;$$

note that  $\mu$  is supported by  $[B_d]$ .

**Theorem 2.1.** *For Brownian occupation measure  $\mu$  in  $R^d$ ,  $d \geq 3$ , the  $AD_2$  of  $\mu$  w.r.t.  $\phi(r) = r^2$  exists at  $\mu$ -a.e.  $x$ , and its value is a constant (depending only on the dimension  $d$ ) multiple of the mean sojourn time of the path in the unit ball.*

Note that the value in the above theorem is non-random (macroscopic, in physical terminology) while the measure is random (microscopic). We also

remark that the sojourn time in the above theorem is infinite for the planar Brownian motion, since  $B_2$  is neighborhood-recurrent. Then Mörters [7] proved that the  $AD_2$  of  $B_2$  does not exist while the  $AD_3$  does exist.

**Theorem 2.2.** *For planar Brownian occupation measure, the  $AD_3$  of  $\mu$  w.r.t.  $\phi(r) = r^2 \log(1/r)$  exists at  $\mu$ -a.e.  $x$ , and its value is 2. Moreover, the  $AD_2$  of  $\mu$  now fails to exist.*

Shieh [11] considered the average density problem of the intersection of two spatial Brownian motions and proved a partial result. The problem is completely solved, both for the planar and the spatial cases, in Mörters-Shieh [8]. Let  $B_d, B'_d$  be two independent Brownian motions in  $R^d$ ,  $d = 2, 3$ . Then the intersection measure is a canonical (random) Borel measure supported by  $I_d = [B_d] \cap [B'_d]$  which is expressed heuristically as

$$\mu(A) = \int_A \int_t \int_{t'} \delta_x(B_d(t)) \delta_x(B'_d(t')) dt dt' dx.$$

We remark that the rigorous definition of the above measure is somewhat involved; see [8] for details (note that we have used a shorter term here rather than a longer one in [8] for the terminology). The next theorem is proved in [8].

**Theorem 2.3.** *Let  $\mu$  be the intersection measure mentioned above. For the spatial case the  $AD_2$  of  $\mu$  w.r.t.  $\phi(r) = r$  exists at  $\mu$ -a.e.  $x$ , and for the planar case the  $AD_3$  of  $\mu$  w.r.t.  $\phi(r) = r^2 \log^2(1/r)$  exists at  $\mu$ -a.e.  $x$ . The value in both cases is  $4/\pi$ . Moreover, the  $AD_2$  in the planar case again fails to exist.*

The proofs of these results break into two parts: to prove the existence of the density and evaluate its value for a typical point, say  $x = 0$ , and then prove the results for generic  $x$ . To prove the reduction from the generic to the typical, in the case of Theorems 2.1 and 2.2 we can proceed with some standard application of the Markov property. However, it is not so easy for the intersection case in Theorem 2.3, since the  $t, t'$  for which  $B_d(t) = B'_d(t')$  cannot be realised as stopping times. This difficulty is overcome by either using a device of Le Gall on “Brownian loops” or by a more analytic approach based on Palm distributions. To prove the order two case for  $x = 0$ , we apply Birkoff’s ergodic theorem to some suitable scaling flow defined on the space of continuous functions, which is naturally associated with Brownian motions. However this scaling approach does not work for the order three case, since the gauge function  $\phi$  now has a slow varying term involved; the difficulty is

overcome by using a certain crossing-number argument for the path oscillating between the annulus with suitably chosen small radii. In the planar case we can consider the multiple intersections rather than just the double intersections shown in the above theorem.

### 3. DIMENSION SPECTRA

Study on the dimension spectra associated with various fractal measures is now the main concern in fractal analysis/geometry. We refer to Falconer [2, Chapter 11] for a convenient reference of such *multifractal analysis*, in which some physical background and some detailed notions and properties are described. In particular, the *upper local dimension*, the *lower local dimension* and the *local dimension* for a locally finite Borel measure  $\mu$  at a point  $x$  are defined. We denote them by  $\bar{d}(\mu, x)$ ,  $\underline{d}(\mu, x)$ , and  $d(\mu, x)$ , respectively. The definitions are

$$\begin{aligned}\bar{d}(\mu, x) &= \limsup_{r \downarrow 0} \frac{\log \mu(B(x, r))}{\log r}; \\ \underline{d}(\mu, x) &= \liminf_{r \downarrow 0} \frac{\log \mu(B(x, r))}{\log r}; \\ d(\mu, x) &= \underline{d}(\mu, x) = \bar{d}(\mu, x).\end{aligned}$$

The above definition is a mathematical view of the physical concern of non-uniform local mass concentrations which appears in the case of oil deposits on a specific region. By (fine) multifractal analysis of  $\mu$ , we seek for an  $f(\alpha)$  curve describing the Hausdorff dimension of the “level set”  $d(\mu, \cdot) = \alpha$  (or that for  $\bar{d}$ ,  $\underline{d}$ ), where  $\alpha$  is in some range  $[\alpha_{\min}, \alpha_{\max}]$ . For Brownian occupation measure, its dimension spectrum, that is  $f(\alpha)$  curve, is trivial, in view of a uniform dimension theorem of Perkins–Taylor. However, recent works of Dembo-Peres-Rosen-Zeitouni show that it does have non-trivial spectrum for “thick” occupations. For Brownian local time, the situation is completely different. We recall that local time is a canonical measure supported by the Brownian zero set  $Z$  (the 1-dimensional case only) which is expressed heuristically as

$$\mu(A) = \int_A \delta_0(B(t)) dt, \quad A \subset R.$$

Hu-Taylor [5] proved that there exists a nontrivial spectrum for the upper  $\bar{d}(\mu)$ , yet the lower  $\underline{d}(\mu)$  has only trivial spectrum. Then Shieh-Taylor [13] continued the study to show that, in the latter case, we do have a nontrivial spectrum in which logarithmic order of magnitude plays a crucial role. Since Brownian local time can be viewed as the occupation measure for a 1/2-stable

subordinator, they proved the results in terms of any stable subordinator. We state their results only for the Brownian local time.

**Theorem 3.1.** *Let  $\mu$  be the Brownian local time measure. For  $\alpha \geq 1/2$ , put*

$$A_\alpha = \{t : \bar{d}(\mu, t) = \alpha\}.$$

*Then*

$$\dim A_\alpha = 1/2\alpha - 1/2, \quad 1/2 \leq \alpha \leq 1,$$

*while  $A_\alpha = \emptyset$  if  $\alpha$  is outside  $[1/2, 1]$ . Moreover, the spectrum for the  $\underline{d}(\mu)$  is trivial; there is only one value  $1/2$  happened at  $\alpha = 1/2$ .*

**Theorem 3.2.** *Let  $\mu$  be the Brownian local time measure. For  $\alpha \geq 0$ , put*

$$B_\alpha = \left\{ t : \limsup_{r \downarrow 0} \frac{\mu(t-r, t+r)}{c\sqrt{r \log(1/r)}} = \alpha \right\},$$

*where the specified constant  $c = 2$ . Then*

$$\dim B_\alpha = (1/2)(1 - \alpha^2), \quad 0 \leq \alpha \leq 1,$$

*while  $B_\alpha = \emptyset$  if  $\alpha$  is outside  $[0, 1]$ .*

The proofs of these results again break into two parts. The easier part is the upper bound estimate for the dimension, in which we apply some Markov property to a certain particular cover of the concerned set. The difficult part is the lower bound estimate; we need to construct a certain Cantor-like random set contained in  $A_\alpha, B_\alpha$  on which there is some measure  $\nu$  supported. Moreover, we need to estimate the energy integral of this  $\nu$  with respect to some potential kernel. It was pointed out that there is a computational error in the lower bound proof; the corrections and addenda are taken up in Shieh-Taylor [14]. Furthermore, it is studied in Shieh-Taylor [15] that the same scenario can hold for the branching measure on a Galton-Watson tree.

We should remark that Theorems 3.1 and 3.2 are very different from “standard” theory of multifractal analysis. The latter one is mainly based on a certain thermodynamical formalism. Whenever the formalism is indeed true, as it is the case of some self-similar measure associated with an iterated function system, there is no distinction for the upper and the lower local dimensions, that is the assertion is true for  $d(\mu, x)$ , and there is usually no logarithmic factor involved. Thus, it seems that Theorems 3.1 and 3.2 are specific to the effect of random fluctuations.

## 4. BROWNIAN MULTIFRACTALS

Brownian paths have rich fractal structures, as we have seen from the previous two sections. However, the path is usually qualified as a monofractal, in view that the Hölder exponent of the path is everywhere  $1/2$  (the variations of the regularity are only of a logarithmic order of magnitude). Thus, it is not perfect to use Brownian path as a curve fitting to those data exhibiting the intermittence. The latter one is very important for the study of, say, turbulences. Mandelbrot introduced the concept of Brownian multifractals in his works on finance theory, and a proposed mathematical theory is proceeded recently by Riedi [10]. Let  $B(t)$  be a real-valued Brownian motion (or a fractional Brownian motion, if one likes to count the long range dependence), and let  $M(t)$  be an increasing process (that is, a process which is pathwise increasing in  $t$ ). Assume that  $B$  and  $M$  are totally independent (quite rough from the viewpoint of practical applications). The composite  $t \rightarrow B(M(t))$  is termed *Brownian motion in multifractal time*. The path of the new process indeed has some multifractal (=intermittent) structure and some dimension spectrum can be computed. In case that  $M$  is a subordinator, then the resulting process is a Lévy process. This case is also known in probability as *Brownian (time) substitution*. We recall that a Lévy process is a stochastic process (real-valued or vector-valued) with stationary and independent increments, and that a subordinator is a real-valued Lévy process with increasing paths. Jaffard [6] proved that the paths of “most” Lévy processes are multifractals and he also determined their spectrum of Hölder exponents.

We specify the general works of Riedi and Jaffard as follows. Let  $B_d(t) = (B^1(t), \dots, B^d(t))$  be a  $d$ -dimensional Brownian motion and let  $\theta^j(t)$ ,  $1 \leq j \leq d$ , be  $d$  stable subordinators with stability index  $\beta_j$ ,  $0 < \beta_j < 1$ , that is, the process  $\theta^j(t)$  is such that it is  $\beta_j$ -stable distributed for all  $t$ . Then we have the composed process  $X(t) = (B^1(\theta^1(t)), \dots, B^d(\theta^d(t)))$ . It is Lévy whenever  $\theta^j$ ,  $1 \leq j \leq d$ , are totally independent on  $B_d$ . We have the following three cases to consider.

1° That  $\beta_j = \beta$  for all  $j$ , and  $(\theta^1, \dots, \theta^d)$  is a  $d$ -dimensional  $\beta$ -stable process. Then  $X$  is a  $d$ -dimensional  $2\beta$ -stable Lévy process (note that  $\beta$  is necessarily  $< 1$ ).

2° That  $\theta^j$  among themselves are independent. Then  $X$  is of independent components. Such processes have been termed as *processes with stable components* in Pruitt-Taylor [9].

3° That we go beyond the first two cases by only assuming that  $X$  is a *d-dimensional self-similar process* with a vector ss index  $H = (H_1, \dots, H_d)$ . See Shieh [12] for the precise definition, where the term *dilation-stable Lévy process* is used. Let  $\gamma_t$  be a diagonal transformation in  $R^d$  whose diagonal

entries are  $t^{H_1}, \dots, t^{H_d}$ . Then the stochastic structure of a dilation-stable Lévy process is determined by the following theorem, which can be seen in Hudson-Mason [4].

**Theorem 4.1.** *The characteristic function of  $X(1)$  of the above dilation-stable Lévy process  $X$  is determined by*

$$E \exp(i(z, X(1))) = \exp \left[ \int_0^\infty \int_S (\exp(i(z, \gamma_r x)) - 1 - i(z, \gamma_r x) 1_D(\gamma_r x)) \frac{\lambda(dx) dr}{r^2} \right],$$

where  $\lambda(dx)$  is a finite Borel measure on  $S = \{x : |x| = 1\}$  and  $D = \{x : |x| \leq 1\}$ . Moreover,  $X$  is of independent components if and only if the measure  $\lambda(dx)$  is concentrated on the coordinate axes.

Multi-dimensional stable Lévy processes are the non-Gaussian counterpart of the Gaussian, that is Brownian, case and have been well studied. Processes with stable components arise from the study of the collisions of two independent stable processes and they are also good examples for showing some significant gaps between the stable and the general Lévy processes. Dilation-stable Lévy processes arise from the study of stochastic flows in which the independent-components assumption is too strong for the purpose. Shieh [12] proved a dimension formula for the multiple points of dilation-stable Lévy processes; the work shows that for such processes we need to proceed with some considerations more complicated than the Brownian and the stable cases.

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