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BLOWUP BEHAVIOR OF MEAN FIELD TYPE EQUATIONS

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Dedicated to Professor Fon-Che Liu on his sixtieth birthday

Abstract. Some recent results on the equation

$$\triangle_g u + \rho \left(\frac{he^u}{\int_M he^u} - 1 \right) = 0 \text{ on } M$$

are reviewed. We focus on the variational structure and the blowup behavior of the solutions.

1. Introduction

Let (M, g) be a compact Riemann surface without boundary and $h(x)$ be a smooth positive function on M. Assume the area of M is 1. We consider the mean field type equation

$$\triangle_g u + \rho \left(\frac{he^u}{\int_M he^u} - 1 \right) = 0 \text{ on } M,$$

where $\triangle_g$ is the Laplace Beltrami operator with respect to g and $\rho > 0$ is a constant. When (M, g) is the standard sphere and $\rho = 8\pi$, the problem to find a solution to equation (1.1) is called "Nirenberg problem". The geometric significance of this problem is that from a solution u, we can obtain a new conformal metric $e^u g$ which has curvature h. This problem has been studied by Moser [32], Kazdan-Warner [27], Hong [24], Chen-Ding [17, 18], Chang-Yang [8, 9, 10], Chang-Gursky-Yang [7], and others. When formulated on bounded domains of R^2 with Dirichlet boundary conditions, equation (1.1)

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was considered as the mean field limit of point vortices for two-dimensional Euler equations in Caglioti, Lions, Marchiore and Pulvirenti [5, 6] and Kiessling [28]. When (M, g) is a flat torus, equation (1.1) is related to the study of condensate solutions of some Chern-Simons-Higgs models; see Taubes [40, 41], Hong, Kim and Pac [25], Jackiw and Weinberg [26], Spruck and Yang [37], Caffarelli and Yang [4], Tarantello [29], Struwe and Tarantello [28], Ding, Jost, Li and Wang [20, 21], and the references therein. We report some recent works on this equation in this paper.

One approach to study the existence of solutions of equation (1.1) is by variational methods. Let

$$E_0 = \left\{ u \in H^1(M) : \int_M u \, dv_g = 0 \right\}.$$

Then solutions of (1.1) correspond to critical points of the functional

$$J_\rho = \frac{1}{2} \int_M |\nabla_g u|^2 \, dv_g - \rho \log \int_M h e^u \, dv_g, u \in E_0.$$

By the Moser-Trudinger inequality

$$(8\pi - \epsilon) \left[\log \int_M e^u \, dv_g - \frac{1}{\mathrm{vol}(M)} \int_M u \, dv_g \right] \leq \frac{1}{2} \int_M |\nabla_g u|^2 \, dv_g + C(\epsilon),$$

J_ρ is bounded from below when $\rho < 8\pi$. It is not difficult to obtain the existence of a solution to (1.1) in this case. When $\rho > 8\pi$, the Moser-Trudinger inequality cannot be applied directly since the coefficient $8\pi - \epsilon$ before the log term is smaller than ρ and J_ρ is no longer bounded from below. Therefore it is interesting to know what happens in the limiting case $\rho = 8\pi$ for J_ρ and $\epsilon = 0$ for the Moser-Trudinger inequality. In Ding, Jost, Li and Wang [20], and Nolasco and Tarantello [33], a sharp inequality and some existence results were obtained for this limiting case.

When (M, g) is a flat torus with fundamental domain $[0, 1] \times [0, 1]$ and $h \equiv 1$, Struwe and Tarantello [38] showed that in the range $8\pi < \rho < 4\pi^2$, J_ρ exhibits a mountain-pass structure around 0. From it, they obtained a nontrivial solution to (1.1).

Another approach to study the existence of solutions of equation (1.1) is by the Leray-Schauder degree as proposed by Li [29]. By the results of Brezis and Merle [3] and Li and Shafrir [30], all solutions of (1.1) stay bounded when ρ lies in compact subsets of $R \setminus \cup_{m \geq 1} \{8m\pi\}$. Therefore if we write equation (1.1) in the form

$$u - \rho(-\triangle_g)^{-1} \left(\frac{h e^u}{\int_M h(x) e^u} - 1 \right) = 0,$$

the corresponding Leray-Schauder degree can be defined when $\rho \neq 8m\pi$. Let d_ρ denote the Leray-Schauder degree. By the homotopy invariance, d_ρ is a constant for ρ in each interval $(8m\pi, 8(m+1)\pi)$. It is well-known that $d_\rho = 1$ for $\rho < 8\pi$. Thus, to obtain a formula for d_ρ, it suffices to know the jump values of d_ρ at $8m\pi$. Since we do not have a priori estimtes as $\rho \to 8m\pi$, the most crucial step in calculating the jump-values is to study the blowup behavior of solutions.

Let $\{u_i\}$ be a sequence of solutions of (1.1) with $\rho = \rho_i \to 8m\pi$ for some $m \geq 1$. A point p on M is called a blowup point if there are $x_i \to p$ such that

$$\lim_{i\to\infty} u_i(x_i) = \infty.$$

Some questions then follow:

(a) How many blowup points can a sequence of solutions have?

(b) What are the locations of the blowup points?

(c) What are the profiles of solutions near a blowup point?

After a suitable scaling, we can approximate a solution near a blowup point by a limiting function. However it is not easy to get a sharp estimate of the difference between these two functions. Li [29] first obtained a uniform estimate for the difference on a given fixed neighborhood of a blowup point. Based on this, the authors showed in [31] and [14] that if the sup-norm of a solution is very large, it exhibits "concentration-induced" symmetry. The second author in [31] further used this "induced symmetry" to get the Leray-Schauder degree for $\rho \in (8\pi, 16\pi) \cup (16\pi, 24\pi)$ in the case $M = S^2$. In [15], the authors further found the dominant terms in the differences between blowup solutions and the corresponding limiting functions when $h(x)$ is not degenerate. This result indicates a possible method to find the Leray-Schauder degree for all $\rho > 8\pi$ and $\rho \neq 8m\pi$.

2. Variational Approach

For $\rho < 8\pi$, as mentioned in Introduction, we can apply the Moser-Trudinger inequality and the assumption $\mathrm{vol}(M) = 1$ to obtain

$$\log \int_M h e^u \, dv_g \leq \log \int_M e^u \, dv_g + \log(\max h)$$

and

$$\begin{aligned} J_\rho(u) &\geq \frac{1}{2}\int_M |\nabla_g u|^2 \, dv_g - \left(\frac{1}{16\pi} + \epsilon\right)\rho \int_M |\nabla_g u|^2 \, dv_g - C(\epsilon) - \log(\max h) \\ &\geq c_1 \int_M |\nabla_g u|^2 \, dv_g - c_2. \end{aligned}$$

Here we assume $\frac{1}{\mathrm{vol}(M)}\int_M u\,dv_g = 0$. If we take a minimizing sequence of J_ρ, then the L^2-norms of their gradients are bounded by the values of J_ρ. Hence some subsequence of the minimizing sequence converges weakly. By a standard argument, the existence of a minimizer of J_ρ follows. We have the following well-know result.

Theorem 2.1. *If $\rho < 8\pi$, then the minimum of J_ρ can be attained and equation* (1.1) *has a solution.*

The case $\rho = 8\pi$ is the limiting case for functional J_ρ. It also corresponds to the limiting case for the Moser-Trudinger inequality. Ding, Jost, Li and Wang [20] studied the behavior of a sequence of minimizers of J_ρ in the space E_o when $\rho \to 8\pi^-$. Nolasco and Tarantello [33] constructed a special minimizing sequence of J_ρ for $\rho = 8\pi$. When (M, g) is a flat torus, both of their results imply the following.

Theorem 2.2. *Let $\rho = 8\pi$ and (M, g) be a flat torus. If there is some $p \in M$ such that $h(p) = \max h$ and*

$$\triangle \log h(p) > -8\pi,$$

then the minimum of J_ρ can be attained, and consequently equation (1.1) *has a solution.*

Let $G(x, y)$ be the Green function satisfying

$$\begin{aligned} &\triangle G = 1 - \delta_y, \\ &\int_M G = 0, \end{aligned} \tag{2.1}$$

where δ_y is the delta function at y. In a normal coordinate system near x, we write

$$8\pi G(x, y) = -4\log r + A(y) + b_1 x_1 + b_2 x_2 + O(r^2), \tag{2.2}$$

where $r(x) = \mathrm{dist}(x, p)$ and $A(y)$ is the regular part of G. Also, we write $\nabla h = (h_1, h_2)$ in this normal coordinate system. For a general compact Riemann surface, Ding, Jost, Li and Wang [20] also obtained

Theorem 2.3. *Let $\rho = 8\pi$ and (M, g) be a compact Riemann surface. Let $K(x)$ be its Gauss curvature. If there is some $p \in M$ such that $A(p) + 2\log h(p) = \max[A(x) + 2\log h(x)]$ and*

$$\triangle h(p) + 2[b_1(p)h_1(p) + b_2(p)h_2(p)] > -h(p)[8\pi + b_1^2(p) + b_2^2(p) - 2K(p)],$$

then the minimum of J_ρ can be attained, and consequently equation (1.1) *has a solution.*

A consequence follows from this result.

Theorem 2.4. *Let (M, g) be a flat torus with fundamental domain $[0,1] \times [0,1]$. Then Moser-Trudinger's inequality holds for the limiting case, that is,*

$$8\pi \left[\log \int_M e^u \, dv_g - \frac{1}{\mathrm{vol}(M)} \int_M u \, dv_g\right] \leq \frac{1}{2} \int_M |\nabla_g u|^2 \, dv_g + C_M$$

for $u \in H^1$.

For $\rho > 8\pi$, J_ρ is not bounded below. When (M, g) is a flat torus with fundamental domain $[0,1] \times [0,1]$ and $h \equiv 1$, Struwe and Tarantello [38] found that in the range $8\pi < \rho < 4\pi^2$, J_ρ exhibits a mountain-pass structure around 0. From it, they obtained a nontrivial solution of (1.1).

Theorem 2.5. *For every $\rho \in (8\pi, 4\pi^2)$, there exits a nontrivial solution u of* (1.1) *satisfying*

$$J_\rho \geq c_0 \left(1 - \frac{\rho}{4\pi^2}\right)$$

for some constant c_0 independent of ρ.

By a result of Ricciardi and Tarantello [34], the corresponding one-dimensional problem

$$u'' + +\rho \left(\frac{e^u}{\int_M e^u} - 1\right) = 0$$

admits a nonconstant solution of period 1 if and only if $\rho > 4\pi^2$. Hence Theorem 2.5 captures the two-dimensional nature of equation (1.1) and this indicates the role played by the value $4\pi^2$ is important.

3. Blowup Behavior

As we mentioned above, a sequence of solutions $\{u_i\}$ of (1.1) with $\rho = \rho_i$ may blow up if $\rho_i \to 8m\pi$ for some positive integer m. Since the cases in Theorems 2.2 and 2.3 in Section 2 correspond to the case $\rho = 8\pi$, some detailed analysis for blowup minimizing sequences is needed in the proofs of these theorems. With these detailed analysis, the limiting value of J_ρ and the limit equation that a minimizing sequence satisfies can be estimated or expressed in simple terms. If the limiting information is not consistent with the assumptions, then the minimizing sequence cannot blow up and some subsequence must converge to a minimizer.

More generally, we consider a sequence of blowup solutions $\{u_i\}$ which may not be a minimizing sequence. Let p be a blowup point of $\{u_i\}$ and x_i be a local maximum point of u_i which tends to p as $i \to \infty$. First, in a normal coordinate system centered at x_i, we can take a suitable scaling to let

$$U_i(y) = u_i\left(\frac{y}{\rho_i^{\frac{1}{2}} h(0)^{\frac{1}{2}} e^{\frac{1}{2}u_i(0)}}\right) - u_i(0).$$

Then after passing to a subsequence, U_i tends to some function U satisfying

$$\begin{aligned} &\triangle U + e^U = 0 \text{ in } R^2, \\ &U(0) = \max U = 0, \quad \text{and } \int_{R^2} e^U < \infty \end{aligned} \tag{3.1}$$

on any compact set of R^2. This is equivalent to, in a normal coordinate system centered at x_i,

$$\max_{|x|<r_i} |u_i(x) - u_i(0) - U(\rho_i^{\frac{1}{2}} h(0)^{\frac{1}{2}} e^{\frac{1}{2}u_i(0)} x)| = o(1), \tag{3.2}$$

where $r_i \geq c\,\rho_i^{-\frac{1}{2}} h(0)^{-\frac{1}{2}} e^{-\frac{1}{2}u_i(0)}$ for any fixed c. By a result of Chen and Li [19], $U(y)$ can be solved uniquely as

$$U(y) = \log \frac{1}{1 + \frac{1}{8}|y|^2}.$$

Another important fact is that blowup solutions accumulate to Green's functions. Assume that $z_1, ..., z_s$ are blowup points of $\{u_i\}$ and

$$\bar{u}_i = \frac{1}{\operatorname{vol} M} \int_M u_i.$$

Then after passing to a subsequence,

$$u_i - \bar{u}_i \to \sum_{j=1}^{j=s} c_j\, G(\cdot, z_j)$$

for some $c_j > 0$ in $C^2_{\text{loc}}(M \setminus \{z_1, ..., z_s\})$, where $G(x, y)$ is the Green function defined in (2.1).

It was conjectured by Brezis and Merle [3] that each c_j can be written as $c_j = 8\pi k_j$ for some positive integer k_j. Although by (3.2), we know the shape of u_i well when $|x| < r_i$, the problem is that r_i may tend to zero as $i \to \infty$. If r_i tends to zero, then the control for u_i in (3.2) is too weak to answer this question. The conjecture was proved by Li and Shafrir [30]. More recently, Li [29] obtainded a uniform estimate for u_i with a non-shrinking r_i. This result implies actually $c_j = 8\pi$. More precisely, we have the following; see [29].

Theorem 3.1. *Let $\{u_i\}$ be a sequence of solutions of* (1.1) *with $\rho = \rho_i \to 8\pi m$ and $\max |u_i| \to \infty$. Then after passing to a subsequence, there are exactly m blowup points $\{z_1, ..., z_m\}$ and m subsequences of points $z_{j,i} \to z_j$ such that:*
(a) *When i is large, $z_{j,i}$ is the unique maximum point of u_i in the neighborhood $\{x | \mathrm{dist}(x, z_j) \le \frac{1}{2} \min_{j \ne l}(z_j, z_l)\}$. Moreover, $u_i(z_{j,i}) \to \infty$.*
(b) *Near each blowup point z_j, we have in a normal coordinate system centered at z_j,*

$$\max_{|x|<r_o} |u_i(x) - u_i(0) - U(\rho_i^{\frac{1}{2}} h(0)^{\frac{1}{2}} e^{\frac{1}{2}u_i(0)} x)| \le C,$$

where $r_o = \frac{1}{4} \min_{j \ne l}(z_j, z_l)$ and $C > 0$ is independent of i.
(c) *There is some $C > 0$ independent of i such that*

$$\max_{1 \le j \le m} |u_i(z_{j,i}) + \bar{u}_i| \le C.$$

(d)

$$u_i - \bar{u}_i \to \sum_{j=1}^{j=m} 8\pi G(\cdot, z_j)$$

in $C^2_{\mathrm{loc}}(M \setminus \{z_1, ..., z_m\})$ and

$$\frac{1}{\int_M \rho_i h e^{u_i}} \rho_i h e^{u_i} \to 8\pi \sum_{j=1}^{m} \delta_{z_j}$$

in the sense of measure, where δ_{z_j} is the delta function at z_j.

It is interesting that the main estimate, part (b) in Theorem 3.1, was proved by the method of moving planes, which was developed by Alexandrov [1], Serrin [36], and Gidas, Ni and Nirenberg [22, 23] and others. This method was used to obtain a priori estimates by Schoen [35], Brezis, Li and Shafrir [2], and Chen and Lin [11, 12, 13].

By parts (c) and (d) of Theorem 3.1, $\bar{u}_i \to -\infty$ as $i \to \infty$ and the mass of $(1/\int_M \rho_i h e^{u_i})\ \rho_i h e^{u_i}$ concentrates to delta functions at the blowup points $\{z_j\}$. In Lin [31] and Chen and Lin [14], it was shown that concentration of mass implies $\{u_i\}$ has some symmetry when i is large. When (M, g) is the standard S^2, Lin [31] obtained.

Theorem 3.2. *Let (M, g) be the standard S^2 and let $\{u_i\}$ be a sequence of solutions of* (1.1) *with $h \equiv 1$, $\rho = \rho_i \to 16\pi$ and $\max |u_i| \to \infty$. Then for* *large i, we have $\rho_i > 16\pi$ and u_i is axially symmetric with respect to some direction.*

When (M, g) is a flat torus with a rectangle fundamental domain, Chen and Lin [14] obtained the following.

Theorem 3.3. *Let (M, g) be a flat torus with a rectangle fundamental domain and let $\{u_i\}$ be a sequence of solutions of* (1.1) *with $h \equiv 1$, $\rho = \rho_i \to 8\pi$ and $\max |u_i| \to \infty$. Then for large i, u_i is symmetric with respect to the maximum point in both x and y directions.*

Theorem 3.4. *Let (M, g) be a flat torus with a rectangle fundamental domain and let $\{u_i\}$ be a sequence of solutions of* (1.1) *with $h \equiv 1$, $\rho = \rho_i \to 16\pi$ and $\max |u_i| \to \infty$. Then for large i, after a translation, there are two local maximum points with one local maximum at the center of the fundamental domain and the other at the corner. Moreover, the solutions are symmetric with respect to the x and y directions.*

As mentioned above, the Leray-Schauder degree d_ρ equals 1 when $\rho < 8\pi$. Using the symmetry of blowup solutions, Lin [31] was able to obtain the degree for S^2 when $8\pi < \rho < 24\pi$.

Theorem 3.5. *Let (M, g) be the standard S^2. Then* (a) *$d_\rho = -1$ for $8\pi < \rho < 16\pi$ and* (b) *$d_\rho = 0$ for $16\pi < \rho < 24\pi$.*

One method to find the value d_ρ for more general situations is to obtain sharper estimates for blowup solutions. We have in [15] the following.

Theorem 3.6. *Let $\{u_i\}$ be a sequence of solutions of* (1.1) *with $\rho = \rho_i \to 8\pi$, $\max |u_i| \to \infty$ and $\int_M u_i = 0$. Let $\lambda_i = \frac{1}{2} \max u_i$, p be the blowup point and $p_i \to p$ be the maximum point of u_i. Then*
(a)

$$|\nabla \left(\log h + \frac{A}{2} \right) (p_i)| = O(\lambda_i e^{-\lambda_i}),$$

where A is the regular part of the Green function defined in (2.2).
(b)

$$|u_i - v_i - \eta_i - \lambda_i| = O(\lambda_i e^{-\lambda_i}),$$

where v_i is a scaling of U in (3.1) *and η_i is a weighted integral of $h - h(p_i)$.*
(c)

$$\rho_i - 8\pi = c(\triangle \log h + 8\pi \hat{g}) \lambda_i e^{-\lambda_i} + O(e^{-\lambda_i})$$

for some $c > 0$, where $\hat{g}$ is the genus of M.

Theorem 3.6 is for the case with one blowup point. For the case with more than one blowup point, that is, $\rho_i \to 8m\pi$ with $m > 1$, we can obtain a similar result. Applying these estimates, we have in [16] the following.

Theorem 3.7.
(a) *Let (M, g) be the standard S^2. Then $d_\rho = 0$ for $\rho > 16\pi$ and $\rho \neq 8m\pi$.*
(b) *Let (M, g) be a flat torus. Then $d_\rho = 1$ for all $\rho \neq 8m\pi$.*

From this therom, we conclude that equation (1.1) always has a solution when M is a torus and $\rho \neq 8m\pi$.

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