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AN INFINITE-DIMENSIONAL HEISENBERG UNCERTAINTY PRINCIPLE

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Abstract. An analogue of the classical Heisenberg inequality is given for an infinite-dimensional space. The proof relies on a commutation relationship and integration by parts formula for Gaussian measure. We also discuss when the equality holds.

1. Introduction

The well-known Heisenberg uncertainty principle [8] says that for any function $f \in L^2(\mathbb{R}^n)$ with $|f|_2 = 1$, we have

$$\int_{\mathbb{R}^n} |xf(x)|^2 dx \cdot \int_{\mathbb{R}^n} |\gamma \hat{f}(\gamma)|^2 d\gamma \geq \frac{n^2}{4(2\pi)^{n-1}}, \tag{1}$$

where $\hat{f}$ is the Fourier transform of f. Since $\lim\limits_{n\to\infty} \frac{n^2}{(2\pi)^{n-1}} = 0$, it appears that there is no such uncertainty principle for the infinite-dimensional case. This is reflected by the fact that the Lebesgue measure does not exist in an infinite-dimensional space. Moreover, the Fourier transform needs to be generalized to such a space.

First we briefly describe the idea to obtain an infinite-dimensional analogue of the above inequality. Take a basic Gel'fand triple $\mathcal{E} \subset E \subset \mathcal{E}'$; e.g., $\mathcal{S}(\mathbb{R}) \subset L^2(\mathbb{R}) \subset \mathcal{S}'(\mathbb{R})$, where $\mathcal{S}(\mathbb{R})$ is the Schwartz space of rapidly decreasing functions on $\mathbb{R}$. Let $|\cdot|_0$ denote the norm on E. The space $\mathbb{R}^n$ is replaced by $\mathcal{E}'$ and the Lebesgue measure on $\mathbb{R}^n$ is replaced by the standard Gaussian

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measure μ on $\mathcal{E}'$. Let (L^2) denote the complex $L^2(\mu)$-space with norm $\|\cdot\|_0$. Let $(\mathcal{E}) \subset (L^2) \subset (\mathcal{E})^*$ be the associated Gel'fand triple (see [3, Section 4.2] for details).

The multiplication by x in (1) is replaced by a multiplication operator $\widetilde{Q}_\eta$ which is continuous from $(\mathcal{E})^*$ into itself [3, Theorem 9.18]. The Fourier transform is replaced by the Fourier-Wiener transform (or the second quantization operator $\Gamma(iI)$ of iI). Thus the infinite-dimensional analogue of the inequality in (1) takes the form

$$\left[\int_{\mathcal{E}'} |\langle x, \eta\rangle \varphi(x)|^2 \, \mu(dx)\right] \left[\int_{\mathcal{E}'} |\langle x, \eta\rangle \mathcal{F}\varphi(x)|^2 \, \mu(dx)\right] \geq |\eta|_0^4 \, \|\varphi\|_0^4, \tag{2}$$

where $\langle\cdot,\cdot\rangle$ denotes the $\mathcal{E}' - \mathcal{E}$ pairing and $\mathcal{F}$ is the Fourier-Wiener transform, i.e.,

$$\mathcal{F}\varphi(x) = \int_{\mathcal{E}'} \varphi(\sqrt{2}y + ix)\mu(dy),$$

for any $\eta \in \mathcal{E}$ and $\varphi \in (L^2)$ (see [5]).

The inequality (1) may be proved directly by integration by parts formula. It can also be shown that the equality in Heisenberg inequality holds if and only if φ is of the form

$$\varphi(x) = e^{\frac{\alpha}{2}\langle x, u_\eta\rangle^2} \varphi({P_\eta}^\perp x),$$

where α is a real number such that $|\alpha| < 1$.

In Section 2, we will provide a brief background concerning the Gel'fand triples $\mathcal{E} \subset E \subset \mathcal{E}'$ and $(\mathcal{E}) \subset (L^2) \subset (\mathcal{E})^*$. The inequality in (2) will be proved in Section 3. We will discuss the equality in (2) in Section 4.

2. Background

2.1. Concept and Notations

Let E be a real separable Hilbert space with norm $|\cdot|_0$. Let A be a densely defined self-adjoint operator on E, whose eigenvalues $\{\lambda\}_{n\geq 1}$ satisfy the following conditions:

- $1 < \lambda_1 < \lambda_2 < \lambda_3 < \cdots$.
- $\sum_{n=1}^{\infty} \lambda_n^{-2} < \infty$. (Hence A^{-1} is a Hilbert-Schmidt operator.)

For any $p \geq 0$, we consider the space $\mathcal{E}_p := \{f \in E \mid |A^p f|_0 < \infty\}$. On the space $\mathcal{E}_p$ we introduce the norm $|f|_p = |A^p f|_0$. Each of these spaces is a Hilbert space and we have the inclusion $\mathcal{E}_q \subset \mathcal{E}_p$ for $p < q$. By the second condition the inclusion $i : \mathcal{E}_{p+1} \longrightarrow \mathcal{E}_p$ is a Hilbert-Schmidt operator. Thus the space $\mathcal{E} = \bigcap_{p\geq 0} \mathcal{E}_p$, equipped with the topology given by the family $\{|\cdot|_p\}_{p\geq 0}$ of seminorms, is a nuclear space.

It can be shown that for all $p \geq 0$, the dual space of $\mathcal{E}_p$ is isomorphic to $\mathcal{E}_{-p}$, which is the completion of the space E with respect to the norm $|f|_{-p} = |A^{-p} f|_0$. Moreover, we have $\mathcal{E}' = \bigcup_{p\geq 0} \mathcal{E}_{-p}$ and for any $0 < p < q$,

$$\mathcal{E} \subset \mathcal{E}_q \subset \mathcal{E}_p \subset \mathcal{E}_0 \subset \mathcal{E}_{-p} \subset \mathcal{E}_{-q} \subset \mathcal{E}'.$$

Equip $\mathcal{E}'$ with the inductive limit topology. The triple $\mathcal{E} \subset E \subset \mathcal{E}'$ becomes a Gel'fand triple.

By Minlos' theorem, there exists a unique probability measure μ on $\mathcal{E}'$ such that for all $f \in \mathcal{E}$, the random variable $\langle \cdot, f\rangle$ is normally distributed with mean 0 and variance $|f|_0^2$. Here $\langle \cdot, \cdot \rangle$ is the duality between $\mathcal{E}'$ and $\mathcal{E}$. Because of the denseness of $\mathcal{E}$ in E, we can define for each $f \in E$, a random variable $\langle \cdot\, f\rangle$ on $\mathcal{E}'$ which is normally distributed with mean 0 and variance $|f|_0^2$.

For $x \in \mathcal{E}'$, we define

$$: x^{\otimes n} := \sum_{k=0}^{[n/2]} \frac{(-1)^k n!}{(n-2k)!k!2^k} \tau^{\widehat{\otimes} k} \widehat{\otimes} x^{\otimes (n-2k)},$$

where $\tau \in (\mathcal{E} \otimes \mathcal{E})'$ is defined by $\langle \tau \ , \xi \otimes \eta \rangle = \langle \xi \ , \eta \rangle$. Let E_c denote the complexification of E. We denote by (L^2) the space of all complex-valued square integrable functions on $\mathcal{E}'$. It can be proved that for each $\varphi \in (L^2)$, there exists a unique sequence $\{f_n\}_{n\geq 0}$, $f_n \in E_c^{\widehat{\otimes} n}$, such that:

$$\varphi(x) = \sum_{n=0}^{\infty} \langle : x^{\otimes n} : , f_n \rangle.$$

Moreover, we have $\|\varphi\|_0^2 = \sum_{n=0}^{\infty} n! \ |f_n|_0^2$.

The second quantization operator $\Gamma(A)$ of A is defined by

$$\Gamma(A)\varphi = \sum_{n=0}^{\infty} \langle : \cdot^{\otimes n} : \ A^{\otimes n} f_n \rangle.$$

By using (L^2) and $\Gamma(A)$ instead of E and A, respectively, we can construct a Gel'fand triple $(\mathcal{E}) \subset (L^2) \subset (\mathcal{E})^*$. The elements in $(\mathcal{E})$ are called test functions on $\mathcal{E}'$. The elements in $(\mathcal{E})^*$ are called generalized functions on $\mathcal{E}'$.

The bilinear pairing between $(\mathcal{E})^*$ and $(\mathcal{E})$ is denoted by $\ll \cdot\, , \cdot \gg$. It must be mentioned that if $\varphi \in (L^2)$ and $\psi \in (\mathcal{E})$, then $\ll \varphi\, , \psi \gg = (\varphi, \bar{\psi})$, where $(\cdot, \cdot)$ is the inner product of the complex Hilbert space (L^2).

Let $\varphi \in (L^2)$ be represented by $\varphi(x) = \sum_{n=0}^{\infty} \langle : x^{\otimes n} : , f_n \rangle$. It can be shown that $\varphi \in (\mathcal{E})$ if and only if for all $p \geq 0$, we have

$$\|\varphi\|_p^2 := \sum_{n=0}^{\infty} n! |f_n|_p^2 < \infty.$$

On the other hand, each $\Phi \in (\mathcal{E})^*$ can be represented as

$$\Phi = \sum_{n=0}^{\infty} \langle : \cdot^{\otimes n} : , F_n \rangle, \quad F_n \in (\mathcal{E}_c')^{\widehat{\otimes} n},$$

and there exists a $p > 0$ depending on Φ such that

$$\|\Phi||_{-p}^2 := \sum_{n=0}^{\infty} n! |F_n|_{-p}^2 < \infty.$$

For $\Phi \in (\mathcal{E})^*$ and $\varphi \in (\mathcal{E})$ from above we have

$$\ll \Phi, \, \varphi \gg = \sum_{n=0}^{\infty} n! \langle F_n \, f_n \rangle.$$

2.2. Differential Operators and the Adjoints

Consider a simple test function $\varphi(x) = \langle : x^{\otimes n} : , f \rangle \in (\mathcal{E})$. Let $y \in \mathcal{E}'$. We can show that

$$\lim_{t \to 0} \frac{\varphi(x + ty) - \varphi(x)}{t} = n \langle : x^{\otimes (n-1)} : , y \widehat{\otimes}_1 f \rangle,$$

where $y \widehat{\otimes}_1 \cdot : E_c^{\widehat{\otimes} n} \longrightarrow E_c^{\widehat{\otimes}(n-1)}$ is the unique continuous and linear map such that

$$y \widehat{\otimes}_1 g^{\otimes n} = \langle y\, , g \rangle g^{\otimes (n-1)}, \quad g \in E_c.$$

This shows that the function φ has Gâteaux derivative $D_y \varphi$. In general, for $\varphi(x) = \sum_{n=0}^{\infty} \langle : x^{\otimes n} : , f_n \rangle \in (\mathcal{E})$, we may define

$$D_y \varphi(x) = \sum_{n=1}^{\infty} n \langle : x^{\otimes (n-1)} : , y \widehat{\otimes}_1 f_n \rangle.$$

It can be checked that D_y is a continuous linear operator on $(\mathcal{E})$ (see [3, Theorem 9.1]).

We can define the adjoint operator D_y^* of D_y by the duality between $(\mathcal{E})^*$ and $(\mathcal{E})$, i.e.,

$$\langle\langle D_y^*\Phi\ , \psi\rangle\rangle = \langle\langle \Phi\ , D_y\psi\rangle\rangle, \quad \Phi \in (\mathcal{E})^*,\ \psi \in (\mathcal{E}).$$

The adjoint D_y^* is a continuous linear operator.

For $\Phi(x) = \sum_{n=0}^{\infty} \langle : x^{\otimes n} : , F_n\rangle \in (\mathcal{E})^*$, we have

$$D_y^*\Phi(x) = \sum_{n=0}^{\infty} \langle : x^{\otimes(n+1)} : , y\widehat{\otimes} F_n\rangle.$$

For $y \in \mathcal{E}$, the differential operator D_y extends by continuity to a continuous linear operator from $(\mathcal{E})^*$ into itself [3, Theorem 9.10]. The extension is denoted by $\widetilde{D}_y$. Moreover, for such $y \in \mathcal{E}$, the restriction of D_y^* to $(\mathcal{E})$ is a continuous linear operator from $(\mathcal{E})$ into itself [3, Corollary 9.14].

2.3. Multiplication Operators

If $\varphi, \psi \in (\mathcal{E})$, then the pointwise multiplication $\varphi \cdot \psi$ is also in $(\mathcal{E})$. Let $\Phi \in (\mathcal{E})^*$ be fixed. For $\varphi \in (\mathcal{E})$, define $\Phi \cdot \varphi \in (\mathcal{E})^*$ by

$$\langle\langle \Phi \cdot \varphi\ ,\ \psi\rangle\rangle = \langle\langle \Phi\ ,\ \varphi \cdot \psi\rangle\rangle, \quad \psi \in (\mathcal{E}).$$

This multiplication operator by Φ is a continuous linear operator from $(\mathcal{E})$ into $(\mathcal{E})^*$.

In particular, if $\eta \in \mathcal{E}$, then the multiplication by $\langle \cdot\ , \eta\rangle$, denoted by Q_η, is a continuous linear operator from $(\mathcal{E})$ into itself and can be extended to a continuous linear operator $\widetilde{Q}_\eta$ from $(\mathcal{E})^*$ into itself. The operators $\widetilde{Q}_\eta$, $\widetilde{D}_\eta$, and D_η^* are related by the formula

$$\widetilde{Q}_\eta = \widetilde{D}_\eta + D_\eta^*$$

(see [3, Theorem 9.18]).

2.4. The exponential Functions

Let $x \in \mathcal{E}_c'$. We define the following function

$$: e^{\langle \cdot, x\rangle} : = \sum_{n=0}^{\infty} \frac{1}{n!} \langle : \cdot^{\otimes n} : , x^{\otimes n}\rangle.$$

It is easy to see that

$$\| : e^{\langle \cdot, x\rangle} : \|_p = e^{|x|_p^2/2}.$$

Thus for all $x \in \mathcal{E}_c'$, we have $: e^{\langle \cdot, x\rangle} :\in (\mathcal{E})^*$. Also $: e^{\langle \cdot, x\rangle} :\in (L^2)$ if and only if $x \in E_c$ and $: e^{\langle \cdot, x\rangle} :\in (\mathcal{E})$ if and only if $x \in \mathcal{E}_c$.

If $x \in \mathcal{E}_c'$ and $\xi \in \mathcal{E}_c$, then we have

$$\langle\langle : e^{\langle \cdot, x\rangle} : , : e^{\langle \cdot, \xi\rangle} : \rangle\rangle = e^{\langle x, \xi\rangle}.$$

The exponential functions $\{: e^{\langle \cdot, \xi\rangle} : | \xi \in \mathcal{E}_c\}$ are linearly independent and span a dense subspace of $(\mathcal{E})$.

2.5. The S-transform

For all $\Phi \in (\mathcal{E})^*$, we define the **S-transform** of Φ to be the function on $\mathcal{E}_c$

$$S\Phi(\xi) = \langle\langle \Phi , : e^{\langle \cdot, \xi \rangle} : \rangle\rangle, \quad \xi \in \mathcal{E}_c.$$

Because the exponential functions span a dense subspace of $(\mathcal{E})$, the S-transform is injective.

2.6. Commutation Relation

For all ξ, $\eta \in \mathcal{E}$, the commutator of $\widetilde{D}_\xi$ and D_η^* is given by

$$[\widetilde{D}_\xi , D_\eta^*] = \langle \xi , \eta \rangle I$$

(see [3, Theorem 9.15]).

3. Heisenberg Uncertainty Principle

It is well-known that every member $\varphi \in (\mathcal{E})$ has an analytic extension $\widetilde{\varphi}$ to $\mathcal{E}_c$ (see [4]) so that every $\varphi \in (\mathcal{E})$ is Fréchet differentiable on $\mathcal{E}'$. Thus $D_\eta \varphi$ is defined for every $\eta \in \mathcal{E}'$. If $\eta \in \mathcal{E}$, we have $D_\eta^* \varphi(x) = (x, \eta)\varphi(x) - D_\eta \varphi(x)$.

Proposition 1. [5] *For $\varphi \in (\mathcal{E})$, let $\mathcal{F}\varphi$ be the Fourier-Wiener transform of φ, i.e., $\mathcal{F}\varphi(x) = \int_{\mathcal{E}'} \widetilde{\varphi}(\sqrt{2}y + ix)\mu(dy)$. Then $\mathcal{F}$ is continuous from $(\mathcal{E})$ onto itself and*

$$\|\mathcal{F}\varphi\|_{L^2} = \|\varphi\|_{L^2}.$$

The inverse transform of $\mathcal{F}$ is given by

$$\mathcal{F}^{-1}\varphi(y) = \int_{\mathcal{E}'} \widetilde{\varphi}(\sqrt{2}x - iy)\mu(dx).$$

Moreover, $\mathcal{F}$ is extended to a continuous operator on $(\mathcal{E})^*$. Denote this extension also by $\mathcal{F}$. Then $\mathcal{F}$ is a unitary operator on (L^2).

Proposition 2. *For $\eta \in \mathcal{E}$ and for $\varphi \in (\mathcal{E})^*$, we have*

$$\mathcal{F}\{(D_\eta^* - \widetilde{D}_\eta)\varphi\} = i\widetilde{Q}_\eta \mathcal{F}\varphi. \tag{3}$$

Proof. By Proposition 1, it is sufficient to verify (3) for $\varphi \in (\mathcal{E})$. Applying integration by parts formula, we obtain, for any $y \in \mathcal{E}'$,

$$\begin{aligned}
\mathcal{F}(D_\eta^* + D_\eta)\varphi(y) &= \mathcal{F}\{\langle \cdot, \eta\rangle \varphi\}(y) \\
&= \int_{\mathcal{E}'} \langle \sqrt{2}x + iy, \eta\rangle \widetilde{\varphi}(\sqrt{2}x + iy)\mu(dx) \\
&= \sqrt{2}\int_{\mathcal{E}'} \langle x, \eta\rangle \widetilde{\varphi}(\sqrt{2}x + iy)\mu(dx) + i\langle y, \eta\rangle \mathcal{F}\varphi(y) \\
&= 2\int_{\mathcal{E}'} \widetilde{D_\eta \varphi}(\sqrt{2}x + iy)\mu(dx) + i\langle y, \eta\rangle \mathcal{F}\varphi(y) \\
&= 2\mathcal{F}(D_\eta \varphi)(y) + i\widetilde{Q}_\eta \mathcal{F}\varphi(y).
\end{aligned}$$

It follows that $\mathcal{F}\{(D_\eta^* - D_\eta)\varphi\}(y) = i\widetilde{Q}_\eta \mathcal{F}\varphi(y)$. ∎

Theorem 3. *For any $\varphi \in (L^2)$ and $\eta \in \mathcal{E}$, we have*

$$\left[\int_{\mathcal{E}'} |\langle x, \eta\rangle \varphi(x)|^2 \mu(dx)\right] \left[\int_{\mathcal{E}'} |\langle x, \eta\rangle \mathcal{F}\varphi(x)|^2 \mu(dx)\right] \geq |\eta|_0^4 \|\varphi\|_0^4.$$

Proof. It is enough to verify the inequality for a real-valued function φ. It follows from the commutation relation $\widetilde{D}_\eta D_\eta^* - D_\eta^* \widetilde{D}_\eta = |\eta|_0^2 I$ that we have

$$\langle\langle \widetilde{Q}\varphi, (D_\eta^* - \widetilde{D}_\eta)\varphi \rangle\rangle = |\eta|_0^2 \|\varphi\|_0^2.$$

Then the theorem follows immediately from Proposition 1, Proposition 2 and Schwarz inequality. ∎

4. Equality in the Heisenberg Uncertainty Principle

Theorem 4. *The equality in the inequality* (2) *holds if and only if there exist real constants K_1 and K_2, not both zero, such that*

$$K_1 \langle \cdot, \eta\rangle \varphi = K_2 D_\eta \varphi. \tag{4}$$

Proof. The well-known criterion in real analysis for the equality in the Schwarz inequality implies that the equality in the inequality (2) holds if and only if there exist constants $A \geq 0$, $B \geq 0$, not both 0, such that, for almost all x with respect to μ,

$$A|(x,\eta)\varphi(x)|^2 = B|D_\eta^*\varphi(x) - D_\eta\varphi(x)|^2. \tag{5}$$

It follows from the identity $(x,\eta)\varphi(x) - 2D_\eta\varphi(x) = D_\eta^*\varphi(x) - D_\eta\varphi(x)$ that

(i) if $A = 0$, $B \neq 0$, then $(x,\eta)\varphi(x) = 2D_\eta\varphi(x)$;

(ii) if $B = 0$, $A \neq 0$, then $(x,\eta)\varphi(x) = 0$;

(iii) if $AB > 0$, then $(x,\eta)\varphi(x) - 2D_\eta\varphi(x) = \text{const.}(x,\eta)\varphi(x)$.

All the above three cases imply that there exist real numbers K_1 and K_2, not both zero, such that

$$K_1\langle x,\eta\rangle\varphi(x) = K_2 D_\eta\varphi(x).$$

Conversely, if condition (4) holds, then condition (5) holds and hence the equality in the inequality (2) holds. ■

Now we solve completely the equation (4). Let $\eta \in \mathcal{E} \setminus \{0\}$ and let $u_\eta = \eta/|\eta|_0$. Denote by P_η the projection $P_\eta(x) = \langle x, u_\eta\rangle u_\eta$ and define ${P_\eta}^\perp = I - P_\eta$.

Theorem 5. *Equality in Theorem* 3 *holds if and only if* φ *is of the form*

$$\varphi(x) = e^{\frac{\alpha}{2}\langle x,u_\eta\rangle^2}\varphi({P_\eta}^\perp x), \tag{6}$$

where α *is a real number such that* $|\alpha| < \frac{1}{2}$.

Proof: Without loss of generality, we may assume that $|\eta|_0 = 1$. It is clear that if $\varphi = 0$, then we have equality in Theorem 3, so we may assume that $\varphi \neq 0$. It is easy to check that the functions of the form (6) satisfies condition (4). Hence by Theorem 4 the equality in Theorem 3 holds.

Now suppose that φ is a function in (L^2) which satisfies the equality in Theorem 3. Then by Theorem 4, φ satisfies condition (4). Since $\varphi \neq 0$, the costant $K_2 \neq 0$. Apply the S-transform to both sides of condition (4). Then $S\varphi$ satisfies the following equation:

$$\alpha\langle\xi,\eta\rangle S\varphi(\xi) = (1-\alpha)S(D_\eta\varphi)(\xi), \tag{7}$$

where $\alpha = \frac{K_1}{K_2}$ and $\xi \in \mathcal{E}$.

The case $\alpha = 1$ implies that $S\varphi(\xi) = 0$ except for $\xi \perp \eta$. If $\xi \perp \eta$, then $\forall t \in R\backslash\{0\}$, $\xi + t\eta$ is not perpendicular to η. Since $S\varphi$ is continuous on $\mathcal{E}_c$, making $t \to 0$ we can see that $S\varphi(\xi) = 0$. Hence $S\varphi(\xi) = 0$ for all $\xi \in \mathcal{E}$ which, in turn, implies that $\varphi = 0$. Therefore $\alpha \neq 1$.

To solve equation (7), for any fixed $\xi \in \mathcal{E}$ define the function f on $\mathbb{R}$ by

$$f(t) = S\varphi(t\eta + P_\eta{}^\perp \xi).$$

Then f is differentiable and

$$f'(\langle \xi, \eta \rangle) = \frac{\alpha}{(1-\alpha)} \langle \xi, \eta \rangle f(\langle \xi, \eta \rangle).$$

Put $t = \langle \xi, \eta \rangle$, and the above equation becomes

$$f'(t) = \frac{\alpha}{(1-\alpha)} t f(t).$$

It is easy to see that the solution f is given by

$$f(t) = f(0) e^{\frac{\alpha}{2(1-\alpha)} t^2}.$$

Observe that $f(\langle \xi, \eta \rangle) = S\varphi(\xi)$ and $f(0) = S\varphi(P_\eta{}^\perp \xi)$. $S\varphi$ is given by

$$S\varphi(\xi) = e^{\frac{\alpha}{2(1-\alpha)} \langle \xi, \eta \rangle^2} S\varphi(P_\eta{}^\perp \xi).$$

Taking the inverse S-transform, we obtain

$$\varphi(x) = e^{\frac{\alpha}{2} \langle x, \eta \rangle^2} \Phi(P_\eta^\perp x), \tag{8}$$

where

$$\Phi(x) = \sqrt{(1-\alpha)} \int_{\mathcal{E}'} \varphi(x + \langle \eta, y \rangle \eta) \mu(dy).$$

Since $\varphi \in (L^2)$, we must have $|\alpha| < \frac{1}{2}$. Finally, if we replace x by $P_\eta^\perp x$ in (8), we find that

$$\varphi(P_\eta^\perp x) = \Phi(P_\eta^\perp x).$$

This proves the theorem. ■

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