TAIWANESE JOURNAL OF MATHEMATICS
Vol. 3, No. 4, pp. 425-474, December 1999

EXPANSION METHODS AND SCALING LIMITS ABOVE CRITICAL DIMENSIONS

Wei-Shih Yang[1] and Aklilu Zeleke[2]

Abstract. In this paper, we give a unified approach to various forms of high temperature expansions. The variants of expansion include Mayer's expansion, cluster expansion and lace expansion. We also give a brief summary of applications of lace expansion to scaling limits of self-avoiding random walks, lattice trees and percolation above their critical dimensions.

1. Introduction

1.1. Motivation

Over the past twenty years, interacting particle systems which exhibit critical phenomena have been intensively studied in the literature of probability theroy. Interesting models in this area of research all have an upper critical dimension, above which the form of limiting distributions do not depend on the dimensions and below which the form of limiting distributions usually depend essentially on the dimensionality of the underlying space. The methods of high temperature expansion used in statistical physics have proved to be rigorous and recently become very successful in dealing with limiting distributions of interacting particle systems above their critical dimensions [11, 21, 31, 32]. The high temperature expansions have changed their names and forms for the past according to the models studied. The names have appeared to be high temperature expansion, low temperature expansion, Mayer's expansion, cluster

Received September 23, 1999.
Communicated by P. Y. Wu.
1991 *Mathematics Subject Classification*: Primary 60J15; Secondary 82A67.
Key words and phrases: Lattice tree, self-avoiding random walk, percolation, oriented percolation, lace expansion, high temperature expansion, mean field.
[1]The work of the first author was partially supported by National Center for Theoretical Sciences, Taiwan.
[2]Reserch partially supported by Alma College Faculty Small Grant.

expansion in quantum field theories, lace expansion and double lace expansion. There is a common feature in all of these expansions, namely, the idea of the "minimally connected graph". The goal of this paper is to give a unified approach to various form of high temperature expansions.

We will start with a few typical models, give definitions, state the conjectures and main results known so far.

1.2. Self-avoiding Random Walks

Self-avoiding random walk is an interacting particle system, where a particle is moving at random on a d-dimensional lattice with the restrictions that it cannot go back to the sites already visited. This particle thus has a self-interaction. This model is originally studied in polymer physics [15]. It is of particular interest because the problem is naturally defined in terms of geometry and is highly non-Markovian from the probabilistic point of view. It also has applications in quantum field theory [6].

The high temperature expansion method is called lace expansion when applied to the problems of self-avoiding random walks. To start, let us look at the problems of self-avoiding random walks. We consider a random walk $X_n = Y_1 + \ldots + Y_n$ on $\mathbb{Z}^d$, where $\{Y_n, n = 1, 2, \ldots\}$ is a family of independently identically distributed random variables on $\mathbb{Z}^d$ with mean 0.

Let $E(\cdot)$ denote the expectation of a random variable, and $P(\cdot)$ the probability of an event.

Definition. (X_n) is in the *domain of attraction* of α-stable distribution if $X_n/n^{1/\alpha} \to Z_\alpha$ in distribution, where $E(e^{ik\cdot Z_\alpha}) = e^{-\xi \sum_{i=1}^{d} |k_i|^\alpha}$ for some constant ξ. In particular, Z_2 is said to be Gaussian distributed and Z_1 is Cauchy distributed.

Example 1 (Nearest-neighbor random walk). If $P(Y_1 = e_j) = P(Y_1 = -e_j) = 1/(2d)$ for all j, then (X_n) belongs to the domain of attraction of Gaussian.

Example 2. It $P(Y_1 = ne_j) = \text{Const}/|n|^{1+\alpha}$ for nonzero integer n and $P(Y_1 = x) = 0$ for any other x, then (X_n) belongs to the domain of attraction of Z_α, $0 < \alpha \leq 2$.

Let $P(A|B) = P(A \cap B)/P(B)$ be the conditional probability of A given B.

Definition. Let $\tilde{P}_n(A) = P(A \mid X_i \neq X_j \text{ for all } 0 \leq i < j \leq n)$. The random walk (X_n) under distribution $\tilde{P}_n$ is called a *self-avoiding random walk*.

The problem is to find v and the limiting distribution of X_n/n^{ν} under $\tilde{P}_n$.

It is conjectured that if (X_n) belongs to the domain of attraction of α-stable process, then as $n \to \infty$, we have

(a) for $d > d_c = 2\alpha$,
$$\tilde{E}_n\left(e^{ik\cdot\frac{X_n}{n^{1/\alpha}}}\right) \to e^{-\xi\Sigma|k_i|^{\alpha}},$$
where $\tilde{E}_n(\cdot)$ is the expectation with respect to the probability measure $\tilde{P}_n$;

(b) for $d < d_c = 2\alpha$,
$\tilde{E}_n\left(e^{ik\cdot\frac{X_n}{n^{\nu}}}\right) \to$ some distribution other than α-stable process;

(c) $\alpha = 2,\ d = 4$,
$$\tilde{E}_n\left(e^{ik\cdot\frac{X_n}{n^{1/2}(\ln n)^{1/8}}}\right) \to e^{-\xi|k|^2}.$$

Here d_c is called the *upper critical dimension.* It is a typical phenomenon in models of statistical mechanics that the limiting distribution of the models does not depend on the dimensions when $d > d_c$.

Remark. For $\alpha = \frac{1}{2},\ 1,\ \frac{3}{2}$, no conjectures have been made on the limiting distribution where $d = d_c$.

Consider a weakly self-avoiding random walk defined by
$$\tilde{E}_n^{\beta}(F) = \frac{E(F\psi[0,n])}{E(\psi[0,n])},$$
where $\psi[0,n] = e^{-\beta\Sigma_{0\le i<j\le n}\delta(X_i - X_j)}$, and $\delta(x) = 1$ if $x = 0$ and $\delta(x) = 0$ if $x \neq 0$. $\tilde{E}_n^{\beta}(\cdot)$ is a self-avoiding random walk if $\beta = \infty$, a simple random walk if $\beta = 0$, and is called a *weakly self-avoiding random walk* if $0 < \beta < \infty$. The following are known results under $\tilde{P}_n^{\beta}$:

(1) (Brydges and Spencer [7]). For nearest-neighbor case, $d \geq 5$, $X_n/\sqrt{n} \to$ *Gaussian* if $0 < \beta$ is sufficiently small.

(2) (D. Klein and W. S. Yang [40]). For $\alpha = 1,\ d \geq 3,\ X_n/n \to$ *Cauchy* distribution if $0 < \beta$ is sufficiently small.

(3) (G. Slade [38]). For nearest-neighbor case, $\beta = \infty$, $X_n/\sqrt{n} \to$ *Gaussian* if d is sufficiently large.

Finally we have the following theorem.

Theorem 1.1. (Hara and Slade [21]) *For nearest-neighbor case, $\beta = \infty$, $d \geq 5$, there exist constants A, D such that*

(a) $C_n = A \cdot \mu^n [1 + O(\frac{1}{n^\epsilon})]$, *for any* $0 < \epsilon < \frac{1}{2}$.

(b) $\tilde{E}_n(|X_n|^2) = Dn[1 + O(\frac{1}{n^\epsilon})]$, *for any* $0 < \epsilon < \frac{1}{4}$.

(c) $\tilde{E}_n\left(e^{ik \cdot \frac{X_n}{\sqrt{n}}}\right) \to e^{-\frac{D}{2d}|k|^2}$, *as* $n \to \infty$.

Here C_n is the number of self-avoiding random walks of length n. Moreover, for $d = 5$, $A \in [1,\ 1.493]$, $D \in [1.098,\ 1.803]$ and $\mu \geq 8.82128$.

In Section 4, we will discuss lace expansion, the expansion method for self-avoiding random walks and the ideas of the proof of Theorem 1.1.

1.3. Lattice Trees

Lattice tree model is a natural generalization of self-avoiding random walks. It can be viewed as a self-avoiding branching random walk. A lattice tree in a d-dimension lattice $\mathbb{Z}^d$ is a finite connected set of lattice bonds containing no cycles. For the nearest-neighbor model, the bonds are nearest-neighbor bonds $\{x, y\}$, $x, y \in \mathbb{Z}^d$, $\|x - y\|_1 = 1$. We will also consider "spread-out" lattice trees constructed from bonds $\{x, y\}$ with $0 < \|x - y\|_\infty \leq L$. The parameter L will be taken to be large but finite. For each $n = 1, 2, \ldots$, let P_n be the uniform distribution over the set of all n-bound lattice trees which contain the origin. For notational convenience, we let P_0 be the probability measure concentrated at the origin.

For an n-bond lattice tree T, we consider a probability measure on $\mathbb{R}^d$ defined by

$$\nu_T(dx) = \frac{1}{n+1} \sum_{i \in T} \delta_{(\frac{i}{D_1 n^{1/4}})}(dx),$$

where D_1 is a constant depending on d and L, which will be determined later in order to get a scaling limit as $n \to \infty$. Here δ_y denotes the probability measure concentrated at y.

Let $M_1(\mathbb{R}^d)$ be the set of all probability measures on $\mathbb{R}^d$. Note that $M_1(\mathbb{R}^d)$ is a topological space with weak topology, i.e., $\nu_n \to \nu$ in $M_1(\mathbb{R}^d)$ if and only if $\lim_{n \to \infty} \int_{\mathbb{R}^d} f d\nu_n = \int_{\mathbb{R}^d} f d\nu$ for all bounded continuous functions f on $\mathbb{R}^d$.

For each n, a probability measure μ_n on $M_1(\mathbb{R}^d)$ is defined by

$$\mu_n(\nu) = P_n\{T;\ \nu_T = \nu\}.$$

It is conjectured by Aldons [2] that for nearest-neighbor model μ_n converges to μ_{ISE} (integrated super-Brownian excursion) for $d > 8$, as $n \to \infty$. In other words, the upper critical dimension for lattice tree should be 8. ISE is super-Brownian motion (see e.g. [9]) conditioned to have total mass 1. The definition of ISE will be given in Section 6.

Theorem 1.2. (E. Derbez and G. Slade [11]) *For nearest-neighbor lattice trees in sufficiently high dimensions $d \geq d_0$, or for spread-out lattice trees with $d > 8$ and L sufficiently large, there exists D_1 such that $\mu_n \to \mu_{\text{ISE}}$, weakly on $M_1(\dot{\mathbb{R}}^d)$, as $n \to \infty$.*

It is generally believed that the upper critical dimensions are the same for nearest neighbor model and for spread-out model. This belief is known as the *hypothesis of universality.* According to the hypothesis of universality, Theorem 1.2 gives strong evidence that the upper critical dimension for lattice trees is 8.

1.4. Percolation

Percolation has been intensively studied in the probability literature as well as in statistical mechanics; see e.g. [19]. In this model, every nearest-neighbor bond in $\mathbb{Z}^d$ is open or closed independently with $P(b \text{ is open }) = p$ and $P(b \text{ is closed }) = 1 - p$, for each nearest-neighbor bond b in $\mathbb{Z}^d$. Here p is a parameter. Given a configuration of open or closed bonds, let $C(0)$ be the set of all sites that are connected to 0 by a path consisting of open bonds. Let $|C(0)|$ denote the cardinality of $C(0)$. It is known that (see e.g. [19]) for $d \geq 2$, there exists $0 < p_c < 1$ such that

$$\begin{aligned} P(|C(0)| < \infty) &= 1, \qquad \text{for } p < p_c, \\ P(|C(0)| < \infty) &< 1, \qquad \text{for } p > p_c. \end{aligned}$$

It is a typical situation in statistical mechanics that the models are easier to be analyzed when p is away from p_c than $p = p_c$. A large number of rigorous results have been obtained for $p \neq p_c$ but only a few rigorous results are known when $p = p_c$. One of the most interesting results known at $p = p_c$ is about critical exponents.

Let $\chi(p) = E(|C(0)|)$, and $\theta(p) = P(|C(0)| = \infty)$. The critical exponents

γ and β are defined by

$$\gamma = -\lim_{p \uparrow p_c} \frac{\ln \chi(p)}{\ln |p - p_c|},$$
$$\beta = \lim_{p \downarrow p_c} \frac{\ln \theta(p)}{\ln |p - p_c|}.$$

It is conjectured in physics literature that γ and β exist and they should take their "mean field" values above the upper critical dimension. The mean field values of γ and β are calculated by replacing $\mathbb{Z}^d$ by a binary tree where the calculation becomes easy and $\gamma = 1$, $\beta = 1$. The upper critical dimension for percolation is conjectured to be $d_c = 6$, In [1], Aizenman and Newman introduced the "triangle condition". They showed that if the triangle condition is satisfied, then $\gamma = \beta = 1$. Then Hara and Slade [20] proved that the triangle condition is satisfied if d is sufficiently large.

For the scaling limit for percolation, given $C(0)$ with $|C(0)| = n$, we define a random measure by

$$\nu_{C(0)} = \frac{1}{n} \sum_{x \in C(0)} \delta_{(\frac{x}{D_2 n^{1/4}})}.$$

Let μ_n be a probability measure on $M_1(\mathbb{R}^d)$ (the set of all probability measures on $\mathbb{R}^d$) with mass

$$\mu_n(\nu) = P(\nu_{C(0)} = \nu \mid |C(0)| = n).$$

It is conjectured in [23] that for $d > 6$ and $p = p_c$, there exists D_2 such that

$$\lim_{n \to \infty} \mu_n = \mu_{\text{ISE}} \text{ on } M_1(\mathbb{R}^d).$$

It has been shown [23] that the first and second moments of μ_n converge to those of μ_{ISE}. We will discuss this in Section 6.4.

1.5. Oriented Percolation

Lace expansion is a very powerful tool. Oriented percolation is another example other than self-avoiding random walk, to which lace expansion can be applied. Consider a $(d+1)$-dimensional lattice $\mathbb{Z}^{d+1}$. Each lattice site is denoted by (x, n), $x \in \mathbb{Z}^d$, $n \in \mathbb{Z}^1$. Let

$$\mathcal{B} = \{b; b \text{ is an oriented bond which goes from } (x, n) \text{ to } (y, n+1)\}.$$

Let $\{\sigma_b,\ b \in \mathcal{B}\}$ be a family of independent Bernoulli random variables such that $P\{\sigma_b = 1\} = p_{x,y}$ and $P\{\sigma_b = 0\} = 1 - p_{x,y}$ for b goes from (x, n) to $(y, n+1)$. We assume that $p_{x,y} = p_{y,x} = p_{0,x-y}$. Two typical cases are considered:

(1) Nearest-neighbor case: $p_{0,x} = p$ if $|x| = 1$, and $p_{0,x} = 0$ for any other$x$.

(2) Spread-out case: $p_{0,x} = pg(\frac{x}{L})\frac{1}{L^d}$, where $g : \mathbb{R}^d \to \mathbb{R}$ is a nonnegative smooth function with compact support, symmetric about coordinate axis and $g(0) > 0$.

Given a configuration $(\sigma_b,\ b \in \mathcal{B})$, we say that b is *open* if $\sigma_b = 1$ and *closed* if $\sigma_b = 0$. Lattice sites (x, n) is said to be *connected* to (y, m) if there exists a path consisting of open oriented bonds, which goes from (x, n) to (y, m). In this case, we write $(x, n) \to (y, m)$. We also say $(x, n) \to (x, n)$. Let

$$C_0 = \{(x, n); (0, 0) \to (x, n)\}.$$

Let $\theta(p) = P\{|C_0| = \infty\}$ and $\chi(p) = E(|C_0|)$. Similar to percolation, it is well-known that for $d \geq 1$, there exists $0 < p_c < 1$ such that $\theta(p) = 0$ for $p < p_c$ and $\theta(p) > 0$ for $p > p_c$. It is conjectured that for all d, there exist constants γ, β such that

$$\begin{aligned} \theta(p) &\sim (p - p_c)^{\beta} && \text{as } p \downarrow p_c, \\ \chi(p) &\sim (p_c - p)^{-\gamma} && \text{as } p \uparrow p_c, \end{aligned}$$

and $\gamma = \beta = 1$ for $d \geq 5$; $\beta,\ \gamma \neq 1$ for $1 \leq d \leq 4$.

Theorem 1.3. (Nguyen and Yang [31])

(a) *For nearest-neighbor model,* $\gamma = \beta = 1$ *if* d *is sufficiently large.*

(b) *For spread-out model,* $d \geq 5$, *we have* $\gamma = \beta = 1$ *if* L *is sufficiently large.*

Note that it is generally believed that nearest-neighbor model and spread-out model have the same β, γ and critical dimensions $d_c = 5$. This is known as both cases belong to the same universality class.

Another important problem one concerns is the limiting distribution of percolation at critical point.

Let $C(x, n) = P((0, 0) \to (x, n))$ and $C_n = \sum_x C(x, n)$.

Theorem 1.4. (Nguyen and Yang [32])

(a) *For nearest-neighbor model, there exists* $d_0 \geq 5$ *such that for all* $d \geq d_0$, $0 < p \leq p_c$, *there exist constants* $A,\ D$ *such that*

  (1) $C_n = A\mu^n[1 + O(\frac{1}{n^\alpha})], \quad 0 < \alpha < \frac{1}{4}$,

  (2) $\frac{1}{C_n}\sum_x C(x, n)|x|^2 = Dn[1 + O(\frac{1}{n^\alpha})], \quad 0 < \alpha < \frac{1}{4}$,

(3) $\frac{1}{C_n}\sum_x e^{ik\cdot\frac{x}{\sqrt{n}}}C(x,n) \to e^{-\frac{D}{2d}|k|^2}$.

Moreover, $\mu \le 1$, and $\mu = 1$ if and only if $p = p_c$.

(b) *For spread-out model, $d \ge 5$, the same hold if L is sufficiently large.*

The methods used in the proofs of Theorem 1.1-1.4 are lace expansion and its generalizations (double lace expansions [11]). Lace expansion was first developed by Brydges and Spencer [7] for weakly self-avoiding random walks. The idea of lace expansion can be traced back to the high temperature expansion for grand canonical ensemble in classical statistical mechanics; see e.g. [33]. There it is called Mayer's expansion. The high temperature expansion for Ising model and models for quantum field theory is called cluster expansion; see e.g. [16]. It is the goal of this paper to explain the transitions of ideas from Mayer's expansion, cluster expansion to lace expansion.

This paper is organized as follows. In Section 2, we discuss Mayer's expansion for grand canonical ensemble. Section 3 is about cluster expansion for Ising model for both high and low temperatures. In Section 4, we discuss lace expansion and the ideas of the proofs of Theorem 1.1. In Section 5, we explain the idea of lace expansion for oriented percolation. In Section 6, we give a brief summary of some of the developments involving lace expansion so far.

2. Mayer's Expansion

2.1. Mayer's Expansion

The grand canonical ensemble of classical particles in $\mathbb{R}^d$ can be described as follows. We consider random number of particles distributed in a bounded measurable subset Λ of $\mathbb{R}^d$. We denote them by $(X_1, \cdots, X_N)$, where $N = 0, 1, 2, \cdots$ is a random variable, $X_i \in \Lambda$. The sample space is

$$\Omega = \left(\bigcup_{n=1}^{\infty} \Lambda^{(n)}\right) \bigcup \{\partial\},$$

where ∂ represents a no particle state, and $\Lambda^{(n)} = \Lambda \times \cdots \times \Lambda$ (n times). Let μ_Λ be a probability measure on Ω such that $\mu_\Lambda|_{\{\partial\}} = \frac{1}{Z_\Lambda}$,

$$\mu_\Lambda|_{\Lambda^{(n)}}(dx_1 \cdots dx_n) = \frac{z^n}{Z_\Lambda n!} e^{-\beta\Sigma_{1\le i<j\le n} V(x_i - x_j)} dx_1 dx_2 \cdots dx_n.$$

Here $V : \mathbb{R}^d \to \mathbb{R}$ is a function such that $V(-x) = V(x)$ and satisfies the stability condition

$$\sum_{1\le i<j\le n} V(x_i - x_j) \ge -cn, \quad \text{for all } x_i \in \mathbb{R}^d \text{ and all } n \ge 1.$$

The parameter z is called *activity*, $\beta = \frac{1}{T}$, T is the *temperature*, and Z_Λ is the normalization constant. Z_Λ is called the partition function and

$$Z_\Lambda = \sum_{n=0}^{\infty} \frac{z^n}{n!} \int_\Lambda \cdots \int_\Lambda e^{-\beta \Sigma_{1\le i<j\le n} V(x_i - x_j)} dx_1 dx_2 \cdots dx_n.$$

Examples of stable V other than $V \ge 0$ can be found, e.g., in [35, §3.2].

Let $|\Lambda|$ denote the Lebesque measure of Λ. For $a > 0$, let

$$\partial_a \Lambda = \{y \in \Lambda; |y - x| \le a \text{ for some boundary point } x \text{ of } \Lambda\}.$$

Definition. The free energy of the grand canonical ensemble is

$$P = -\lim_{\Lambda \nearrow \mathbb{R}^d} \frac{\ln Z_\Lambda}{|\Lambda|\beta}, \text{ in van Hove sense.}$$

Here $\Lambda \nearrow \mathbb{R}^d$ in van Hove sense means that $\Lambda \nearrow \mathbb{R}^d$ and $|\partial_a \Lambda|/|\Lambda| \to 0$ for all $a > 0$.

One of the most important problems in statistical mechanics is the following

Problem: Determine whether P is an analytic function of z and find the radius of convergence.

To solve this problem, we will give an expansion formula for $\ln Z_\Lambda$. To this end, let

$$\psi[1, \cdots, n] = \prod_{1\le i<j\le n} e^{-\beta V_{ij}},$$

where $V_{ij} = V(x_i - x_j)$. Then

$$\begin{aligned}
\psi[1, \cdots, n] &= \prod_{1\le i<j\le n} [(e^{-\beta V_{ij}} - 1) + 1] \\
&= \sum_{G \text{ graph on } \{1,\cdots,n\}} \prod_{ij\in G} (e^{-\beta V_{ij}} - 1), \ n \ge 2,
\end{aligned}$$

where a subset of $\{(i,j); 1 \le i < j \le n\}$ is called a graph on $\{1, 2, \cdots, n\}$. $(i,j) \in G$ is called an edge of G. Given a graph G, i and j are said to be connected, $i \leftrightarrow j$, if $(i,j) \in G$. For any $1 \le i < j \le n$, given G, i and j are also said to be connected if there exist $i_0, i_1, \cdots, i_m$ such that $i = i_0 \leftrightarrow i_1 \leftrightarrow i_2 \leftrightarrow \cdots \leftrightarrow i_m = j$. In this case, we also denote $i \leftrightarrow j$. Let $S \subseteq \{1, 2, \cdots, n\}$. The graphs on S, and their connectedness are defined analogously. We say that G is a connected graph on S if for every i, $j \in S$, $i \neq j$, we have $i \leftrightarrow j$ by G.

Definition. Let $S \subseteq \{1, \cdots, n\}$. The connected part of ψ is defined by

$$\begin{aligned} \psi_c(S) &= 1, && \text{if } |S| = 1, \\ \psi_c(S) &= \sum_{\substack{\Gamma:\text{connected} \\ \text{graph on S}}} \prod_{ij \in \Gamma} (e^{-\beta V_{ij}} - 1), && \text{if } |S| > 1. \end{aligned}$$

Theorem 2.1. (Mayer's Expansion) *For all z inside the radius of convergence of the expansion, we have*

$$\ln Z_\Lambda = \sum_{n=1}^{\infty} \frac{z^n}{n!} \int_\Lambda \cdots \int_\Lambda \psi_c(1, 2, \ldots, n) dx_1 dx_2 \cdots dx_n.$$

Proof.

$$\begin{aligned} Z_\Lambda &= 1 + \sum_{n=1}^{\infty} \frac{z^n}{n!} \int_\Lambda \cdots \int_\Lambda \sum_G \prod_{ij \in G} (e^{-\beta V_{ij}} - 1) dx_1 \cdots dx_n \\ &= + \sum_{n=1}^{\infty} \frac{z^n}{n!} \int_\Lambda \cdots \int_\Lambda \sum_{l=1}^{\infty} \sum_{\substack{\{S_1, \cdots, S_l\} \\ \text{partition} \\ \text{of } \{1, \ldots, n\}}} \psi_c(S_1) \cdots \psi_c(S_l) \prod_{i=1}^{n} dx_i \\ &= 1 + \sum_{n=1}^{\infty} \frac{z^n}{n!} \int \cdots \int \sum_{l=1}^{\infty} \frac{1}{l!} \sum_{\substack{(S_1, \cdots, S_l) \\ S_i \bigcap S_j = \emptyset \\ \bigcup_i S_i = \{1, \cdots, n\}}} \psi_c(S_1) \cdots \psi_c(S_l) \prod_{i=1}^{n} dx_i \\ &= 1 + \sum_{n=1}^{\infty} \frac{z^n}{n!} \sum_{l=1}^{\infty} \frac{1}{l!} \sum_{\substack{n_1, \cdots, n_l \\ n_1 \geq 1 \\ n_1 + \cdots + n_l = n}} \frac{n!}{n_1! n_2! \ldots n_l!} \prod_{i=1}^{l} K(n_i), \end{aligned}$$

where

$$K(n_i) = K(|S_i|) = \int_\Lambda \cdots \int_\Lambda \psi_c(S_i) \prod_{j \in S_i} dx_j.$$

Note that this integral depends only on the number of elements in S_i and therefore the above equals

$$\begin{aligned} &1 + \sum_{l=1}^{\infty} \frac{1}{l!} \sum_{n_i \geq 1} \frac{z^{n_1 + \cdots + n_l}}{n_1! \cdots n_l!} \prod_{i=1}^{l} K(n_i) \\ &= \sum_{l=0}^{\infty} \frac{1}{l!} \left(\sum_{n=1}^{\infty} \frac{z^n}{n!} K(n) \right)^l = \exp \left(\sum_{n=1}^{\infty} \frac{z^n}{n!} K(n) \right). \end{aligned}$$

■

Definition. Let $\Gamma \in \mathcal{C}(1, \cdots, n)$, the set of connected graphs on $\{1, 2, \cdots, n\}$. Γ is called a *tree graph* if Γ is minimally connected.

Let $T(1, \cdots, n)$ denote the set of all tree graphs on $\{1, 2, \cdots, n\}$. Given $\Gamma \in \mathcal{C}(1, \cdots, n)$, we define a tree graph $T(\Gamma) \subseteq \Gamma$ as follows: We order all the edges by $(i, j) < (i', j')$ if $i < i'$ or $j < j'$ whenever $i = i'$. We throw away edges in Γ according to this order, as many as possible. As long as it is still connected, then the result is a tree graph. We call it $T(\Gamma)$. Given a tree graph T, let $e(T) = \{\Gamma; T(\Gamma) = T\}$. We claim:

(2.1) $\Gamma_1, \Gamma_2 \in e(T) \Rightarrow \Gamma_1 \bigcup \Gamma_2 \in e(T)$.

(2.2) There is a maximal element in $e(T)$, called $m(T)$.

(2.3) Let Γ be such that $T \subseteq \Gamma \subseteq m(T) \Rightarrow \Gamma \in e(T)$.

It follows from (2.1), (2.2), (2.3) that the mapping

(2.4) $A \subseteq m(T) \setminus T \leftrightarrow A \bigcup T \in e(T)$ is one-to-one.

(2.2) follows easily from (2.1) by taking unions of all elements in $e(T)$. We leave (2.1), (2.3) for the reader to check.

Theorem 2.2. (Tree Graph Formula) *For z inside the radius of convergence of Mayer's expansion of* $\ln Z_\Lambda$, *we have*

$$\ln Z_\Lambda = \sum_{n=1}^{\infty} \frac{z^n}{n!} \int_\Lambda \cdots \int_\Lambda (dx_1) \cdots (dx_n) \sum_{T \in T(1,\cdots,n)} \prod_{ij \in T} (e^{-\beta V_{ij}} - 1) \prod_{m(T)\setminus T} (e^{-\beta V_{ij}}).$$

Proof. Since

$$\begin{aligned}\prod_{m(T)\setminus T} e^{-\beta V_{ij}} &= \prod_{m(T)\setminus T} [(e^{-\beta V_{ij}} - 1) + 1] \\ &= \sum_{A \subseteq m(T)\setminus T} \prod_{ij \in A} (e^{-\beta V_{ij}} - 1),\end{aligned}$$

$$\begin{aligned}\text{RHS} &= \sum_{n=1}^{\infty} \frac{z^n}{n!} \int_\Lambda \cdots \int_\Lambda dx_1 \cdots dx_n \sum_{T \in T(1,\cdots,n)} \sum_{A \in m(T)\setminus T} \prod_{ij \in T \bigcup A} (e^{-\beta V_{ij}} - 1) \\ &= \sum_{n=1}^{\infty} \frac{z^n}{n!} \int_\Lambda \cdots \int_\Lambda dx_1 \cdots dx_n \sum_{T} \sum_{\Gamma \in e(T)} \prod_{ij \in \Gamma} (e^{-\beta V_{ij}} - 1) \\ &= \sum_{n=1}^{\infty} \frac{z^n}{n!} \int_\Lambda \cdots \int_\Lambda dx_1 \cdots dx_n \sum_{\Gamma \in \mathcal{C}(1,\cdots,n)} \prod_{ij \in \Gamma} (e^{-\beta V_{ij}} - 1) \\ &= \log Z_\Lambda, \quad \text{by Theorem 2.1.}\end{aligned}$$

■

2.2. Convergence of Mayer's Expansion

Corollary 2.3. (Penrose Tree Graph Bound [33]) *If* $V \geq 0$ (*repulsive potential*), *then*

$$|\psi_c(1,\cdots,n)| \leq \sum_{T\in T(1,\cdots,n)} \prod_{ij\in T} |e^{-\beta V_{ij}} - 1|.$$

Proof. It follows from the proof of Theorem 2.2 that

$$\psi_c(1,2,\ldots,n) = \sum_{T\in T(1,2,\ldots,n)} \prod_{ij\in T} (e^{-\beta V_{ij}} - 1) \prod_{m(T)\setminus T} e^{-\beta V_{ij}},$$

and therefore the inequality holds. ■

Remark. It is well-known (see e.g. Brydges [4]) that for stable potential,

$$|\psi_c(1,\cdots,n)| \leq \sum_{T\in T(1,\cdots,n)} \prod_{ij\in T} |\beta V_{ij}|.$$

2.3. Convergence of Mayer's Expansion

We shall discuss the convergence of Mayer's expansion for repulsive potential only as the case of stable potential can be treated similarly using the above Remark.

By Penrose Tree Graph Bound, for $V \geq 0$, $\log Z_\Lambda$ can be estimated by

$$\sum_{n=1}^{\infty} \frac{|z|^n}{n!} \sum_T \int_\Lambda \cdots \int_\Lambda dx_1 \cdots dx_n \prod_{ij\in T} |e^{-\beta V_{ij}} - 1|. \tag{2.5}$$

If we integrate it starting from the variables corresponding to extremal points of T, and by Cayley's Theorem, the number of T is bounded by n^{n-2}, we get an upper bound for the above

$$\sum_{n=1}^{\infty} \frac{|z|^n}{n!} n^{n-2} |\Lambda| \alpha^{n-1} \tag{2.6}$$

with

$$\alpha = \int_{\mathbb{R}^d} |e^{-\beta V(x)} - 1| dx.$$

So $\frac{1}{|\Lambda|} \ln Z_\Lambda$ is absolutely convergent, uniformly in Λ, if

$$|z| e \int_{\mathbb{R}^d} |e^{-\beta V(x)} - 1| dx < 1.$$

Since $\psi_c(1,2,\dots n)$ is a function of $x_1, x_2, \dots, x_n$, we will also write $\psi_c(x_1, x_2, \dots, x_n)$ for $\psi_c(1,2,\dots n)$.

Theorem 2.4. *If $e\alpha|z| < 1$, then*

$$P = -\frac{1}{\beta}\sum_{n=1}^{\infty}\frac{z^n}{n!}\int_{\mathbb{R}^d}\cdots\int_{\mathbb{R}^d}\psi_c(0, x_2, \cdots, x_n)dx_2\cdots dx_n.$$

Proof. Since $e\alpha|z| < 1$, (2.5) and (2-6) hold. Using upper bounds (2.5), (2.6) and Theorem 2.1, to prove Theorem 2.4, by Dominated Convergence Theorem, it is sufficient to prove

$$\begin{aligned}&\lim_{\Lambda\nearrow R^d}\frac{1}{|\Lambda|}\int_{\Lambda}\cdots\int_{\Lambda}\psi_c(x_1, x_2, \cdots, x_n)dx_1\cdots dx_n\\ &=\int_{\mathbb{R}^d}\cdots\int_{\mathbb{R}^d}\psi_c(0, x_2, \cdots, x_n)dx_2\cdots dx_n\end{aligned}\tag{2.7}$$

for all $n \geq 2$.

Note that in (2.5), when we integrate the variables corresponding to extremal points of T to get (2.6), we may always let x_1 be the last variable to be integrated. Therefore, for all $n \geq 2$, fixed T,

$$\int_{\mathbb{R}^d}\cdots\int_{\mathbb{R}^d}dx_2\cdots dx_n\prod_{ij\in T}|e^{-\beta V_{ij}}-1| \leq \alpha^{n-1} < \infty,\tag{2.8}$$

for all $x_1 \in \mathbb{R}^d$. Using Penrose Tree Graph Bound and (2.8), we have for all $x_1 \in \mathbb{R}^d$,

$$\begin{aligned}c&=\int_{\mathbb{R}^d}\dots\int_{\mathbb{R}_d}|\psi_c(x_1, x_2, \dots, x_n)|dx_2\dots dx_n\\ &=\int_{\mathbb{R}^d}\dots\int_{\mathbb{R}^d}|\psi_c(0, x_2, \dots, x_n)|dx_2\dots dx_n \leq \alpha^{n-1}\cdot n^{n-2} < \infty.\end{aligned}\tag{2.9}$$

By the Dominated Convergence Theorem, for all $\epsilon > 0$, there exists $r > 0$ such that

$$\int_{\mathbb{R}^d}\dots\int_{|x_i|\geq r}\dots\int_{\mathbb{R}^d}|\psi_c(0, x_2, \dots, x_n)|dx_2\dots dx_n < \epsilon\tag{2.10}$$

for all $i = 2, \dots, n$.

By translational invarance and (2.10),

$$\int_{\mathbb{R}^d}\dots\int_{|x_i-x_1|\geq r}\dots\int_{\mathbb{R}^d}|\psi_c(x_1, x_2, \dots x_n)|dx_2\dots dx_n < \epsilon\tag{2.11}$$

for all $i = 2, \dots, n$. Now

$$\begin{aligned}
& |\frac{1}{|\Lambda|}\int_\Lambda \cdots \int_\Lambda \psi_c(x_1, \dots, x_n)dx_1 \dots dx_n \\
& -\int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \psi_c(0, x_2, \dots, x_n)dx_2 \dots dx_n| \\
= \; & \frac{1}{|\Lambda|}|\int_\Lambda \cdots \int_\Lambda \psi_c(x_1, \dots, x_n)dx_1 \dots dx_n \\
& -\int_{x_1\in\Lambda}\int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \psi_c(x_1, x_2, \dots, x_n)dx_1 \dots dx_n| \\
\leq \; & \sum_{i=2}^n \frac{1}{|\Lambda|}\int_{x_1\in\Lambda}\int_{\mathbb{R}^d} \cdots \int_{x_i\in\Lambda^c} \cdots \int_{\mathbb{R}^d} |\psi_c(x_1, \dots, x_n)|dx_1 \dots dx_n \\
= \; & \sum_{i=2}^n \frac{1}{|\Lambda|}[\int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} 1_{x_1\in\Lambda, d(x_1,\partial\Lambda)>r} 1_{x_i\in\Lambda^c}|\psi_c(x_1 \dots x_n)|dx_1 \dots dx_n \\
& +\int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} 1_{x_1\in\Lambda, d(x_1,\partial\Lambda)\leq r} 1_{x_i\in\Lambda^c}|\psi_c(x_1 \dots x_n)|dx_1 \dots dx_n],
\end{aligned}$$

by (2.9) and (2.11), which is bounded by

$$\sum_{i=2}^n \left[\frac{\epsilon}{|\Lambda|}|\Lambda| + \frac{1}{|\Lambda|}|\partial_r\Lambda| \cdot c\right]$$

$\leq (n-1)\epsilon$, as $\Lambda \nearrow \mathbb{R}^d$ in van Hove sense. Since ϵ is arbitrary, (2.7) follows. ∎

3. Expansions for Ising Model

3.1. High Temperature Expansion for Ising Model

In this section, we consider the cluster expansion method. This expansion method is very useful for spin systems in statistical mechanics and quantum field theory. We will use Ising model as an example to illustrate this method.

Let $\mathbb{Z}^d$ be a d-dimensional lattice. Let $\sigma = (\sigma_i,\, i \in \mathbb{Z}^d)$, $\sigma_i = -1$ or 1, denote a configuration. The set of all configurations on $\mathbb{Z}^d$ is denoted by χ. Then $\chi = \{-1, 1\}^{\mathbb{Z}^d}$.

Let Λ be a finite subset of $\mathbb{Z}^d$, and $\chi_\Lambda = \{-1, 1\}^\Lambda$. Let $\xi \in \chi$. The Hamiltonian in Λ with 0 external field and boundary condition ξ is defined by

$$H_\Lambda^\xi(\sigma) = -\sum_{\langle i,j\rangle} \sigma_i\sigma_j \tag{3.1}$$

for $\sigma \in \chi_\Lambda$, where $\langle i, j\rangle$ runs over all nearest-neighbor pairs such that either i or j is in Λ. If $i \in \Lambda^c$, then σ_i is understood as ξ_i. The partition function in Λ with temperature $T = \beta^{-1}$ is

$$Z_\Lambda^\xi = \int_{\chi_\Lambda} e^{-\beta H_\Lambda^\xi(\sigma)} d\mu_{0,\Lambda}\,, \tag{3.2}$$

where

$$d\mu_{0,\Lambda}(\sigma) = \prod_{i\in\Lambda} d\mu_0(\sigma_i)\,,$$

and $\mu_0(1) = \mu_0(-1) = \frac{1}{2}$. We also consider the case of 0-boundary conditions where $\xi_i = 0$ for all i. It is well-known (see e.g. [36]) that the pressure

$$p = \lim_{\Lambda\nearrow\mathbb{Z}^v} \frac{1}{|\Lambda|} \ln Z_\Lambda^\xi, \tag{3.3}$$

exists as $\Lambda \nearrow \mathbb{Z}^v$ in van Hove sense and is independent of ξ. The high temperature expansion gives the following

Theoremm 3.1. *There exists $\beta_0 > 0$ such that p is an analytic function of complex-valued β in the region $|\beta| < \beta_0$.*

The proof of Theorem 3.1 will be proceeded in the following steps.

Step 1. Polymer Expansion.

By (2.2), with 0-boundary conditions,

$$\begin{aligned} Z_\Lambda &= \int_{\chi_\Lambda} e^{\beta\Sigma_{\langle i,j\rangle}\sigma_i\sigma_j} d\mu_{0,\Lambda} \\ &= \int_{\chi_\Lambda} \prod_{\langle i,j\rangle} \Big[(e^{\beta\sigma_i\sigma_j} - 1) + 1\Big] d\mu_{0,\Lambda} \\ &= \sum_X \int_{\chi_\Lambda} \prod_{\langle i,j\rangle\in X} (e^{\beta\sigma_i\sigma_j} - 1) d\mu_{0,\Lambda}\,, \end{aligned} \tag{3.4}$$

where X runs over all subsets of bonds inside Λ. Let $\{\gamma_1, \gamma_2, \cdots, \gamma_n\}$ be the connected components of X. Then (3.4) equals

$$\sum_X \prod_{l=1}^n K(\gamma_l)\,, \tag{3.5}$$

where

$$K(\gamma) = \int_{\chi_{S(\gamma)}} \prod_{\langle i,j\rangle\in\gamma} (e^{\beta\sigma_i\sigma_j} - 1) d\mu_{0,S(\gamma)}$$

and $S(\gamma)$ is the set of lattice sites i which belong to any end points of bonds in γ. Since the set $\{\gamma_1, \gamma_2, \cdots, \gamma_n\}$ and X are in one-to-one correspondence, (3.5) can be rewritten as

$$1 + \sum_{n=1}^{\infty} \sum_{\{\gamma_1, \gamma_2, \cdots, \gamma_n\}} \prod_{l=1}^{n} K(\gamma_l), \tag{3.6}$$

where γ_i's are connected subsets of bonds in Λ and $S(\gamma_i) \cap S(\gamma_j) = \emptyset$, for any $i \neq j$. If we order the elements in $\{\gamma_1, \gamma_2, \cdots, \gamma_n\}$, by (3.6), then

$$Z_\Lambda = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\gamma_1, \gamma_2, \cdots, \gamma_n} e^{-\Sigma_{1 \le i < j \le n} V(\sigma_i, \sigma_j)} \prod_{i=1}^{n} K(\gamma_i), \tag{3.7}$$

where

$$V(\gamma_i, \gamma_j) = 0 \quad \text{if} \quad S(\gamma_i) \cap S(\gamma_j) = \emptyset$$

and

$$V(\gamma_i, \gamma_j) = \infty \quad \text{if} \quad S(\gamma_i) \cap S(\gamma_j) \neq \emptyset.$$

Equation (3.7) is called the polymer expansion for Z_Λ. If we call γ a "polymer", then the right-hand side of (3.7) is a grand canonical ensemble in Λ for polymer gas with repulsive pair potential $V(\gamma_i, \gamma_j)$ and activity measure $K(\gamma)$.

Step 2. Mayer Expansion for Polymer Gas.

Let

$$\psi[1, 2, \cdots, n] = \prod_{1 \le i < j \le n} e^{-V_{ij}},$$

where $V_{ij} = V(\gamma_i, \gamma_j)$. Let $\psi_c(n)$ be the connected part of ψ as defined in Section 2. By Mayer's Expansion (Theorem 2.1),

$$\frac{1}{|\Lambda|} \ln Z_\Lambda = \frac{1}{|\Lambda|} \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\gamma_1, \gamma_2, \cdots, \gamma_n} \psi_c(n) K(\gamma_1) \cdots K(\gamma_n), \tag{3.8}$$

if the series is convergent. By Penrose Tree Graph Bound (Corollary 2.3),

$$\begin{aligned} |\psi_c(n)| &\le \sum_{T \in T(1, \cdots, n)} \prod_{ij \in T} |e^{-\beta V_{ij}} - 1| \\ &= \sum_{T \in T(1, \cdots, n)} \prod_{ij \in T} \delta_{ij}, \end{aligned} \tag{3.9}$$

where

$$\delta_{ij} = 0 \quad \text{if} \quad S(\gamma_i) \cap S(\gamma_j) = \emptyset,$$

and

$$\delta_{ij} = 1 \quad \text{if} \quad S(\gamma_i) \cap S(\gamma_j) \neq \emptyset.$$

Step 3. Convergence of $|\Lambda|^{-1} \ln Z_\Lambda$.

Let $T \in T(1, \cdots, n)$ be the set of tree graphs on $\{1, 2, \cdots, n\}$. Let $d_1, d_2, \cdots, d_n$ be the incident of T, i.e., d_i is te number of edges in T which meet node i. Let

$$Q(d) = \sum_{\gamma: 0 \in \gamma} |K(\gamma)||\gamma|^{d-1},$$

where $|\gamma|$ is the number of lattice sites in γ. By (3.8), (3.9) and starting summing over γ_i where i is an extreme point of T, the series (3.8) is bounded by

$$\begin{aligned} &\sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\substack{d_1, d_2, \cdots, d_n \\ d_i \geq 1}} \sum_{T \in T(1, \cdots, n)} 1_{\text{incident } T = \{d_1, \cdots, d_n\}} Q_{d_1+1} Q_{d_2} \cdots Q_{d_n} \qquad (3.10) \\ &\leq \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\substack{d_1, d_2, \cdots, d_n \\ d_i \geq 1}} \frac{(n-2)!}{d_1! \cdots d_n!} Q(d_1+1) \cdots Q(d_n+1) \\ &\leq \sum_{n=1}^{\infty} \Big[\sum_{d=1}^{\infty} \frac{1}{d!} Q(d+1) \Big]^n, \qquad (3.11) \end{aligned}$$

where we have used Cayley's Theorem which says that the number of tree graphs T with incident $\{d_1, d_2, \cdots, d_n\}$ is equal to $\frac{(n-2)!}{d_1! \cdots d_n!}$. By (3.11), the series (3.8) is uniformly convergent if

$$Q = \sum_{d=1}^{\infty} \frac{1}{d!} Q(d+1) < 1. \qquad (3.12)$$

Step 4. Estimate β_0.

$$\begin{aligned} Q &= \sum_{d=1}^{\infty} \frac{1}{d!} Q(d+1) \\ &= \sum_{d=1}^{\infty} \frac{1}{d!} \sum_{\gamma \ni 0} |K(\gamma)||\gamma|^d \\ &\leq \sum_{\gamma \ni 0} |K(\gamma)| e^{|\gamma|} \\ &= \sum_{\gamma \ni 0} e^{|\gamma|} \int_{\chi_{s(\gamma)}} \prod_{\langle i,j \rangle \in \gamma} |e^{\beta \sigma_i \sigma_i} - 1| d\mu_{0,s(\gamma)}. \end{aligned}$$

If $|x| \leq 1$, then there exists a constant c such that $|e^x - 1| \leq c|x|$. Now we estimate the number of γ's such that $0 \in \gamma$ and $|\gamma| = n$. We consider sites in γ

as islands and bonds in γ as bridges. Starting from $0 \in \gamma$, we can go through all bridges and come back to 0 in a way that every bridge is gone through at most twice. This is a solution of Königsberg Bridge Problem. Therefore, the number of such γ's is bounded by the number of paths of length $2m$, where m is the number of bridges in γ. Since $m \le n$, the number of such γ's is bounded by $(2d)^{2n}$. Therefore, (3.13) is bounded by

$$\sum_{n=1}^{\infty} e^n c^n |\beta|^{n/2} (2d)^{2n} < 1,$$

if $\beta < \beta_0$, where $\frac{ec\beta_0^{1/2}(2d)^2}{1-ec\beta_0^{1/2}(2d)^2} = 1$.

Step 5. Analyticity of p.

We shall use the following theorem from complex analysis.

Theorem 3.2. *Let* $f_\Lambda(\beta) = \sum_{n=0}^{\infty} g_{\Lambda,n}(\beta)$. *Suppose*

(i) $g_{\Lambda,n}(\beta)$ *is analytic in* β, $|\beta| < \beta_0$,

(ii) $\lim_{\Lambda\to\infty} g_{\Lambda,n}(\beta) = g_{\infty,n}(\beta)$ *for each* β, $|\beta| < \beta_0$,

(iii) $\sup_{\Lambda, |\beta|<\beta_0} |g_{\Lambda,n}(\beta)| \le G_n$,

(iv) $\sum_{n=0}^{\infty} G_n < \infty$.

Then $\lim_{\Lambda\to\infty} f_\Lambda = f_\infty$ *exists,* f_∞ *is analytic in* β, $|\beta| < \beta_0$, *and*

$$f_\infty(\beta) = \sum_{n=0}^{\infty} g_{\infty,n}(\beta), \quad |\beta| < \beta_0. \tag{3.14}$$

Proof of Theorem 3.2. The existence of f_∞ and (3.14) follow from Lebesgue's Dominated Convergence Theorem. To show analyticity of f_∞, let C be any closed contour in the region $|\beta| < \beta_0$, and

$$\int_C f_\infty dz = \sum_{n=0}^{\infty} \int_C g_{\infty,n} dz = \sum_{n=0}^{\infty} \lim_{\Lambda\to\infty} \int_C g_{\Lambda,n} dz = 0$$

by the Dominated Convergence Theorem and Cauchy's Theorem. Hence f is analytic in $|\beta| < \beta_0$ by Morera's Theorem. ∎

We apply Theorem 3.2 to (3.8). In the right-hand side of (3.8),

$$g_{\Lambda,n} = \frac{1}{|\Lambda|} \sum_{\substack{\gamma_1,\gamma_2,\cdots,\gamma_n \\ \gamma_i \subseteq \Lambda}} \psi_c(n) K(\gamma_1)\cdots K(\gamma_n) \tag{3.15}$$

is analytic for all β, for all n , Λ. We claim that

$$\lim_{\Lambda \nearrow \mathbb{Z}^v} g_{\Lambda,n} = \sum_{\substack{\gamma_1,\gamma_2,\cdots,\gamma_n \\ \circ(\gamma_1)=0}} \psi_c(n) K(\gamma_1)\cdots K(\gamma_n) \equiv g_{\infty,n}, \tag{3.16}$$

where $\circ(\gamma_1)$ is the first site in γ_1 in lexicographical order.

To prove (3.16), note that by translational invariance,

$$g_{\infty,n} = \frac{1}{|\Lambda|} \sum_{x\in\Lambda} \sum_{\gamma_1:\circ(\gamma_1)=x} \sum_{\gamma_2} \cdots \sum_{\gamma_n} \psi_c(n) K(\gamma_1)\cdots K(\gamma_n).$$

We also have

$$\begin{aligned} g_{\Lambda,n} =& \frac{1}{|\Lambda|} \sum_{x\in\Lambda} \sum_{\gamma_1:\circ(\gamma_1)=x} \\ & \sum_{\gamma_2} \cdots \sum_{\gamma_n} \Big(\prod_{i=1}^{n} 1_{\gamma_i \subseteq \Lambda} \Big) \psi_c(n) K(\gamma_1)\cdots K(\gamma_n). \end{aligned} \tag{3.17}$$

If $|\beta| < \beta_0$, then there exists $\alpha > 0$ such that $e^{\alpha}|\beta|^{\frac{1}{2}} < \beta_0^{\frac{1}{2}}$ and therefore

$$Q' = \sum_{n=1}^{\infty} (e^{\alpha})^n c^n |\beta|^{\frac{n}{2}} (2d)^{2n} < \sum_{n=1}^{\infty} e^n c^n \beta_0^{\frac{n}{2}} (2d)^{2n} = 1. \tag{3.18}$$

Then we have

$$\begin{aligned} & \sum_{\gamma_1:o(\gamma_1)=0} \sum_{\gamma_2} \cdots \sum_{\gamma_n} 1_{\gamma_j \not\subseteq \Lambda} \, |\psi_c(n)| \, |K(\gamma_1)| \, |K(\gamma_n)| \\ = & \sum_{\gamma_1:o(\gamma_1)=0} \sum_{\gamma_2} \cdots \sum_{\gamma_n} 1_{\gamma_j \not\subseteq \Lambda} \, |\psi_c(n)| \, e^{-\alpha \Sigma_{i=1}^n |\gamma_i|} \prod_{i=1}^{n} |K(\gamma_i)| e^{\alpha|\gamma_i|} \\ \leq & \sum_{\gamma_1:o(\gamma_1)=0} \sum_{\gamma_2} \cdots \sum_{\gamma_n} 1_{\gamma_j \not\subseteq \Lambda} \sum_{T\in T(1,\ldots,n)} \prod_{ij\in T} \delta_{ij} \prod_{i=1}^{n} |K(\gamma_i)| e^{\alpha|\gamma_i|} e^{-\alpha \Sigma_{i=1}^n |\gamma_i|}, \end{aligned}$$

by (3.9). Note that in the above expansion, for each fixed T, $\gamma_1, \gamma_2, \ldots, \gamma_n$ are connected through the relation T. Therefore $\sum_{i=1}^{n} |\gamma_i| \geq d(0, \partial\Lambda)$ if $\gamma_j \not\subseteq \Lambda$ for some $j = 1, 2, \ldots, n$. Thus the above is bounded by

$$\begin{aligned} & e^{-\alpha d(0,\partial\Lambda)} \sum_{\gamma_1:o(\gamma_1)=0} \sum_{\gamma_2} \cdots \sum_{\gamma_n} \sum_{T\in T(1,\ldots,n)} \prod_{ij\in\delta_{ij}} \prod_{i=1}^{n} |K(\gamma_i)| e^{\alpha|\gamma_i|} \\ & \leq e^{-\alpha d(0,\partial\Lambda)} \sum_{n=1}^{\infty} (Q')^n < \infty, \end{aligned}$$

using the same arguments as those in (3.10) and (3.11). By translational invariance, we have obtained, for fixed x and $0 \in \Lambda$,

$$\sum_{\gamma_1 : o(\gamma_1)=x} \sum_{\gamma_2} \cdots \sum_{\gamma_n} 1_{\gamma_j \not\subseteq \Lambda + x} |\psi_c(n)| \prod_{i=1}^{n} |K(\gamma_i)| \le e^{-\alpha d(0,\partial\Lambda)} \frac{Q'}{1-Q'}. \tag{3.19}$$

It follows that for all $\epsilon > 0$, there exists $r > 0$ such that

$$\sum_{\gamma_1 : o(\gamma_1)=x} \sum_{\gamma_2} \cdots \sum_{\gamma_n} 1_{\gamma_j \not\subseteq \Lambda + x} |\psi_c(n)| \prod_{i=1}^{n} |K(\gamma_i)| \le \epsilon \tag{3.20}$$

if $d(0, \partial\Lambda) \ge r$.

Now by (3.17),

$$\begin{aligned} |g_{\infty n} - g_{\Lambda,n}| \le & \sum_{j=1}^{n} \frac{1}{|\Lambda|} \sum_{x\in\Lambda} \sum_{\gamma_1 : o(\gamma_1)=x} \\ & \sum_{\gamma_2} \cdots \sum_{\gamma_n} 1_{\gamma_j \not\subseteq \Lambda} |\psi_c(n)| \, |K(\gamma_1)| \ldots |K(\gamma_n)|. \end{aligned} \tag{3.21}$$

We split the above sum $\sum_{x\in\Lambda}$ into two terms. The first term is $\sum_{x\in\Lambda, d(x,\partial\Lambda)\le r}$ and the second term is $\sum_{x\in\Lambda, d(x,\partial\Lambda)>r}$. Then the second term is bounded by

$$\begin{aligned} & \sum_{j=1}^{n} \frac{1}{|\Lambda|} \sum_{x\in\Lambda} \sum_{\gamma_1 : o(\gamma_1)=x} \\ & \sum_{\gamma_2} \cdots \sum_{\gamma_n} 1_{\gamma_j \not\subseteq \Lambda' + x} |\psi_c(n)| \, |k(\gamma_1)| \ldots |k(\gamma_n)|, \end{aligned} \tag{3.22}$$

where $\Lambda' = \{y \in Z^\nu; \, d(0,y) \le r\}$. By (3.20) and (3.22), the second term is bounded by ϵn. Using (3.10), (3.11) and (3.12), the first term is bounded by

$$\frac{Q}{1-Q} \frac{1}{|\Lambda|} |\partial_r \Lambda| = \frac{Q}{1-Q} \frac{1}{|\Lambda|} \, r \, |\partial\Lambda|,$$

which goes to zero as $\Lambda \nearrow \mathbb{Z}^\nu$ in van Hove sense.

The uniform bounds (iii) and (iv) in Theorem 3.2 follow from uniform estimates in Step 3 and Step 4. Therefore, $p = \sum_{n=1}^{\infty} g_{\infty,n}$ is analytic if $|\beta| < \beta_0$ with sufficiently small β_0. ■

3.2. Low Temperature Expansion for Ising Model

We will continue to study the expansion method for Ising model in low temperature region. We consider the Hamiltonian of nearest-neighbor Ising model given by (3.1) in Section 3.1. Using the same notations as those in Section 3.1, we have

Theorem 3.3. *There exists* $\beta_1 > 0$ *such that the pressure* p *is an analytic function of complex-valued* β *in the region* $|\beta| > \beta_1$.

Proof. Since the pressure does not depend on the boundary conditions, we may take $\xi_i = 1$ for all i in (3.3) and get

$$p = \lim_{\Lambda \nearrow \mathbb{Z}^\vee} \frac{1}{|\Lambda|} \ln Z_\Lambda^+, \tag{3.23}$$

where Z_Λ^+ is the partition in Λ with boundary conditions $\xi_i = 1$ for all $i \in \mathbb{Z}^\vee$. We may also consider Λ as a square centered at 0. We will follow similar ideas in the expansion for high temperature case. However, the first step before polymer expansion is to change the model into contour gas.

Step 1. Polymer expansion for contour gas.

Given a configuration $\sigma \in \{-1, 1\}^\Lambda$ with $\sigma_x = 1$ for all $x \notin \Lambda$, we draw vertical and horizontal lines of unit length between adjacent sites which have opposite signs given by σ. Let $\Gamma(\sigma)$ be the set of all such vertical and horizontal lines. Note that $\Gamma(\sigma)$ is a polygonal curve which may have self-intersection. By using north-east rounding the corner, the $\Gamma(\sigma)$ is a union of disjoint contours, where a contour is a non-self-intersecting polygonal curve; see Figure 1.

Figure 1. An example of $\Gamma(\sigma)$ where it is rounded at the self-intersection in the north-east direction. The resulting $\Gamma(\sigma)$ is non-self-intersecting.

Note that given a Γ, the configuration σ can be reconstructed with positive boundary conditions. Therefore, Γ and σ are in one-to-one correspondence. The Hamiltonian

$$
\begin{aligned}
H_\Lambda^+(\sigma) &= -\sum_{\langle x,y\rangle\subseteq\Lambda} \beta\sigma_x\sigma_y = \frac{\beta}{2}\sum_{\langle x,y\rangle\subseteq\Lambda}(\sigma_x-\sigma_y)^2 - \beta C_\Lambda \\
&= 2\beta|\Gamma| - \beta C_\Lambda\,,
\end{aligned}
\tag{3.24}
$$

where C_Λ is the number of $\langle x,y\rangle \subseteq \Lambda$ and $|\Gamma|$ denotes the length of Γ. Then

$$
Z_\Lambda^+ = \sum_{\Gamma\subseteq\Lambda} e^{-2\beta|\Gamma|}\cdot e^{c_\Lambda\beta}. \tag{3.25}
$$

Let $\{\gamma_1,\gamma_2,\cdots,\gamma_n\}$ be the connected components of Γ. γ_i's are called contours. Then

$$
e^{-c_\Lambda\beta}Z_\Lambda^+ = \sum_{n=0}^{\infty}\sum_{\{\gamma_1,\gamma_2,\cdots,\gamma_n\}} e^{-2\beta\Sigma_{i=1}^n|\gamma_i|}, \tag{3.26}
$$

where

$$
\gamma_i\cap\gamma_j=\emptyset \quad \text{for all } i\neq j.
$$

If we order the elements in $\{\gamma_1,\gamma_2,\cdots,\gamma_n\}$, then

$$
e^{-c_\Lambda\beta}Z_\Lambda^+ = \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{\gamma_1,\gamma_2,\cdots,\gamma_n} e^{-\sum\limits_{1\le i<j\le n}V(\gamma_i,\gamma_j)}\prod_{i=1}^{n} e^{-2\beta|\gamma_i|}, \tag{3.27}
$$

where

$$
V(\gamma_i,\gamma_j)=\infty \quad \text{if } \ \gamma_i\cap\gamma_j\neq\emptyset.
$$

and

$$V(\gamma_i, \gamma_j) = 0 \quad \text{if} \ \ \gamma_i \cap \gamma_j = \emptyset.$$

Equation (3.27) turns the model into a grand canonical ensemble of polymer gas (γ's are polymers now, compared to high temperature case).

It is easy to see that $\ln \frac{e^{c_\Lambda \beta}}{|\Lambda|} \to d\beta$. Therefore, it is sufficient to consider the expansion for the logarithm of the right-hand side of (3.27). The rest of expansions, Mayer expansion and its criterion for convergence are the same as in the high temperature case. Using the same argument as in the proof of Theorem 3.1, to prove Theorem 3.3, it is sufficient to show

$$Q = \sum_{d=1}^{\infty} \frac{1}{d!} Q(d+1) < 1, \tag{3.28}$$

where

$$Q(d) = \sum_{\gamma \ni 0} |\gamma|^{d-1} e^{-2\beta|\gamma|}.$$

Since the number of γ's such that $|\gamma| = n$ is bounded by $(2d)^n$, we have

$$Q \le \sum_{\gamma \ni 0} e^{|\gamma|} e^{-2\beta|\gamma|} \le \sum_{n=1} e^n e^{-2\beta n} (2d)^n < 1$$

if β is sufficiently large. ∎

4. Lace Expansion

In this section, we consider self-avoiding random walks on $\mathbb{Z}^d$ as defined in Section 1.2. Following the notations in Section 1.2, we let $C(x, n) = E(1_x(X_n)\psi[0, n])$, and $\hat{C}(k, n) = \sum_x C(x, n) e^{ik\cdot x}$. Then we are interested in $\lim_{n\to\infty} \hat{C}(\frac{k}{\sqrt{n}}, n) \Big/ \hat{C}(0, n)$. Consider the Fourier-Laplace transform

$$\hat{C}(k, z) = \sum_{n=0}^{\infty} z^n \hat{C}(k, n)$$

with radius of convergence $r(k)$. We put $r = r(0)$. Note that $1 \le r(k) \le r$. Define the perturbation term $\pi(k, z)$ by

$$\hat{C}(k, z) = \frac{1}{1 - D(k)z - \pi(k, z)}, \quad |z| < r(k), \tag{4.1}$$

where $D(k) = \frac{1}{d} \sum_{i=1}^{d} \cos k_i$. We also put

$$C(x, z) = \sum_{n=0}^{\infty} z^n C(x, n).$$

If we have a good control of $\pi(k,z)$, then we can compute $\hat{C}(k,n)$ by Cauchy contour integral and then obtain the limit. The goal of lace expansion is to give an expansion foumula for $\pi(k,z)$.

4.1. Lace Expansion ($\beta=\infty$, nearest-neighbor case only)

We will consider nearest-neighbor case with $\beta=\infty$ only, since the other cases can be treated similarly. Then

$$\psi[0,n]=\prod_{0\le i<j\le n}(1-\delta_{ij}),\quad \text{where } \delta_{ij}=\delta(X_i-X_j).$$

Expand the product, we get

$$\psi[0,n]=\sum_{G:\text{graph on }[0,n]}\prod_{ij\in G}(-\delta_{ij}),\ \ n\ge 1. \tag{4.2}$$

Here a graph G on $[0,n]$ is a graph with vertices $\{0,1,2,\ldots,n\}$ and edges where each edge connects two distinct vertices. It is convenient to arrange the vertices in linear order, so a typical graph looks like

Definition. A graph G on $[0,n]$ is said to be *connected* if 0 and n are connected to some vertices and each vertex other than 0 or n is strictly underneath an edge.

For example, G_1 is connected and G_2 is disconnected.

Definition. The *connected part* ψ_c of ψ is defined by

$$\begin{aligned}\psi_C[0,n]&=0, && n=1,\\ \psi_C[0,n]&=\sum_{\substack{\Gamma:\text{connected}\\ \text{graph on }[0,n]}}\prod_{ij\in\Gamma}(-\delta_{ij}), && n>1.\end{aligned}\tag{4.3}$$

Theorem 4.1. (Connected Formula for $\pi(k, z)$)

$$\pi(k,z) = \sum_{n=1}^{\infty} z^n E(e^{ik\cdot X_n}\psi_c[0,n]), \tag{4.4}$$

for all k and for all z inside both of the radii of convergence of $\hat{C}(k, z)$ and the above series.

Proof of Theorem 4.1. By definition of $C(k, z)$ and (4.2),

$$\hat{C}(k,z) = 1 + \sum_{n=1}^{\infty} z^n \sum_{G:\ \text{graph on } [0,n]} E[e^{ikX_n} \prod_{ij\in G}(-\delta_{ij})].$$

Let $a < b$ be nonnegative integers. For an interval $I = [a, b]$, let

$$\begin{aligned} \tilde{\psi}_C[a,b] \ &= 1 && \text{if } b-a=1, \\ &= \psi_c[a,b] && \text{if } b-a>1. \end{aligned}$$

Note that $\delta_{i,i+1} = 0$ for all i. Then we have

$$\hat{C}(k,z) = 1 + \sum_{n=1}^{\infty} z^n \sum_{l=1}^{\infty} \sum_{I_1,I_2,\ldots,I_l} E[e^{ikX_n} \prod_{j=1}^{l} \tilde{\psi}_c(I_j)],$$

where $I_1 \cup I_2 \cup \ldots \cup I_l = [0, n]$ with nonoverlapping interiors, and I_i is to the left of I_{i+1} for all $i = 1, \ldots, l-1$.

Let $X_I = X_b - X_a$ if $I = [a, b]$. Write

$$X_n = \sum_{j=1}^{l} X_{I_j}, \quad n = \sum_{j=1}^{l} \mid I_j \mid,$$

and note that $e^{ikX_{I_j}}\tilde{\psi}_c(I_j)$, $j = 1, 2, \ldots, l$, are independent. We have

$$\hat{C}(k,z) = 1 + \sum_{l=1}^{\infty} \sum_{I_1,I_2,\ldots,I_l} \prod_{j=1}^{l} z^{|I_j|} E(e^{ikX_{I_j}}\tilde{\psi}_c(I_j)).$$

Since

$$z^{|I|}E(e^{ikX_I}\tilde{\psi}_c(I)) = z^{|I|}E(e^{ikX_{|I|}}\tilde{\psi}_c[0,|I|]),$$

we have

$$\begin{aligned} \hat{C}(k,z) &= 1 + \sum_{l=1}^{\infty} \sum_{\substack{n_1,n_2,\ldots,n_l \\ n_i\ge 1}} \prod_{j=1}^{l} z^{n_j} E(e^{ikX_{n_j}}\tilde{\psi}_c[0,n_j]) \\ &= 1 + \sum_{l=1}^{\infty} \Big(\sum_{n=1}^{\infty} z^n E(e^{ikX_n}\tilde{\psi}_c[0,n])\Big)^l \\ &= \frac{1}{1-\sum_{n=1}^{\infty} z^n E(e^{ikX_n}\tilde{\psi}_c[0,n])}. \end{aligned}$$

Now the theorem follows from

$$\begin{aligned}\sum_{n=1}^{\infty} z^n E(e^{ikX_n}\tilde{\psi}_c[0,n]) &= zE(e^{ikX_1}\tilde{\psi}_c[0,1]) + \pi(k,z) \\ &= zD(k) + \pi(k,z).\end{aligned}$$

■

A minimally connected graph is called a *lace graph*. Given a connected graph Γ on $[0,n]$, we define a lace graph $L(\Gamma)$ whose vertices are $\{0,1,2,\ldots,n\}$ and edges are chosen from Γ as follows: Choose edges $t_1s_1, t_2s_2, t_3s_3, \ldots, t_ls_l$, where

$$\begin{aligned} t_1 = 0, \quad & s_1 = \max\{i : 0i \in \Gamma\} \\ & s_2 = \max\{i : ji \in \Gamma \text{ for some } j \text{ and } (j,i)\cap(t_1,s_1) \neq \emptyset\}, \\ & t_2 = \min\{j : js_2 \in \Gamma\}, \\ & s_3 = \max\{i : ji \in \Gamma \text{ for some } j \text{ and } (j,i)\cap(t_2,s_2) \neq \emptyset\}, \\ & t_3 = \min\{j : js_3 \in \Gamma\}, \ldots, \text{ and so forth.}\end{aligned}$$

Given a lace graph L on $[0,n]$, let $e(L) = \{\Gamma; L(\Gamma) = L\}$. *Note that the relationship between a connected graph* Γ *and a lace graph is the same as that between a connected graph and a tree graph in Mayer's expansion.* The fact is that there exists a maximal element in $e(L)$, called $m(L)$. Let $o(L)$ be the number of edges in L. Then using exactly the same argument as that in Mayer's expansion, we have

Theorem 4.2. (Lace Expansion for π) *Inside the radius of convergence,*

$$\pi(k,z) = \sum_{l=1}^{\infty} \pi^{(l)}(k,z), \tag{4.5}$$

$$\pi^{(l)}(k,z) = \sum_{n=1}^{\infty} z^n E\left\{ e^{ik\cdot X_n} \sum_{\substack{L:\text{ lace on } [0,n] \\ o(L)=l}} \prod_{ij\in L}(-\delta_{ij}) \prod_{ij\in m(L)\setminus L}(1-\delta_{ij})\right\}. \tag{4.6}$$

4.2. Convergence of Lace Expansion

We put

$$\begin{aligned} g(x,y) &= \sum_{n=0}^{\infty} p_n(x,y), \\ g_1(x,y) &= \sum_{n=1}^{\infty} p_n(x,y),\end{aligned}$$

where $p_n(x,y) = P(X_n = y \mid X_0 = x)$ is the transition probability of random walk (X_n).

To explain the basic idea on the convergence of $\pi(k, z)$, we first consider the case $|z| = 1$. By definition,

$$\pi^{(1)}(k, z) = \sum_{n=1}^{\infty} z^n E\Big(e^{ikX_n}(-\delta_{0n})\Big), \tag{4.7}$$

which can be estimated by

$$\sum_{n=1}^{\infty} E(\delta_{0n}) = \sum_{n=1}^{\infty} p_n(0, 0) \equiv g_1(0, 0).$$

If we drop $1 - \delta_{ij}$ in $\pi^{(2)}$, then

$$\begin{aligned}
|\pi^{(2)}(k, z)| &\le \sum_{n=1}^{\infty} |z|^n \sum_{0<s<t<n} E\Big(|e^{ikX_n}|\delta(X_0 - X_t)\delta(X_s - X_n)\Big) \\
&= \sum_{0<s<t<n} \sum_x p_s(0, x)p_{t-s}(x, 0)p_{n-t}(0, x) \\
&= \sum_x g_1(0, x)g_1(x, 0)g_1(0, x) \\
&=_0 \qquad\qquad .
\end{aligned}$$

The last expression is a Feynman diagram, where each edge corresponds to a Green function g_1. In general, $\pi^l(k, z)$ can be estimated by a Feynman diagram:

The slashed edge corresponds to Green function g and the unslashed corresponds to g_1. This diagram has l vertices and $2l - 1$ edges. By Young and Hölder inequalities, this diagram is bounded by $\|g_1\|_2^{l-2}\|g_1\|^l\|g_1\|_\infty$. It is well-known that $\|g_1\|_2 = O(\frac{1}{d})$, so $\pi(k, z)$ is convergent in $|z| \le 1$ if d is sufficiently large. To obtain convergence for $d = 5$, we need detailed estimate on $\|g_1\|_2$ and we also need factors $(1 - \delta_{ij})$ in $\pi^{(l)}$. This has been done in [21] by assistance of computer estimation.

The derivative $\partial^u_{k_\mu}\pi(k, z)$ can be estimated in a similar way. We can bound the result by introducing a factor $|X_\mu|^u$. Similarly, a z-derivative $\partial_z\pi(k, z)$ gives a factor n. For example,

$$
\begin{aligned}
|\partial_{k_1}^2 \pi^{(2)}(k,z)| &\le \sum_{0<s<t<n} \sum_x |x_1|^2 p_s(0,x) p_{t-s}(x,0) p_{n-t}(0,x) \\
&= \sum_x |x_1|^2 g_1(0,x) g_1(x,0) g_1(0,x) \\
&\le \Big\| |x_1|^2 g_1(x) \Big\|_\infty \ \Big\| g_1 \Big\|_2^2,
\end{aligned}
$$

$$
\begin{aligned}
|\partial_z \pi^{(2)}(k,z)| &\le \sum_{0<s<t<n} n \sum_x p_s(0,x) p_{t-s}(x,0) p_{n-t}(0,x) \\
&= \sum_x \sum_{0<s<t<n} \Big[(n-t)+(t-s)+s\Big] p_s(0,x) p_{t-s}(x,0) p_{n-t}(0,x) \\
&\le 3 \Big\| \sum_{n=1}^\infty n p_n(0,x) \Big\|_\infty \ \Big\| g_1(0,x) \Big\|_2^2.
\end{aligned}
$$

If we also include the factors $(1-\delta_{ij})$ inside each subwalk, we have the following

Lemma 4.3. *Suppose that in each term of* (4.5)*, the factors* $|x_\mu|^u$ *or* n *is included in* (4.6)*. The series for* $\partial_{k_\mu}^u \pi(k,z)$ *and* $\partial_z \pi(k,z)$ *are absolutely convergent if*

$$\Big\| |x_\mu|^u c(x,|z|) \Big\|_\infty < \infty\,,$$

$$\Big\| \sum_{n=1}^\infty n c(x,n) |z|^n \Big\|_\infty < \infty$$

and

$$\Big\| c(x,|z|) \Big\|_2^2 \ \Big\| c^{(1)}(x,|z|) \Big\|_2^2 < 1,$$

where

$$c^{(1)}(x,z) = \sum_{n=1}^\infty c(x,n) z^n.$$

The main results in [21] are the following Lemmas.

Lemma 4.4.

(a) *Let* $d \ge 5$*. There are constants* c_1*,* c_2 *such that for any* z *with* $|z| \le r(0)$*,*

$$\Big\| |x|^2 C(x,z) \Big\|_\infty \le c_1,$$

$$\Big\| C^{(1)}(x,z) \Big\|_2^2 \le c_2$$

with $c_2(1+c_2) < 1$.

(b) *Let $d \geq 5$ and $u \in \{0, 1, 2\}$. The quantities $|\partial_z \pi(k, z)|$ and $|\partial_{k_\mu}^u \pi(k, z)|$ are finite and bounded uniformly in $k \in [-\pi, \pi]^d$ and $|z| \leq r$. In fact, the series representations of these quantities are bounded absolutely* (*absolute values inside sums over x, n*) *and uniformly.*

It follows from Lemma 4.3(b) and the Dominated Convergence Theorem that $\partial_z \pi(k, z)$ and $\partial_{k_\mu}^u \pi(k, z)$ $(u = 0, 1, 2)$ are continuous on the closed disk $|z| \leq r$. In particular, since $[1 - z - \pi(0, z)]^{-1}$ diverges at r, we have

$$1 - r - \pi(0, r) = 0. \tag{4.8}$$

Lemma 4.5. *Let $d \geq 5$. For $z = re^{i\theta}$ with $\theta \neq 0$, we have*

$$1 - z - \pi(0, z) \neq 0. \tag{4.9}$$

Lemma 4.6.*Let $d \geq 5$. There are positive constants c_3, c_4, ϵ_1 such that for any $p \in [r - \epsilon_1, r]$,*

$$1 + \partial_z \pi(0, p) \geq c_3 > 0, \tag{4.10}$$

$$p - \nabla_k^2 \pi(0, p) \geq c_4 > 0. \tag{4.11}$$

Also, there is a positive constant c_5 such that for any $p \in [0, r]$,

$$\pi(0, p) - \pi(k, p) \geq -c_5[1 - D(k)], \tag{4.12}$$

with $r - c_5 > 0$. In particular,

$$1 - rD(k) - \pi(k, r) \geq 0. \tag{4.13}$$

4.3. Fractional Derivatives

To prove Theorem 1.1, we will need fractional derivatives and a Tauberian theorem [21].

For a power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ with radius of convergence R, we define the fractional derivative

$$\delta_z^\epsilon f(z) = \sum_{n=0}^{\infty} n^\epsilon a_n z^n, \qquad \epsilon \geq 0. \tag{4.14}$$

$$\delta_z^{-\alpha} f(z) = \sum_{n=1}^{\infty} n^{-\alpha} a_n z^n, \quad \alpha > 0. \tag{4.15}$$

The fractional derivatives are finite at least strictly inside the circle of convergence of f.

The following lemma is useful for estimating fractional derivatives.

Lemma 4.7. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ with radius of convergence R. Then for all $|z| < R$,*

$$\delta_z^{-\alpha} f(z) = c_\alpha \int_0^\infty [f(ze^{-\lambda^{\frac{1}{\alpha}}}) - f(0)] d\lambda, \quad \alpha > 0, \tag{4.16}$$

$$\delta_z^{\epsilon} f(z) = c_{1-\epsilon} z \int_0^\infty f'(ze^{-\lambda^{\frac{1}{1-\epsilon}}}) e^{-\lambda^{\frac{1}{1-\epsilon}}} d\lambda, \quad 0 < \epsilon < 1, \tag{4.17}$$

where $c_\alpha = [\alpha\Gamma(\alpha)]^{-1}$. The identities (4.16) and (4.17) also hold for $z = R$ if $a_n \geq 0$.

The following lemma is analogous to the error estimate in Taylor's theorem. In applications of the lemma, R will be the radius of convergence of f.

Lemma 4.8. *Let $0 < \epsilon < 1$ and $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Let $R > 0$.*

(i) *Suppose $A_\epsilon = \sum_{n=0}^{\infty} n^\epsilon |a_n| R^{n-\epsilon} < \infty$, so in particular $f(z)$ converges for $|z| \leq R$. Then for any z with $|z| \leq R$,*

$$|f(z) - f(R)| \leq 2^{1-\epsilon} A_\epsilon |R - z|^\epsilon. \tag{4.18}$$

(ii) *Suppose that $B_\epsilon = \sum_{n=1}^{\infty} n^{1+\epsilon} |a_n| R^{n-1-\epsilon} < \infty$, so in particular $f'(z) = \sum_{n=0}^{\infty} n a_n z^{n-1}$ converges for $|z| \leq R$. Then for any $|z| \leq R$,*

$$|f(z) - f(R) - f'(R)(z - R)| \leq \frac{2^{1-\epsilon}}{1+\epsilon} B_\epsilon |R - z|^{1+\epsilon}. \tag{4.19}$$

Lemma 4.9. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ have radius of convergence greater than or equal to $R > 0$.*

(i) *Suppose that for $|z| < R$, $|f(z)| \leq \text{const} \cdot |R - z|^{-b}$, for some $b \geq 1$. Then $|a_n| \leq O(R^{-n} n^\alpha)$ for any $\alpha > b - 1$.*

(ii) *If for some $b \geq 1$, a bound on the derivative of the form $|f'(z)| \leq \text{const} \cdot |R - z|^{-b}$ holds for every $|z| < R$, then $|a_n| \leq O(R^{-n} n^{-\alpha})$ for any $\alpha < 2 - b$.*

The following lemma is a kind of Tauberian theorems.

Lemma 4.10. *Let*

$$f(z) = \frac{1}{R - z - \varphi(z)} = \sum_{n=0}^{\infty} b_n z^n,$$

where $\varphi(z) = \sum_{n=0}^{\infty} a_n z^n$. Suppose that for some $\epsilon \in (0,1)$, $\sum_{n=0}^{\infty} n^{1+\epsilon}|a_n|R^n < \infty$, so in particular $\varphi(z)$ and $\varphi'(z)$ are both finite for $|z| = R$. Assume in addition that $\varphi'(R) \neq -1$. Suppose that $\varphi(R) = 0$ but that $R - z - \varphi(z) \neq 0$ for $|z| \leq R$, $z \neq R$. Then

$$f(z) = \frac{1}{1+\varphi'(R)} \frac{1}{R-z} + O(|R-z|^{\epsilon-1}) \tag{4.20}$$

uniformly in $|z| \leq R$, and

$$b_n = R^{-n-1}\Big[\frac{1}{1+\varphi'(R)} + O(n^{-\alpha})\Big], \quad \text{as } n \to \infty, \tag{4.21}$$

for every $\alpha < \epsilon$.

4.4. Fractional Derivatives of $\pi(k,z)$ and $\hat{c}(k,z)$

By Lemma 4.10, to prove Theorem 1.1 (a), it is sufficient to show that $\partial_z^{1+\epsilon}\pi(0,z)$ converges at $z = r$, for $d \geq 5$. The main results on the fractional derivatives of $\pi(k,z)$ and $\hat{c}(k,z)$ are the following theorem. Let $i(d) = (d-4)/2$ if d is even and $i(d) = (d-3)/2$ if d is odd.

Theorem 4.11. [21] *Let $d \geq 5$. There are positive (dimension-dependent) constants $K_1(\epsilon)$, $K_2(\epsilon)$, $K_3(\epsilon)$ such that for any $p \in (0,r]$,*

$$\|\delta_p^\epsilon \partial_p^{i(d)} c(x,p)\|_\infty \leq K_1(\epsilon), \qquad if\ \epsilon \in \left(0, \frac{d-2}{2} - i(d)\right), \tag{4.22}$$

$$\|\delta_p^\epsilon c(x,p)\|_2 \leq K_2(\epsilon), \qquad if\ \epsilon \in \left(0, \frac{1}{4}\right), \tag{4.23}$$

$$\||x|^2 \delta_p^\epsilon c(x,p)\|_\infty \leq K_3(\epsilon), \qquad f\ \epsilon \in \left(0, \frac{1}{2}\right). \tag{4.24}$$

The idea of the proof is to apply Lemma 4.7 to convert the fractional derivaties to derivatives. See [21] for details.

Corollary 4.12. *Let $d \geq 5$. There is a finite positive constant $K_4(\epsilon)$ such that for any $k \in [-\pi, \pi]^d$ and $|z| \leq r$,*

$$|\delta_z^\epsilon \partial_z \pi(k, z)|, \quad |\delta_z^\epsilon \partial_{k_\mu}^u \pi(k, z)| \leq K_4(\epsilon),$$

for $u = 0, 1, 2$, where the first bound holds for any positive $\epsilon < \frac{1}{2}$ and the second for any positive $\epsilon < \frac{1}{4}$. The series representation of the left-hand sides are bounded by $K_4(\epsilon)$.

Corollary 4.13. *Let $d \geq 5$. There is a constant $c > 0$ such that for any z with $|z| \leq r$,*

$$F(0, z) \geq c|r - z|,$$

where $F(k, z) = 1 - zD(k) - \pi(k, z)$.

4.5. Proof of Theorem 1.1

We will explain the proofs of (a), (b) only and refer the proof of (c) to [21].

Proof of Theorem 1.1 (a).

By definition,

$$c_n = (2d)^n \sum_x c(x, n) = (2d)^n \hat{c}(0, n).$$

Let $\chi(z) = \sum_{n=0}^\infty c_n z^n$. Then

$$\chi(z) = \hat{c}(0, 2dz) = \frac{1}{1 - 2dz - \pi(0, 2dz)}.$$

By (4.8), $1 - r - \pi(0, r) = 0$ and

$$\chi(z) = \frac{1}{2d}\Big[\frac{1}{(\frac{r}{2d} - z) - \varphi(z)}\Big], \tag{4.25}$$

where $\varphi(z) = \frac{1}{2d}[\pi(0, 2dz) - \pi(0, r)]$.

By Corollary 4.12, for any $\epsilon < \frac{1}{2}$,

$$\sum_{n=1}^\infty n^{1+\epsilon} |\pi_n| z^n < \infty,$$

where π_n is the coefficient of z^n in the power series representation of $\pi(0, z)$. Moreover, by (4.10),

$$\varphi'\Big(\frac{r}{2d}\Big) = \frac{\partial}{\partial z}\pi(0, r) \neq -1. \tag{4.26}$$

By (4.9), the only zero of $\frac{r}{2d} - z - \varphi(z) = (1 - 2dz - \pi(0, 2dz))/(2d)$ for $|z| \le r/(2d)$ is $z = r/(2d)$. By (4.21),

$$\begin{aligned} c_n &= \frac{1}{2d}\Big(\frac{r}{2d}\Big)^{-n-1}\Big[\frac{1}{1 + \frac{\partial}{\partial_z}\pi(0, r)} + O(n^{-\alpha})\Big] \\ &= A\mu^n[1 + O(n^{-\alpha})] \end{aligned} \tag{4.27}$$

for all $0 < \alpha < \epsilon < 1/2$, where

$$\begin{aligned} \mu &= \frac{2d}{r}, \\ A &= \frac{1}{r}\frac{1}{1 + \frac{\partial}{\partial_z}\pi(0, r)}. \end{aligned} \tag{4.28}$$

Since $\epsilon < 1/2$ is arbitrary, (4.27) gives the desired result. ■

Proof of Theorem 1.1 (b).

By definition,

$$\tilde{E}(|X_n|^2) = \frac{-\nabla_k^2 \hat{c}(0, n)}{\hat{c}(0, n)}. \tag{4.29}$$

Since $\hat{c}(0, n) = c_n/(2d)^n$, the asymptotic behavior of the denominator is obtained in (4.27). Since $\hat{c}(0, n)$ is the coefficient of z^n in $\hat{c}(k, z)$,

$$\begin{aligned} -\nabla_k^2 \hat{c}(0, n) &= -\frac{1}{2\pi i}\oint \frac{\nabla_k^2 \hat{c}(0, z)}{z^{n+1}} dz \\ &= \frac{1}{2\pi i}\oint \frac{\nabla_k^2 F(0, z)}{F(0, z)^2}\frac{dz}{z^{n+1}}, \end{aligned} \tag{4.30}$$

where $F(k, z) = 1 - zD(k) - \pi(k, z)$, and the integrals are over a contour around a small circle centered at the origin. Define the error term $E(z)$ by

$$\frac{\nabla_k^2 F(0, z)}{F(0, z)^2} = \frac{\nabla_k^2 F(0, r)}{[\partial_z F(0, r)]^2 (r - z)^2} + E(z). \tag{4.31}$$

Substituting (4.31) into (4.30), we get

$$-\nabla_k^2 \hat{c}(0, n) = \frac{\nabla_k^2 F(0, r)}{[\partial_z F(0, r)]^2}(n + 1)\frac{1}{r^{n+2}} + \frac{1}{2\pi i}\oint \frac{E(z)}{z^{n+1}} dz. \tag{4.32}$$

By Lemma 4.9 (i), if for every $\epsilon < 1/4$, $|E(z)| \leq$ const $\cdot\, |r - z|^{-2+\epsilon}$ for all $|z| \leq r$, then the second term on the right-hand side of (4.32) is $O(r^{-n} n^{\alpha})$ for every $\alpha > 3/4$. Therefore, to obtain the desired result, it is sufficient to show

$$|E(z)| \leq \text{ const} \cdot |r - z|^{-2+\epsilon} \tag{4.33}$$

for every $\epsilon < 1/4$.

By the definition of $E(z)$, we have

$$E(z) = T_1 + T_2, \tag{4.34}$$

where

$$T_1 = [\partial_z F(0, r)]^{-2} \frac{\nabla_k^2 F(0, z) - \nabla_k^2 F(0, r)}{(r - z)^2}, \tag{4.35}$$

$$T_2 = \frac{-\nabla_k^2 F(0, z) \Big\{ F(0, z)^2 - [\partial_z F(0, r)]^2 (r - z)^2 \Big\}}{[\partial_z F(0, r)]^2 F(0, z)^2 (r - z)^2}. \tag{4.36}$$

For a fixed $\epsilon < 1/4$, by Corollary 4.12 and (4.18),

$$|T_1| \leq O(|r - z|^{\epsilon - 2}). \tag{4.37}$$

By Lemma 4.13,

$$\begin{aligned} |T_2| \leq \text{const} \cdot |r - z|^{-4} &|F(0, z) + \partial_z F(0, r)(r - z)| \\ &|F(0, z) - \partial_z F(0, r)(r - z)|. \end{aligned} \tag{4.38}$$

By Corollary 4.12, $|\delta_z^{\epsilon} \partial_z F(k, z)|$ is absolutely convergent uniformly in k, $|z| \leq r$. By (4.19),

$$|F(0, z) - \partial_z F(0, r)(z - r)| \leq B |r - z|^{1+\epsilon} \tag{4.39}$$

for all $|z| \leq r$.

Since $F(0, r) = 0$, the middle factor of (4.38),

$$F(0, z) + \partial_z F(0, r)(r - z) = F(0, z) - F(0, r) + \partial_z F(0, r)(r - z),$$

is $O(|r - z|)$ because $\partial_z F(0, r)$ exists. Therefore, $|T_2| \leq O(|r - z|^{\epsilon - 2})$. This proves (4.33). ∎

5. Limiting Distribution of Critical Oriented Percolation Above Critical Dimensions

5.1. Motivation of Expansion

Following the definitions and the notations introduced in Section 1.5, we first explain the motivation for the expansion.

The triangle condition for oriented percolation is the following

Theorem 5.1. (Barsky and Aizenman [3])
Let $\nabla_p = \sum_{(x,n)(y,m)} C(x,n)C(y-x,m-n)C(y,m)$. *If* $\nabla_{p_c} < \infty$, *then* $\beta = \gamma = 1$.

Let

$$\hat{C}(k,z) = \sum_{n=0}^{\infty} z^n \sum_{x \in Z^d} e^{ik\cdot x} C(x,n)$$

with radius of convergence $r(k)$. Let

$$\hat{C}(k,n) = \sum_{x \in Z^d} e^{ik\cdot x} C(x,n).$$

Define $\pi(k,z)$ by the formula

$$\hat{C}(k,z) = \frac{1+\pi(k,z)}{1-2dzpD(k)-2dzpD(k)\pi(k,z)}, \tag{5.1}$$

where $D(k) = (\sum_{i=1}^{d} \cos k_i)/d$. The idea in the proof of Theorem 1.3 is to show that $\hat{C}(k,z)$ has "infrared bound": There exists a universal constant Q such that

$$|\hat{C}(k,e^{it})| \le \frac{Q}{|k|^2+|t|} \tag{5.2}$$

for all $k \in [-\pi,\pi]^d$ and $t \in [-\pi,\pi]$ and for all $0 < p < p_c$. Let $\tilde{c}(x,n) = c(x,-n)$. Then $\hat{\tilde{c}}(k,e^{it}) = \hat{c}(k,e^{-it})$ and

$$\begin{aligned}
\nabla_{p_c} &= \lim_{p\uparrow p_c} (C * C * \tilde{C})(0,0) \\
&\le \lim_{p\uparrow p_c} \text{const} \cdot \int\int |\hat{C}(k,e^{it})|^2 |\hat{c}(k,e^{-it})| dkdt \\
&\le \lim_{p\uparrow p_c} \text{const} \cdot \int\int \Big(\frac{1}{|k|^2+|t|}\Big)^3 dtdk < \infty, \qquad \text{if } d \ge 5.
\end{aligned}$$

Therefore Theorem 1.3 follows from Theorem 5.1 and the infrared bound. We can obtain infrared bound if we can show that π and its derivatives are small. Therefore a convergent expansion for π will be very useful. By a similar idea as that in self-avoiding random walk discussed in Sections 4, a good control of π can also give Theorem 1.4.

5.2. Expansion for π

Given a configuration σ, b is called a *pivotal bond* for $(0,0) \to (x,n)$ if $(0,0) \to (x,n)$ when σ_b is replaced by 1 and $(0,0) \not\to (x,n)$ when σ_b is replaced by 0. Suppose $\sigma \in \{(0,0) \to (x,n)\}$. Then we can find out all pivotal bonds $b_1, b_2, \ldots, b_m$, and put them in order $b_1 < b_2 < \ldots < b_m$, which means b_i is below b_{i+1}. So

$$\begin{aligned}
\hat{C}(k,z) &= \sum_{n=0}^{\infty} z^n \sum_{x\in Z^d} e^{ik\cdot x} P((0,0) \to (x,n)) \\
&= \sum_{n=0}^{\infty} z^n \sum_{x\in Z^d} e^{ik\cdot x} \sum_{m=0}^{\infty} \sum_{b_1<b_2<\ldots<b_m} \\
&\qquad P((0,0) \to (x,n) \text{ and } b_i'\text{s are exactly pivotal bonds}) \\
&= \sum_{n=0}^{\infty} z^n \sum_{x\in Z^d} e^{ik\cdot x} \sum_{m=0}^{\infty} \sum_{b_1<b_2<\ldots b_m} \\
&\qquad E\left\{ \prod_{i=0}^{m} 1_{B_i} \prod_{i=1}^{m} 1_{b_i \text{ open}} \prod_{0\le i<j\le n} (1-\delta_{ij}) \right\}.
\end{aligned} \tag{5.3}$$

Here B_0 is the event that $(0,0)$ and the bottom of b_1 are doubly connected, B_m is the event that the top of b_m and (x,n) are doubly connected and

B_i is the event that the top of b_i is doubly connected to the bottom of b_{i+1}, for $i = 1, \ldots, m-1$. $\delta_{ij} = 1$ if the top of b_i is connect to the bottom of b_j without using any $b_{i+1}, \ldots b_j$, and $\delta_{ij} = 0$ otherwise. Two lattice sites are said to be *doubly connected* if there exist two paths consisting of open bounds which connect the lattice sites and the two paths share no common bonds. The same lattice site is also said to be doubly connected to itself. Like what we have in Section 4, let

$$\psi[0,n] = \prod_{0\le i<j\le n} (1-\delta_{ij}).$$

Then

$$\psi[0,n] = \sum_{G:\text{graph on }[0,n]} \prod_{ij\in G} (-\delta_{ij}). \tag{5.4}$$

Let

$$\psi_c[0,n] = \sum_{\Gamma:\text{connected graph on } [o,n]} \prod_{ij\in\Gamma} (-\delta_{ij}), \quad n \geq 1, \tag{5.5}$$

$$\psi_c[0,0] = 1. \tag{5.6}$$

Here the definition of a connected graph is slightly different from the one used in Section 4.1. In the present case, a graph Γ on $[0,n]$ is said to be *connected* if $\cup_{ij\in\Gamma}[i,j] = [0,n]$.

Theorem 5.2. (Connected Formula for $\pi(k,z)$)

$$\begin{aligned}\pi(k,z) = &\sum_{n=0}^{\infty} z^n \sum_{x\in Z^d} e^{ik\cdot x} \\ &\sum_{m=0}^{\infty} \sum_{b_1<b_2<\ldots<b_m} E\left\{\prod_{i=0}^{m} 1_{B_i} \prod_{i=1}^{m} 1_{b_i \text{ open}}\, \psi_c[0,m]\right\}\end{aligned} \tag{5.7}$$

for all $|z| < r(k)$ and inside the radius of convergence of the above series.

A minimally connected graph is called a *lace graph.* The number of edges in a lace graph L is called the *order* of L and denoted by $o(L)$. Using a similar argument as that in Section 4, given a connected graph Γ, we can define a lace graph $L(\Gamma)$. Let $m(L)$ be the maximal connected graph Γ such that $L(\Gamma) = L$. Then we have

Theorem 5.3. (Lace Expansion for $\pi(k,z)$)

$$\pi(k,z) = \sum_{l=0}^{\infty} \pi^{(l)}(k,z), \tag{5.8}$$

$$\begin{aligned}\pi^{(0)}(k,z) = &\sum_{n=0}^{\infty} z^n \sum_{x\in Z^d} e^{ik\cdot x} \\ &P\{(0,0) \text{ and } (x,n) \text{ are doubly connected}\},\end{aligned} \tag{5.9}$$

$$\pi^{(l)}(k,z) = \sum_{n=0}^{\infty} z^n \sum_{x\in Z^d} e^{ik\cdot x} \sum_{m=1}^{\infty} \sum_{b_1<b_2<\ldots<b_m} \tag{5.10}$$

$$E\left\{\prod_{i=0}^{n} 1_{B_i} \prod_{i=1}^{m} 1_{b_i} \text{ open} \sum_{\substack{L:lace on [0,n]\\ o(L)=l}} \prod_{ij\in L}(-\delta_{ij}) \prod_{ij\in m(L)\setminus L} (1-\delta_{ij})\right\}, \; l \geq 1.$$

5.3. Convergence of $\pi(k,z)$

If $|z| \le 1$, then $|\pi^{(0)}(k,z)| \le \sum_{n=0}^{\infty}\sum_{x\in\mathbb{Z}^d} C^2(x,n)$ by v.d. Berg-Kesten inequality; see e.g. [19]. If $p = 1/(2d)$, then $C(x,n) \le p_n(0,x)$, where $p_n(0,x)$ is the transition function for simple random walk. So $|\pi^{(0)}(k,z)| \le \sum_{n=0}^{\infty} p_{2n}(0,0)$, by Markov property. Therefore $|\pi^{(0)}(k,z)| \le g_1(0,0)$, the Green function defined in Section 4.

By a more complicated argument (see [31, 32] for details),

$$|\pi^{(l)}(k,z)| \le \sum_x \sum_{n=0}^{\infty} {}^{(0,0)} \qquad\qquad {}^{(x,n)};$$

here the right-hand side is a Feynman diagram, where each edge represents a connectivity function. Again for $p = 1/(2d)$, the connectivity function can be estimated by transition function of simple random walk. Then one can prove that $|\pi(k,z)| \le O(1/d)$ for $|z| \le 1$, $p = 1/(2d)$. The technique for extension of estimate to $|z| \le \gamma(0)$ and $0 < p < p_c$ is similar to the one used in self-avoiding random walks in high dimensions. Complete arguments are in Nguyen and Yang [31, 32].

6. Integrated super-Brownian Excursions (ISE), Lattice Trees and Percolation.

In this section, we give the definition of ISE and a brief summary of the recent developments of scaling limits of lattice trees, percolation and oriented percolation above their upper critical dimensions. A survey in this topic can be found in [39].

6.1. Probability Measures on $M_1(\mathbb{R}^d)$

Let $\mathbb{R}^d$ be the d-dimensional Euclidean space and $\dot{\mathbb{R}}^d$ its one-point compactification. Let $M_1(\mathbb{R}^d)$ and $M_1(\dot{\mathbb{R}}^d)$ be the set of all probability measures on $\mathbb{R}^d$ and $\dot{\mathbb{R}}^d$, respectively. $M_1(\mathbb{R}^d)$ and $M_1(\dot{\mathbb{R}}^d)$ are topological spaces with weak topologies. Under the weak topology, for ν_n, $\nu \in M_1(\mathbb{R}^d)$, $\nu_n \to \nu$ if and only if $\nu_n(f) \to \nu$ for all bounded continuous functions f on $\mathbb{R}^d$. The same statement holds for $\mathbb{R}^d$ replaced by $\dot{\mathbb{R}}^d$.

With the weak topology, $M_1(\dot{\mathbb{R}}^d)$ is a compact metric space and $M_1(\mathbb{R}^d)$ can be regarded as an embedded subspace in $M_1(\dot{\mathbb{R}}^d)$.

Let $\mathcal{F}$ and $\dot{\mathcal{F}}$ be the Borel σ-algebra of $M_1(\mathbb{R}^d)$ and $M_1(\dot{\mathbb{R}}^d)$, respectively. It can be easily seen that $M_1(\mathbb{R}^d) \in \dot{\mathcal{F}}$.

Let μ be a probability measure defined on $(M_1(\dot{\mathbb{R}}^d), \dot{\mathcal{F}})$. For $l = 1, 2, \ldots$, the l^{th} moment measure of μ is defined by a probability measure M^l_μ on $(\dot{\mathbb{R}}^d)^l$ such that

$$M^l_\mu(f) = \int_{M_1(\dot{\mathbb{R}}^d)} \nu^l(f) d\mu(\nu) \tag{6.1}$$

for all bounded continuous functions f on $(\dot{\mathbb{R}}^d)^l$, where $\nu^l = \nu \times \nu \times \ldots \times \nu$ is the product measure of ν of l times on $(\dot{\mathbb{R}}^d)^l$.

Note that for every bounded continuous function f on $(\dot{\mathbb{R}}^d)^l$, $\nu^l(f)$ is a continuous function on $\nu \in M(\dot{\mathbb{R}}^d)$, and therefore (6.1) makes sense. It follows from Riesz-Markov Theorem that M^l_μ exists.

By Stone-Weierstrass Theorem and Compactness of $M_1(\dot{\mathbb{R}}^d)$, we have

Proposition 6.1.

(a) *Let μ_1, μ_2 be probability measures on $(M_1(\dot{\mathbb{R}}^d), \dot{\mathcal{F}})$. Suppose $M^l_{\mu_1} = M^l_{\mu_2}$ for all $l = 1, 2, \ldots,$ then $\mu_1 = \mu_2$.*

(b) *Let μ_n be probability measure on $M_1(\dot{\mathbb{R}}^d)$. If $M^l_{\mu_n}$ converges, as $n \to \infty$, for all $l = 1, 2, \ldots,$ then there exists a probability measure μ on $M_1(\dot{\mathbb{R}}^d)$ such that $\mu_n \to \mu$, as $n \to \infty$.*

6.2. Definition of Integrated super-Brownian Excursion (ISE)

Let

$$p_t(x) = \frac{1}{(2\pi t)^{d/2}} e^{-x^2/2t},\ x \in \mathbb{R}^d,\ t > 0. \tag{6.2}$$

The function $p_t(x)$ is known as the transition function for Brownian motion in $\mathbb{R}^d$. Let

$$a^2(x,t) = te^{-\frac{t^2}{2}} p_t(x) \tag{6.3}$$

and

$$A^2(x) = \int_0^\infty a^{(2)}(x,t)dt = \int_0^\infty te^{-\frac{t^2}{2}} p_t(x)dx. \tag{6.4}$$

To define A^l for $l \geq 3$, we need the notion of shapes, which is defined as follows. An m-skeleton is a tree having m unlabelled external vertices of degree 1, and $m - 2$ unlabelled internal vertices of degree 3, and no other vertices. An m-shape is a tree having m labeled external vertices of degree 1, and $m-2$ unlabeled internal vertices of degree 3, and no other vertices. In other words, an m-shape is a labeling of an m-skeleton's external vertices by the labels $0, 1, \ldots, m-1$. For notational convenience, we associate to each m-shape an arbitrary labeling of its $2m-3$ edges, with labels $1, 2, \ldots, 2m-3$. This choice of edge labeling is arbitrary but fixed; see Figure 2.

Let $\sum_m$ be the set of all m-shapes. Let $|\sum_m|$ be the number of m-shapes. Then we have

Figure 2. The shapes for $m = 2, 3, 4$ and two examples of $m = 5$. The shapes' oriented edge labeling are arbitrary but fixed.

$$|\sum_2| = 1, \tag{6.5}$$

$$|\sum_3| = 1, \tag{6.6}$$

$$|\sum_m| = (2m-5)!!, \tag{6.7}$$

where $(-1)!! = 1$, $(2j+1)!! = (2j+1)(2j-1)!!$ for $j \geq 0$. (6.5)-(6.6) are obvious from the definition and a proof of (6.7) can be found (e.g. (5.96) of [19]).

Let $m \geq 2$ and $\sigma \in \sum_m$. To each edge j of σ, oriented away from vertex 0, there is associated (t_j, y_j). Here t_j is a nonnegative real number and $y_j \in \mathbb{R}^d$.

Let $\overrightarrow{y} = (y_1, y_2, \ldots, y_{2m-3})$ and $\overrightarrow{t} = (t_1, t_2, \ldots, t_{2m-3})$. Define

$$a^m(\sigma; \overrightarrow{y}, \overrightarrow{t}) = \left(\sum_{i=1}^{2m-3} t_i\right) e^{-\frac{(\Sigma_{i=1}^{2m-3} t_i)^2}{2}} \prod_{i=1}^{2m-3} p_{t_i}(y_i), \tag{6.8}$$

$$A^m(\sigma; \overrightarrow{y}) = \int_0^\infty dt_1 \ldots \int_0^\infty dt_{2m-3} a^{(m)}(\sigma; \overrightarrow{y}, \overrightarrow{t}). \tag{6.9}$$

We have the following properties:

$$\int_{\mathbb{R}^{d(2m-3)}} A^m(\sigma; \overrightarrow{y}) d\overrightarrow{y} = \frac{1}{(2m-5)!!} \tag{6.10}$$

and

$$\sum_{\sigma \in \Sigma_m} \int_{\mathbb{R}^{d(2m-3)}} A^m(\sigma; \overrightarrow{y}) d\overrightarrow{y} = 1. \tag{6.11}$$

The Fourier transform of a^m and A^m are given by

$$\hat{a}^{(m)}(\sigma; \overrightarrow{k}, \overrightarrow{t}) \left(\sum_{i=1}^{2m-3} t_i\right) e^{-\frac{(\Sigma_{i=1}^{2m-3} t_i)^2}{2}} \prod_{i=1}^{2m-3} e^{-\frac{k_i^2 t_i^2}{2}} \tag{6.12}$$

and

$$\hat{A}^{(m)}(\sigma; \overrightarrow{k}) \int_0^\infty dt_1 \ldots \int_0^\infty dt_{2m-3} \hat{a}^{(m)}(\sigma; \overrightarrow{k}, \overrightarrow{t}).$$

The integrated super-Brownian excursion (ISE) is a probability measure μ on $M_1(\mathbb{R}^d)$ such that its moments are given by

$$dM_\mu^{(1)}(x) = A^{(2)}(x) dx, \tag{6.13}$$

$$dM_\mu^{(2)}(x_1, x_2) = \left[\int_{\mathbb{R}^d} A^{(3)}(y, x_1 - y, x_2 - y) dy\right] dx_1 dx_2, \tag{6.14}$$

$$dM_\mu^{(l)}(x_1, \ldots, x_l) = \left[\sum_{\sigma \in \Sigma_{l+1}} \int_{\mathbb{R}^{d(l-1)}} A^{(l+1)}(\sigma; \overrightarrow{y}) \prod_{j \in J} dy_j\right] \prod_{i=1}^{l} dx_i, \tag{6.15}$$

where an oriented edge j is in J if the end point of j has degree 3. Note that $|J| = l - 1$. The other variables y_a, $a \notin J$, satisfy the constraint that for each external vertex $i = 1, 2, \ldots, l$ of σ, $\sum y_l = x_i$, where the summation is taking over all y_l on the path from 0 to i. For example, the contribution to the $dM^4(x_1, x_2, x_3, x_4)$ by σ_4 of Figure 2 is

$$\begin{aligned}\int A^{(5)}(\sigma_4; y_1, x_1 - y_1, y_3, x_3 - y_3 - y_1, y_5, \\ x_2 - y_5 - y_3 - y_1, x_4 - y_5 - y_3 - y_1) dy_5 dy_3 dy_1.\end{aligned} \tag{6.16}$$

The expression (6.16) is a function

$$F = F_{\sigma_4}(x_1, x_2, x_3, x_4).$$

Let

$$\hat{F}_{\sigma_4}(\bar{k}) = \sum_{\bar{x}} F_{\sigma_4}(\bar{x}) e^{i\bar{k}\cdot\bar{x}}$$

be its Fourier transform, with $\bar{x} = (x_1, \ldots, x_4)$, $\bar{k} = (k_1, \ldots, k_4)$ and $\bar{k} \cdot \bar{x} = \sum_{i=1}^{4} k_i \cdot x_i$. Using (6.16), we have

$$\begin{aligned} \hat{F}_{\sigma_4}(\bar{k}) = \hat{A}^5(&\sigma_4; k_1 + k_2 + k_3 + k_4, k_1, \\ & k_2 + k_3 + k_4, k_3, k_2 + k_4, k_4, k_2). \end{aligned} \tag{6.17}$$

Note that the diagram for $\hat{F}_{\sigma_4}(\bar{k})$ is

(6.18)

This diagram satisfies the "conservation of momentum".

By (6.15), the characteristic function of $M_\mu^{(l)}$ is

$$\hat{M}_\mu^{(l)}(\bar{k}) = \int e^{i\bar{k}\cdot\bar{x}} dM_\mu^{(l)}(\bar{x}) \tag{6.19}$$

$$= \sum_{\sigma \in \Sigma_{l+1}} \hat{F}_\sigma(\bar{k}), \tag{6.20}$$

$$\hat{F}_\sigma(\bar{k}) = \hat{A}^{(l+1)}(\sigma; \vec{p}), \tag{6.21}$$

where $p_j = k_i$ if j is an oriented edge of σ with external end point i, and all other p_l's satisfy the "conservation of momentum"; see (6.18).

6.3. Lattice Trees

In this subsection, we consider lattice trees defined in Section 1.3. We will look at the main idea of the proof of Theorem 1.2 and discuss some open problems. Following the definitions and notations in Section 1.3, we let

$$S_n^{(l+1)}(x_1,\ldots,x_n) = \sum_{T;|T|=n+1} \sum_{i_1,\ldots,i_l \in T} \prod_{j=1}^{l} 1_{x_j}(i_j). \tag{6.22}$$

Note that $S_n^{(l+1)}(x_1,\ldots,x_n)$ is the number of n-bond lattice trees containing $\{0,x_1,\ldots,x_n\}$. Then we have

$$M_{\mu_n}^{(l)}(x_1,\ldots,x_n) = \frac{1}{(n+1)^l S_n^{(0)}} S_n^{(l+1)}(x_1,\ldots,x_n). \tag{6.23}$$

Moreover, we have

$$\hat{S}_n^{(l+1)}(0) = (n+1)^l S_n^{(0)}. \tag{6.24}$$

Therefore

$$\hat{M}_{\mu_n}^{(l)}(\bar{k}) = \frac{\hat{S}_n^{(l+1)}(\bar{k}D_1^{-1}n^{-\frac{1}{4}})}{\hat{S}_n^{l+1}(0)}. \tag{6.25}$$

By (6.19)-(6.21) and Proposition 6.1 (b), to prove Theorem 1.2 it is sufficient to show

$$\lim_{n\to\infty} \frac{\hat{S}_n^{(l+1)}(\bar{k}D_1^{-1}n^{-\frac{1}{4}})}{\hat{S}_n^{l+1}(0)} = \sum_{\sigma\in\Sigma_{l+1}} \hat{A}^{(l+1)}(\sigma;\vec{p}), \tag{6.26}$$

where the right-hand side satisfies the constraints given in (6.21).

We will relate $S_n^{(l)}$ to l-point function $t_n^{(l)}$ defined as follows. The one-point function $t_n^{(1)}$ is defined to be the number of n-bond lattice trees containing the origin, with $t_0^{(1)} = 1$. For the definition of m-point function $t_n^{(m)}(\sigma;\vec{y},\vec{s}), m \geq 2$, we will need the notions of shapes and backbonds. Let $\sigma \in \Sigma_m$ be an m-shape (see Section 6.2 for the definition), and associated to each j in σ, let $y_j \in \mathbb{Z}^d$ and let s_j be a nonnegative integer for $j = 1,2,\ldots,2m-3$. Given a lattice tree T containing the sites $0,x_1,\ldots,x_{m-1}$, we define the backbone $B = B(T;0,x_1,\ldots,x_{m-1})$ to be the subtree T spanning $0,x_1,\ldots,x_{m-1}$. There is an induced labeling of the external vertices of the backbone, in which vertex x_l is labeled l. Ignoring vertices of degree 2 in B, this backbone is equivalent to a shape σ_B or to a subshape of σ_B (note that in the latter case, σ_B is not uniquely dertermined). Restoring vertices of degree 2 in B, let b_j denote the length of the backbone path corresponding to edge j of σ_B, with $b_j = 0$ for any contracted edge in a subshape. We say that $(T;0,x_1,\ldots,x_{m-1})$ is compatible with $(\sigma;\vec{y},\vec{s})$ if σ_B can be chosen such that $\sigma_B = \sigma$, $b_j = s_j$ for all edges j of σ, and if the backbone path corresponding to j undergoes the displacement y_j for all edges j of σ.

We define $t_n^{(m)}(\sigma;\, \vec{y}, \vec{s})$ to be the number of n-bond lattice trees T, containing $0, x_1, x_2, \ldots, x_{m-1}$, such that $(T;\, 0, x_1, \ldots, x_{m-1})$ is compatible with $(\sigma;\, \vec{y}, \vec{s})$. Let

$$t_n^{(m)}(\sigma;\, \vec{y}) = \sum_{\vec{s}} t_n^{(m)}(\sigma;\, \vec{y}, \vec{s}), \tag{6.27}$$

where $\vec{s} = (s_1, s_2, \ldots, s_{2m-3})$, $s_j = 0, 1, 2, \ldots$, for all j, and

$$\hat{t}_n^{(m)}(\sigma;\, \vec{k}) = \sum_{\vec{y}} t_n^{(m)}(\sigma;\, \vec{y}) e^{i\vec{k}\cdot\vec{y}},\ k_j \in [-\pi, \pi]^d. \tag{6.28}$$

The proof of Theorem 1.2 follows from the following two theorems. For $m \geq 2$, let

$$p_n^{(m)}(\sigma, \vec{y}) = \frac{t_n^{(m)}(\sigma;\, \vec{y})}{\sum_{\sigma\in\Sigma_m} \hat{t}_n^{(m)}(\sigma;\, \vec{0})}. \tag{6.29}$$

Theorem 6.2. [11] *Let $m \geq 2$, and $k_j \in \mathbb{R}^d$ $(j = 1, 2, \ldots, 2m-3)$. For nearest-neighbor lattice trees in sufficiently high dimensions $d \geq d_0$, and for spread-out lattice trees with $d > 8$ and L sufficiently large depending on d, there are constants z_c, C_1, D_1 depending on d and L such that, as $n \to \infty$,*

$$\hat{t}_n^{(m)}(\sigma;\, \vec{k} D_1^{-1} n^{-\frac{1}{4}}) \sim c_1 n^{m-\frac{5}{2}} z_c^{-n} \hat{A}^{(m)}(\sigma;\, \vec{k}), \tag{6.30}$$

$$[\hat{t}_n^{(1)}]^{\frac{1}{n}} \sim z_c. \tag{6.31}$$

In particular,

$$\lim_{n\to\infty} p_n^{(m)}(\sigma;\, \vec{k} D_1^{-1} n^{-\frac{1}{4}}) = \hat{A}^{(m)}(\sigma;\, \vec{k}). \tag{6.32}$$

In view of (6.26) and Theorem 6.2, to prove Theorem 1.2, it remains to show that the differences of $\hat{S}_n^{(l+1)}$ and $\sum_{\sigma\in\Sigma_{l+1}} \hat{t}_n^{(l+1)}$ are small.

Note that

$$\hat{S}_n^{(2)}(k) = \hat{t}_n^{(2)}(k), \tag{6.33}$$

$$\hat{S}_n^{(3)}(k_1, k_2) = \hat{t}_n^{(3)}(k_1 + k_2, k_1, k_2). \tag{6.34}$$

By Theorem 6.2, (6.25) and (6.26), convergence of the first and second moments follows. Convergence of higher moments follows from Theorem 6.2, (6.25) and (6.26) and the following theorem.

Theorem 6.3. [39] *Under the same assumptions as in Theorem* 6.2*, for* $l \geq 3$*,*

$$|\hat{S}_n^{(l+1)}(k_1,\dots,k_l) - \sum_{\sigma\in\Sigma_{(l+1)}} \hat{t}_n^{(l+1)}(\sigma,\vec{p})| \leq O(n^{l-2}z_c^{-n}), \tag{6.35}$$

where $\vec{p} = (p_1,\dots,p_{2l-1})$ *satisfies "conservation of momentum" with external vertices* k_i*,* $i = 1,2,\dots,l$*.*

Now, we discuss the convergence of backbone functions. We have

Theorem 6.4. [11] *Let* $m = 2$*, or* $m = 3$*,* $k_j \in \mathbb{R}^d$ *and* $t_j \in [0,\infty)$ *for* $j = 1,2,\dots,2m-3$*. For nearest-neighbor lattice trees in sufficiently high dimensions* $d \geq d_0$*, and for spread-out lattice trees with* $d > 8$ *and* L *sufficiently large depending on* d*, there exists a constant* T_1 *depending on* d *and* L*, such that, as* $n \to \infty$*,*

$$\hat{t}_n^{(m)}(\sigma;\,\vec{k}D_1^{-1}n^{-\frac{1}{4}},\,[T_1 n^{\frac{1}{2}}]) \sim C_1 T_1^{-(2m-3)} n^{-1} z_c^{-n} \hat{a}^{(m)}(\sigma;\,\vec{k},\,\vec{t}). \tag{6.36}$$

In particular,

$$\lim_{n\to\infty} (T_1 n^{\frac{1}{2}})^{2m-3} \hat{p}_n^{(m)}(\sigma;\,\vec{k}D_1^{-1}n^{-\frac{1}{4}},\,[\vec{t}\,T_1 n^{\frac{1}{2}}]) = \hat{a}^{(m)}(\sigma;\,\vec{k},\,\vec{t}). \tag{6.37}$$

It is conjectured in [39] that Theorem **??** should hold for all $m \geq 2$.

The idea of proofs of Theorem 6.2 and Theorem 6.4 is as follows.
Let

$$G_{z,\vec{\xi}}^{(m)}(\sigma;\,\vec{y}) = \sum_{n=0}^{\infty}\sum_{\vec{s}} t_n^{(m)}(\sigma;\,\vec{y},\vec{s}) z^n \prod_{j=1}^{2m-3} \xi_j^{s_j}, \tag{6.38}$$

$|z| < z_c$ and $|\xi_j| \leq 1$. By using double lace expansion, it is proved in [11] that

$$\hat{G}_{z,\xi}^{(2)}(k) = \frac{C_1}{D_1^2 k^2 + 2^{3/2}(1-\frac{z}{z_c})^{1/2} + 2T_1(1-\xi)} + \text{error}, \tag{6.39}$$

and

$$\hat{G}_{z,\xi}^{(m)}(\sigma;\,\vec{k}) = \nu_1^{m-2} \prod_{j=1}^{2m-3} \hat{G}_{z,\xi_j}^{(2)}(k_j) + \text{error}, \tag{6.40}$$

for sufficiently high d, and for spread-out models for $d > 8$ with sufficiently large L. The error terms in (6.39) and (6.40) are controlled for all $m \geq 2$ if $\vec{\xi} = \vec{1}$ and for $m = 2,3$ for general $\vec{\xi}$. This gives the asymptotic behavior in

Theorems 6.2 and 6.4.

The control of error terms for $m \geq 2$ for general $\vec{\xi}$ is still an open problem.

6.4. Percolation

We will follow the definition and notations in Section 1.4. The 2-point and 3-point functions are defined by

$$\tau^{(2)}(x;\, n) = P_{p_c}(x \in C(0);\, |C(0)| = n), \quad x \in \mathbb{Z}^d. \tag{6.41}$$

and

$$\tau^{(3)}(x, y;\, n) = P_{p_c}(x, y \in C(0);\, |C(0)| = n), \quad x, y \in \mathbb{Z}^d. \tag{6.42}$$

Let $\hat{\tau}^{(2)}(k, n)$ and $\hat{\tau}^{(3)}(k, l;\, n)$ denote their Fourier transforms.

The limits of $\hat{\tau}^{(2)}$ and $\hat{\tau}^{(3)}$ are given by the following theorem.

Theorem 6.5. [23] *Fix $k, l \in \mathbb{R}^d$ and any $\epsilon \in (0, 1/2)$. There is a d_0 such that for nearest-neighbor percolation with $d \geq d_0$, there are constants C_2, D_2 (depending on d) such that as $n \to \infty$,*

$$\hat{\tau}^{(2)}(kD_2^{-1}n^{-\frac{1}{4}};\, n) = \frac{C_2}{\sqrt{8\pi n}}\hat{A}^{(2)}(k)[1 + O(n^{-\epsilon})], \tag{6.43}$$

$$\hat{\tau}^{(3)}(kD_2^{-1}n^{-\frac{1}{4}}, lD_2^{-1}n^{-\frac{1}{4}};n) = \frac{C_2}{\sqrt{8\pi n}}n^{\frac{1}{2}}\hat{A}^{(3)}(k + l, k, l)[1 + O(n^{-\epsilon})]. \tag{6.44}$$

It follows that the first and second moments of μ_n converge to μ_{ISE}. In fact, the characteristic functions $\hat{N}_n^{(1)}(k)$ and $\hat{N}_n^{(2)}(k, l)$ of the first and second moments of μ_n are given by

$$\hat{N}_n^{(1)}(k) = \frac{\hat{\tau}^{(2)}(kD_2^{-1}n^{-\frac{1}{4}};\, n)}{\hat{\tau}^{(2)}(0;\, n)}, \tag{6.45}$$

$$\hat{N}_n^{(2)}(k, l) = \frac{\hat{\tau}^{(3)}(kD_2^{-1}n^{-\frac{1}{4}}, lD_2^{-1}n^{-\frac{1}{4}};\, n)}{\hat{\tau}^{(3)}(0, 0;\, n)}, \tag{6.46}$$

and these converge respectively to the characteristic functions $\hat{A}^{(2)}(k)$ and $\hat{A}^{(3)}(k + l, k, l)$ of the first and second moments of μ_{ISE}, in high dimensions.

The problem of convergence of μ_n to μ_{ISE} for $d > 6$ remains open even for sufficiently large d.

6.5. Oriented Percolation

Using the notations defined in Section 1.5, we let

$$\sigma^{(2)}((x,n);\,N) = P_{p_c}(C(0,0) \ni (x,n),\, |C(0,0)| = N).$$

The following theorem is essentially Theorem 1.4.

Theorem 6.6. (Nguyen and Yang [32]) *For oriented percolation, there exist constants C_3, T_3, D_3 such that*

$$\lim_{n\to\infty} 2C_3^{-1}T_3 \sum_{N=1}^{\infty} \hat{\sigma}^{(2)}(kT_3^{\frac{1}{2}}D_3^{-1}n^{-\frac{1}{2}}, [tn]) = e^{-k^2t/2},$$

for sufficiently high d or for $d+1>5$ with sufficiently large L.

There is work in progress by E. Derbez, R. van der Hofstad and G. Slade showing that the scaling limit of oriented percolation is a super-Brownian motion, if d is sufficiently large or for $d+1>5$ with sufficiently large L. Their method is based on the inductive method of [25].

References

1. M. Aizenman and C. M. Newman, Tree graph inequalities and critical behavior in percolation models, *J. Statist. Phys.* **36** (1984), 107-143.
2. D. Aldous, Tree-based models for random distribution of mass, *J. Statist. Phys.* **73** (1993), 625-641.
3. D. J. Barsky and M. Aizenman, Percolation critical exponents under the triangle condition, *Ann. Probab.* **19** (1991), 1520-1536.
4. D. Brydges, A short course on cluster expansions, in: *Critical Phenomena*, *Random Systems*, *Gauge Theories*, Les Houches Section XLIII, K. Osterwalder, B. Stora, eds., North Holland, Amsterdam, 1984.
5. D. Brydges and P. Federbush, A new form of the Mayer expansion in classical statistical mechanics, *J. Math. Phys.* **19** (1978), 2064-2067.
6. D. C. Brydges, J. Frölich and T. Spencer, The random walk representation of classical spin systems and correlation inequalities, *Comm. Math. Phys.* **83** (1982), 123-150.
7. D. C. Brydges and T. Spencer, Self-avoiding walk in 5 or more dimensions, *Comm. Math. Phys.* **97** (1985), 125-148.
8. R. M. Corless, G. H. Gonnet, E. D. G. Hare, D. J. Jeffrey and D. E. Knuth, On the Lambert W function, *Adv. Comput. Math.* **5** (1996), 329-359.

9. D. A. Dawson, *Measure-valued Markov Processes*, Lecture Notes in Mathematics, vol. 1541, Springer, Berlin, 1993.

10. E Derbez and G. Slade, Lattice trees and super-Brownian motion, *Canad. Math. Bull.* **40** (1997), 19-38.

11. E. Derbez and G. Slade, The scaling limit of lattice trees in high dimensions, *Comm. Math. Phys.* **193** (1998), 69-104.

12. R. L. Dobrushin, Investigation of the conditions of the asymptotic existence of the configuration integral of the Gibbs distribution, *Teor. Veroyatnost. i Primenen.* **9** (1964), 626-643.

13. M. E. Fisher and D. Ruelle, The stability of many-particle systems, *J. Math. Phys.* **7** (1966), 260-270.

14. P. Flajolet and A Odlyzko, Singularity analysis of generating functions, *SIAM J. Discrete Math.* **3** (1990), 216-240.

15. P.-G. de Gennes, *Scaling Concepts in Polymer Physics*, Cornell University Press, Ithaca, 1979.

16. J. Glimm and A. Jaffe, *Quantum Physics*, *A Functional Integral Point of View.* Springer-Verlag, New York, 1981.

17. J. Glimm, A. Jaffe and T. Spencer, The particle structure of the weakly coupled $P(\phi)_2$ model and other applications of high temperature expansions, in: *Constructive Field Theory*, G. Velo, A.S. Wightman, eds., Lecture Notes in Physics, vol. **25**, Springer-Verlag, New York, 1973, pp.133-242.

18. S. Golowich and J. Z. Imbrie, A new approach to the long-time behavior of self-avoiding random walks, *Ann. Phys.* **217** (1992) 142-169.

19. G. Grimmett, *Percolation*, Springer-Verlag, New York, 1989.

20. T. Hara and G. Slade, Mean-field critical behavior for percolation in high dimensions, *Comm. Math. Phys.* **128** (1990), 333-391.

21. T. Hara and G. Slade, Self-avoiding walk in five or more dimensions. I. The critical behavior, *Comm. Math. Phys.* **147** (1992), 101-136. The lace expansion for self-avoiding walk in five or more dimensions, *Rev. Math. Phys.* **4** (1992), 235-327.

22. T. Hara and G. Slade, The incipient infinite cluster in high-dimensional percolation, *Electron. Res. Announc. Amer. Math. Soc.* **4** (1998).

23. T. Hara and G. Slade, The scaling limit of the incipient infinite cluster in high-dimensional percolation, I. Critical exponents, II. Integrated super-Brownian excursion, Department of Mathematics, Tokyo Institute of Technology, Tokyo, Japan, preprint, 1999.

24. F. Harary and E. M. Palmer, *Graphical Enumeration*, Academic Press, New York, 1973.

25. R. van der Hofstad, F. den Hollander and G. Slade, A new inductive approach to the lace expansion for self-avoiding random walks, *Probab. Theory Related Fields* **111** (1998), 253-286.

26. T. Kennedy, Ballistic behavior in a 1D weakly self-avoiding walk with decaying energy penalty, *J. Statist. Phys.* **77** (1994), 565-579.

27. K. M. Khanin, J. L. Lebowitz, A. E. Mazel and Ya. G. Sinai, Self-avoiding walks in five or more dimensions: polymer expansion approach, *Russian Math. Surveys* **50** (1995), 403-434.

28. D. Klein and W.-S. Yang, Absense of first order phase transition in 2-dimensional antiferromagnetic systems, *J. Statist. Phys.* **70** (1993), 1391-1400.

29. T. M. Liggett, *Interacting Particle Systems*, Springer-Verlag, New York, 1985.

30. N. Madras and G. Slade, *The Self-Avoiding Walk*, Birkhäuser, Boston, 1993.

31. B. G. Nguyen and W-S. Yang, Triangle condition for oriented percolation in high dimensions, *Ann. Probab.* **21** (1993), 1809-1844.

32. B. G. Nguyen and W-S. Yang, Gaussian limit for critical oriented percolation in high dimension, *J. Statist. Phys.* **78** (1995), 841-876.

33. O. Penrose, Convergence of fugacity expansions for fluids and lattice gases, *J. Math. Phys.* **4** (1963), 1312-20.

34. M. Reed and B. Simon, *Methods of Modern Mathematical Physics I: Functional Analysis*, Academic Press, New York, 1972.

35. D. Ruelle, *Statistical Mechanics: Rigorous Results*, Addison-Wesley, 1989.

36. B. Simon, *The Statistical Mechanics of Lattice Gases*, V.I., Princeton University Press, 1993.

37. G. Slade, The diffusion of self-avoiding random walk in high dimensions, *Comm. Math. Phys.* **110** (1987), 661-683.

38. G. Slade, The scaling limit of self-avoiding random walk in high dimensions, *Ann. Probab.* **17** (1989), 91-107.

39. G. Slade, Lattice trees, percolation and super-Brownian motion, in: *Perplexing Problems in Probability: Papers in Honor of Harry Kesten*, M. Bramson, R. Durrett, eds., Birkhäuser, Basel, 1999.

40. W.-S. Yang and D. Klein, A note on the critical dimension for weakly self-avoiding walks, *Probab. Theory Related Fields* **79** (1988), 99-114.

Wei-Shih Yang
Department of Mathematics, Temple University
Philadelphia, PA 19122, U.S.A.
E-mail: yang@math.temple.edu

Aklilu Zeleke
Department of Mathematics and Computer Sciences, Alma College
Alma, MI 48801, U.S.A.

$G:\ n\ G_1:\ G_2\ n\ n$

$m=2\ m=3\ \sigma_1\ \sigma_2\ \sigma_3\ \sigma_4\ \sigma_5$

$K_3\ K_2+K_3+K_4\ K_2+K_4\ K_4$

$K_2\ K_1\ K_1+K_2+K_3+K_4$