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AN ALGORITHM FOR CALCULATING BETTI NUMBERS OF MANAGEABLE MODULES

Shou-Te Chang

Abstract. In this paper we provide an algorithm for computing Betti numbers and ranks of syzygy of manageable modules with respect to an R-regular sequence of a Noetherian local ring R when certain conditions are met. This method can be applied to many interesting cases and used to verify Horrocks' conjecture. The problem of calculating Betti numbers is reduced to calculating ranks of certain matrices with a special form over a field. Our approach is characteristic free.

1. Introduction

Throughout this paper the ring (R, m, k) is always a Noetherian local ring.

Write $\beta_i^R(M)$, or $\beta_i(M)$ if R is not emphasized, for the ith Betti number of the R-module M, and write $r_i^R(M)$, or $r_i(M)$ if R is not emphasized, for the rank of the ith syzygy of M. The Betti numbers of M may be defined as the ranks of the free modules in a minimal free resolution; it follows easily that $\beta_i^R(M)$ can be simply calculated as $\dim_k \operatorname{Tor}_i^R(M, k)$.

Let (R, m, k) be a regular local ring of dimension n and let M be a finite length module over R. Horrocks' question asks must $\beta_i(M)$, the ith Betti number of M, be at least $\binom{n}{i}$, where i is an integer between 0 and n; these numbers are achieved when $M = k = R/m$. In fact since $\beta_i^R(M) = r_i^R(M) + r_{i+1}^R(M)$, in [1] Buchsbaum and Eisenbud conjectured even more strongly that $r_i^R(M)$ is at least $\binom{n-1}{i-1}$. For a detailed account on the history, background and a problem list please see the interesting and informative paper by Charalambous and Evans [7].

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In the author's previous papers [4, 5] we developed methods for calculating $\beta_i^R(M)$ and $r_i^R(M)$ in various cases which confirms Horrocks' question regardingly. In [4] we dealt with modules of exponent two and in [5] we dealt with modules of essentially monomial type. There we utilized the same method of constructing (not minimal) free resolutions for modules with sufficiently simple structure. If certain conditions are met these free resolutions can be constructed with particular elegant form, and from there $\beta_i(M)$ and $r_i(M)$ can be computed, and sometimes it is only a matter of pencil and paper (see [4, 5] for some useful techniques). This method can be applied to a wider range of modules. Furthermore it has the advantage of being characteristic free. We will describe this method in general in §1 and we will give examples in §2.

§1

Throughout this paper we let our matrices act on the right so that the cokernel of the map represented by a matrix is obtained by killing the row space.

Lemma 1.1. *Let (R, m, k) be a regular local ring of dimension n. Let $x_1, \ldots, x_n$ be a minimal set of generators for m. Write x for the column vector $(x_1, \ldots, x_n)^{tr}$. Let M be a finite length R-module and let ℓ be the length of M. Then there exist column n-vectors r_{st} for $1 \le s \le \ell - 1$ and $2 \le t \le \ell$ such that M is isomorphic to the cokernel of*

$$\begin{pmatrix} \mathbf{x} & \mathbf{r}_{12} & \mathbf{r}_{13} & \ldots & \mathbf{r}_{1\ell} \\ & \mathbf{x} & \mathbf{r}_{23} & \ldots & \mathbf{r}_{2\ell} \\ & & \ddots & & \vdots \\ & & & & \mathbf{x} \end{pmatrix}. \tag{1-1}$$

Proof. This lemma is trivial when $\ell = 1$. In general find an element a in M such that $N = Ra \simeq k$. Then by the induction hypothesis on the length of M we may assume that the lemma holds for M/N as well. Now the lemma follows from the following lemma. ∎

Lemma 1.2 is very elementary but we include its proof here for reader's convenience.

Lemma 1.2. *Let R be a ring (not necessarily regular local). Let M be a finitely generated R-module and N be a submodule of M. Suppose M/N is isomorphic to the cokernel of the matrix $A_{s' \times s}$ while N is isomorphic to the cokernel of the matrix $B_{t' \times t}$. Then there exists an $s' \times t$ matrix C such that M is isomorphic to the cokernel of $\left(\begin{smallmatrix} A & C \\ \mathbf{0} & B \end{smallmatrix}\right)$.*

Proof. Find $m_1, \ldots, m_s$ in M such that the sequence

$$R^{s'} \xrightarrow{A} R^s \to M/N \to 0,$$

where $(a_1, \ldots, a_s) \in R^s$ is sent to $\sum_j a_j \overline{m_j}$, is exact. Find as well $n_1, \ldots, n_t$ in N such that the sequence

$$R^{t'} \xrightarrow{B} R^t \to N \to 0,$$

where $(b_1, \ldots, b_t) \in R^t$ is sent to $\sum_k b_k n_k$, is exact. Let $A = (a_{ij})$. Since for each i the sum $\sum_j a_{ij} m_j$ is in N, we can find $c_{ik} \in R$ such that $\sum_j a_{ij} m_j + \sum_k c_{ik} n_k = 0$. Let $C = (c_{ik})$. We claim that C is what we want.

Now map R^{s+t} to M by sending $(a_j, b_k)_{\substack{j=1,\ldots s \\ k=1,\ldots,t}}$ to $\sum_j a_j m_j + \sum_k b_k n_k$. Obviously the image of $\left(\begin{smallmatrix} A & C \\ \mathbf{0} & B \end{smallmatrix}\right)$ is a subset of the kernel of this map. If $(a_j, b_k)_{j,k}$ is inside the kernel of this map, it suffices to check that it is also in the image of $\left(\begin{smallmatrix} A & C \\ \mathbf{0} & B \end{smallmatrix}\right)$. Since $\sum_j a_j \overline{m_j} = 0$, we have $(a_j)_j = \alpha A$ for some $\alpha \in R^{s'}$. Let $\alpha C = (c_k)_k$. Then $\sum_j a_j m_j + \sum_k b_k n_k = \sum_k (-c_k + b_k) n_k = 0$. Thus we can find $\beta \in R^{t'}$ such that $-\alpha C + (b_k)_k = (-c_k + b_k)_k = \beta B$. Note that

$$(\alpha, \beta) \begin{pmatrix} A & C \\ \mathbf{0} & B \end{pmatrix} = (\alpha A, \alpha C + \beta B) = (a_j, b_k)_{j,k}.$$

Our claim is established. ∎

We give a brief description of Koszul complexes; readers are referred to [17] for further details. Let R be any ring. If $\boldsymbol{x} = x_1, \ldots, x_n$ is any sequence of elements of R, the Koszul complex $\mathcal{K}_\bullet(\boldsymbol{x};\ R)$ may be defined in the following fashion. One may identify $\mathcal{K}_1(\boldsymbol{x};\ R)$ with a free module G on n generators u_i, where the differential n maps u_i to x_i in $\mathcal{K}_0(\boldsymbol{x};\ R) = R$, and then the entire complex may be identified with the exterior algebra $\wedge G$, where the map d is extended to $\wedge G$ in the unique way that makes it a *derivation* of degree -1 (so that if $v \in \wedge^i G$ and $w \in \wedge^j G$ are homogeneous elements of $\wedge G$ of respective degrees i and j, then $d(u \wedge v) = (du) \wedge v + (-1)^i u \wedge (dv)$).

To be more specific, let T_i be the set of the i-element subsets $\mathbf{t}$ of $\{1, \ldots, n\}$. Let $\langle \mathbf{t} \rangle$ be the increasing sequence $t_1 < t_2 < \cdots < t_i$ of elements of $\mathbf{t}$. Write $u_{\langle \mathbf{t} \rangle}$ for $u_{t_1} \wedge \cdots \wedge u_{t_i}$ and if $1 \le j \le i$ we write $\langle \mathbf{t} \rangle - j$ for this sequence with its jth term omitted, a sequence of length $i - 1$. With these conventions we have the explicit formula

$$d_i(u_{\langle t \rangle}) = \sum_{j=1}^{i} (-1)^{j-1} x_{t_j} u_{\langle t \rangle - j}.$$

Write u_t for $u_{\langle t\rangle}$ for short. Order T_i according to the lexicographical order of $\langle \mathbf{t}\rangle$. Denote by $\mathbf{A}_i^n(x_1,\dots,x_n)$ the associated matrix of the differential map from $\mathcal{K}_i(\boldsymbol{x};\ R)$ to $\mathcal{K}_{i-1}(\boldsymbol{x};\ R)$ with respect to these bases. For instance, we have

$$\mathbf{A}_2^3(x_1,x_2,x_3)=\begin{pmatrix} -x_2 & x_1 & \\ -x_3 & & x_1 \\ & -x_3 & x_2 \end{pmatrix}$$

In this fashion, it is easy to check that

$$\mathbf{A}_i^n(x_1,\dots,x_n)=\begin{pmatrix} -\mathbf{A}_{i-1}^{n-1}(x_2,\dots,x_n) & x_1\mathbf{I}_{\binom{n-1}{i-1}} \\ \mathbf{0} & \mathbf{A}_i^{n-1}(x_2,\dots,x_n) \end{pmatrix}.$$

For the rest of the paper we will write $\mathbf{x}$ for the column n-vector $(x_1,\dots,x_n)^{tr}$ and we will also write $\mathbf{A}_i^n(\mathbf{x})$ for the matrix $\mathbf{A}_i^n(x_1,\dots,x_n)$.

Let A be any commutative ring. Let $a\in A$ and $\mathbf{x},\mathbf{y}\in A^n$. The Koszul matrices have the following nice properties:

$$\text{(1-2)}\qquad \begin{cases} \mathbf{A}_i^n(a\mathbf{x})=a\mathbf{A}_i^n(\mathbf{x}), \\ \mathbf{A}_i^n(\mathbf{x}+\mathbf{y})=\mathbf{A}_i^n(\mathbf{x})+\mathbf{A}_i^n(\mathbf{y}), \\ \mathbf{A}_i^n(\mathbf{x})\mathbf{A}_{i-1}^n(\mathbf{x})=\mathbf{0}. \end{cases}$$

Now let's come back to the original problem. Conversely, suppose M is isomorphic to the cokernel of a matrix of the form in (1–1), where ℓ is not necessarily the length of M. Sometimes the $\mathbf{r}_{jk}$'s can be carefully chosen so that it satisfies the condition

$$(*)\qquad A_iA_{i-1}=\mathbf{0}\qquad \text{for } 2\le i\le n,$$

where

$$\text{(1-3)}\qquad A_i=\begin{pmatrix} \mathbf{A}_i^n(\mathbf{x}) & \mathbf{A}_i^n(\mathbf{r}_{12}) & \mathbf{A}_i^n(\mathbf{r}_{13}) & \cdots & \mathbf{A}_i^n(\mathbf{r}_{1\ell}) \\ & \mathbf{A}_i^n(\mathbf{x}) & \mathbf{A}_i^n(\mathbf{r}_{23}) & \cdots & \mathbf{A}_i^n(\mathbf{r}_{2\ell}) \\ & & \ddots & & \vdots \\ & & & & \mathbf{A}_i^n(\mathbf{x}) \end{pmatrix}.$$

This condition may look scary, but thanks to Lemma 1.4 and the properties in (1–2) it is often quite easy to check and holds in quite a few interesting cases, for example, like the modules of exponent two or of essentially monomial type. The following two lemmas can be used to ease the pain of checking the condition $(*)$.

Lemma 1.3. *Checking the condition* $(*)$ *is equivalent to checking*

$$\sum_{\substack{2\le r\le \ell-1 \\ r\neq s,t}} \mathbf{A}_i^n(\mathbf{r}_{sr})\mathbf{A}_{i-1}^n(\mathbf{r}_{rt})=\mathbf{0}$$

for each s, t. Here we use the convention that $\mathbf{r}_{st} = \mathbf{0}$ for $s > t$.

Proof. We can think of the A_i's as having block entries of the form $\mathbf{A}_i^n(\mathbf{x})$, $\mathbf{A}_i^n(\mathbf{r}_{st})$ or 0. When $s \neq t$, the (s,t)-th block entry of $A_i A_{i-1}$ is

$$\mathbf{A}_i^n(\mathbf{x})\mathbf{A}_{i-1}^n(\mathbf{r}_{st}) + \mathbf{A}_i^n(\mathbf{r}_{st})\mathbf{A}_{i-1}^n(\mathbf{x}) + \sum_{\substack{2 \le r \le \ell-1 \\ r \neq s,t}} \mathbf{A}_i^n(\mathbf{r}_{sr})\mathbf{A}_{i-1}^n(\mathbf{r}_{rt}).$$

When $s = t$, the (s,t)-th block entry of $A_i A_{i-1}$ is

$$\mathbf{A}_i^n(\mathbf{x})\mathbf{A}_{i-1}^n(\mathbf{x}) + \sum_{\substack{2 \le r \le \ell-1 \\ r \neq s,t}} \mathbf{A}_i^n(\mathbf{r}_{sr})\mathbf{A}_{i-1}^n(\mathbf{r}_{rt}).$$

Now the assertion follows from the next lemma. ∎

Lemma 1.4. *Let $\boldsymbol{X} = X_1, \dots, X_n$ and $\boldsymbol{Y} = Y_1, \dots, Y_n$ be two sets of indeterminates. Then*

$$\mathbf{A}_{i+1}^n(\boldsymbol{X})\mathbf{A}_i^n(\boldsymbol{Y}) + \mathbf{A}_{i+1}^n(\boldsymbol{Y})\mathbf{A}_i^n(\boldsymbol{X}) = \mathbf{0}.$$

In particular, when $\boldsymbol{X}$ and $\boldsymbol{Y}$ are specialized by elements in a given ring this equation still holds.

Proof. See [5, Lemma 2.2]. ∎

Once the condition $(*)$ holds it makes

$$K_\bullet = (\cdots \to R^{\binom{n}{i}\ell} \xrightarrow{A_i} R^{\binom{n}{i-1}\ell} \to \cdots) \tag{1-4}$$

into a complex. It can be shown using Buchsbaum-Eisenbud's acyclicity criterion that $K_\bullet$ is a free resolution for the module M when R is a regular local ring and $x_1, \dots, x_n$ form a minimal generating set for m. In fact we don't need such a strong condition, thanks to Proposition 1.5.

Proposition 1.5. *Let (R, m, k) be a local ring and let $x_1, \dots, x_n$ form an R-regular sequence. Let M be the cokernel of the matrix in $(1-1)$. (We do not require ℓ to be the length of M.) Let S be the subring of R generated by the entries of the $\mathbf{r}_{st}$'s. Suppose that S is a domain and that the condition $(*)$ holds for this matrix. Then the complex $K_\bullet$ in $(1-4)$ is a free resolution for M.*

Proof. We need to show that $K_\bullet$ is acyclic. Remember that the rank of $\mathbf{A}_i^n(\mathbf{x})$ is $\binom{n-1}{i-1}$ and the rank ideal of $\mathbf{A}_i^n(\mathbf{x})$ contains $x_k^{\binom{n-1}{i-1}}$ for each k.

Observe the position of $\mathbf{A}_i^n(\mathbf{x})$ in the matrix $(1-3)$. We can see that the rank of A_i is at least $\binom{n-1}{i-1}\ell$ and the ideal generated by its $\binom{n-1}{i-1}\ell$-minors contains $x_k^{\binom{n-1}{i-1}\ell}$ for each k. Now replace the x_k's by distinct indeterminates. Then the A_i's have entries inside $S[x_1,\ldots,x_n]$, which is a domain. Since in this case we still have $A_iA_{i-1}=0$ by Lemma 1.3, we should have $rkA_i + rkA_{i-1} \le \binom{n}{i-1}\ell$. Since the rank of A_i cannot increase if $\boldsymbol{x}$ are specialized by any sequence of elements in R, we have

$$\mathrm{rk}A_i = \binom{n-1}{i-1}\ell$$

even if we drop the condition that $\boldsymbol{x}$ are distinct indeterminates. It is now obvious that the rank ideal of A_i contains $x_k^{\binom{n-1}{i-1}\ell}$ for each k. Thus the depth of the rank ideal of $A_i = n \ge i$ for $1 \le i \le n$. By Eisenbud-Buchsbaum acyclicity criterion, the complex $K_\bullet$ is acyclic. ■

Remark. The condition of S being a domain can be loosened slightly. Examining the proof one can see that all we need is that $rkA_i \le \binom{n-1}{i-1}\ell$, or equivalently that $rkA_i + rkA_{i-1} \le \binom{n}{i-1}\ell$.

It follows that

$$r_i^R(M) = \binom{n-1}{i-1}\ell - \mathrm{rk}\overline{A}_i,$$

where $\overline{A}_i$ is the image of A_i modulo m. Thus the problem of finding $r_i^R(M)$ is reduced to computing the rank of $\overline{A}_i$.

§2

As we have said, the method here can be applied to situations other than what we have studied as long as it satisfies the condition $(*)$. We will supply an example here. The reader is encouraged to try out the method case-by-case since our method applies in a larger context.

Definition 2.1. Let $u_1,\ u_2,\ldots$ be distinct indeterminates and let Λ be the set of monomials in these indeterminates. Let $\boldsymbol{u} = u_1,\ldots,u_n$. Let $f,\ g \in \Lambda$. We say f *$\boldsymbol{u}$-divides* g, denoted $f|_{\mathbf{u}}g$, if g/f is a monomial solely in elements of $\boldsymbol{u}$. We use $f|_{\mathbf{u}}g|_{\mathbf{u}}h$ to denote the situation f $\boldsymbol{u}$-divides g and g $\boldsymbol{u}$-divides h. A finite subset T of Λ is called a u-*segment* if every $g \in \Lambda$ with $f|_u g|_u f'$ for some $f,\ f' \in T$ is in T. We say an element g is related to f if $g = u_k f$ for some $k = 1,\ldots,n$. We say an element g is secondly related to f if $g = u_k u_l f$ for some $k, l = 1,\ldots,n$.

Let $\boldsymbol{x} = x_1, \ldots, x_n$ be an R-regular sequence and let $\boldsymbol{u} = u_1, \ldots, u_n$. Let T be a $\boldsymbol{u}$-segment and T_0 be the subset of elements f in T such that $u_k f \notin T$ for all $k = 1, \ldots, n$. Suppose we are given a subset S of T and $\alpha_{fg} \in R$ with $f \in S$ and $g \in T_0$. If M is isomorphic to the quotient module of the free module on the generators $\{w_f\}_{f \in T}$ modulo the submodule N generated by

$$\{x_k w_f\}_{\substack{k=1,\ldots,n \\ f \in T \\ u_k f \notin T}} \cup \{x_k w_f - w_{u_k f}\}_{\substack{k=1,\ldots,n \\ f \in T \\ u_k f \in T}}$$
$$\cup \{w_f + \sum_{g \in T_0} \alpha_{fg} w_g\}_{f \in S \setminus T_0} \cup \{w_f + \sum_{g \in T_0 \setminus S} \alpha_{fg} w_g\}_{f \in T_0 \cap S},$$

we say M is *manageable with respect to* $\boldsymbol{x}$.

We may replace the generator $w_f + \sum_{g \in T_0} \alpha_{fg} w_g$ for $f \in S \setminus T_0$ by $w_f + \sum_{g \in T_0} \alpha_{fg} w_g - \sum_{h \in T_0 \cap S} \alpha_{fh}(w_h + \sum_{g \in T_0 \setminus S} \alpha_{hg} w_g)$. Thus by using the new set of generators we may assume that N is generated by a set of the form

$$\{x_k w_f\}_{\substack{k=1,\ldots,n \\ f \in T \\ u_k f \notin T}} \cup \{x_k w_f - w_{u_k f}\}_{\substack{k=1,\ldots,n \\ f \in T \\ u_k f \in T}} \cup \{w_f + \sum_{g \in T_0 \setminus S} \alpha_{fg} w_g\}_{f \in S}.$$

Let $f \in S$. Suppose $u_k f \in T$ for some k. Then $w_{u_k f} = (w_{u_k f} - x_k w_f) + x_k(w_f + \sum_{g \in T_0} \alpha_{fg} w_g) - \sum_{g \in T_0} \alpha_{fg} x_k w_g \in N$. Thus we may drop all such generators (it won't affect T being a $\boldsymbol{u}$-segment) and rephrase the manageable module M with respect to $\boldsymbol{x}$ as follows. Let T be a $\boldsymbol{u}$-segment and let T_0 be the subset of elements f in T such that $u_k f \notin T$ for all $k = 1, \ldots, n$. Let S and T_{socle} be disjoint subsets of T_0 where T_{socle} is nonempty. Then M is manageable with respect to x if it is isomorphic to the quotient module of the free module on the free generators $\{w_f\}_{f \in T}$ modulo the submodule generated by

$$\{x_k w_f\}_{\substack{k=1,\ldots,n \\ f \in T \\ u_k f \notin T}} \cup \{x_k w_f - w_{u_k f}\}_{\substack{k=1,\ldots,n \\ f \in T \\ u_k f \in T}} \cup \{w_f + \sum_{g \in T_{\mathrm{socle}}} \alpha_{fg} w_g\}_{f \in S}$$

for some $\alpha_{fg} \in R$ where $f \in S$ and $g \in T_{socle}$. Now replace the generators $\{w_f\}_{f \in T}$ of F by the new set of generators

$$\{w_f\}_{f \in T \setminus S} \cup \{w'_f = w_f + \sum_{g \in T_{\mathrm{socle}}} \alpha_{fg} w_g\}_{f \in S}.$$

The module M becomes the quotient module of F on the aforementioned set of generators modulo the submodule N generated by

$$\{x_k w_f\}_{\substack{k=1,\ldots,n \\ f \in T \\ u_k f \notin T}} \cup \{x_k w_f - w_{u_k f}\}_{\substack{k=1,\ldots,n \\ f \in T \\ u_k f \in T \setminus S}}$$
$$\cup \{x_k w_f - w'_{u_k f} + \sum_{g \in T_{\mathrm{socle}}} \alpha_{u_k f, g} w_g\}_{\substack{k=1,\ldots,n \\ f \in T \\ u_k f \in S}} \cup \{w'_f\}_{f \in S}.$$

Again, dropping the generators w'_f for $f \in S$ we can rephrase M in the following manner.

Suppose we are given a $\boldsymbol{u}$-segment T. Let T and S be disjoint finite subsets of Λ such that the set T_{socle} of $f \in T$ with $u_k f \notin T \cup S$ for all $k = 1, \dots, n$ is not empty. Then M is manageable with respect to $\boldsymbol{x}$ if M is isomorphic to the quotient module of the free module on the generators $\{w_f\}_{f \in T}$ modulo the submodule generated by

$$\{x_k w_f\}_{\substack{k=1,\dots,n\\ f\in T\\ u_k f \notin T\cup S}} \cup \{x_k w_f - w_{u_k f}\}_{\substack{k=1,\dots,n\\ f\in T\\ u_k f\in T}} \cup \{x_k w_f + \sum_{g \in T_{\text{socle}}} \vartheta u_k f, g w_g\}_{\substack{k=1,\dots,n\\ f\in T\\ u_k f\in S}}.$$

Order the elements of T so that the elements of $T \setminus T_{\text{socle}}$ and T_{socle} are ordered lexicographically and that the elements of $T \setminus T_{\text{socle}}$ precede those of T_{socle}. With respect to this order we can write M as the cokernel of a certain matrix of the form (1–1).

We claim that the condition $(*)$ holds if $|S| = 1$.

Think of the matrix as having its column and row blocks indexed by the elements of T in the given order. Let $S = \{f\}$. Let $f, g \in T$. Then

$$\mathbf{r}_{fg} = \begin{cases} -\mathbf{e}_k, & \text{if } g = u_k f \in T \text{ and either f is unrelated to } f \\ & \text{or } g \notin T_{\text{socle}}, \\ -\mathbf{e}_l + \alpha_{fg}\mathbf{e}_k, & \text{if f} = u_k f \textit{ and } g = u_l f \in T_{\text{socle}}, \\ \alpha_{fg}\mathbf{e}_k, & \text{if f} = u_k f, g \in T_{\text{socle}} \text{ and } g \text{ is unrelated to } f, \\ \mathbf{0}, & \text{otherwise.} \end{cases}$$

By Lemma 1.3, to check the condition $(*)$ it suffices to check that whether

$$\sum_{h \neq f, g} \mathbf{A}_i^n(\mathbf{r}_{fh})\mathbf{A}_{i-1}^n(\mathbf{r}_{hg}) \tag{2-1}$$

is $\mathbf{0}$. Let $f, g \in T$. We divide the problem into the following cases.

Case 1. Suppose f is not related or secondly related to f. Then the summation in (2–1) becomes

$$\begin{aligned} &\sum_{u_k f \in T} \mathbf{A}_i^n(\mathbf{r}_{f, u_k f})\mathbf{A}_{i-1}^n(\mathbf{r}_{u_k f, g}) \\ &= \sum_{u_k f \in T} \mathbf{A}_i^n(-\boldsymbol{e}_k)\mathbf{A}_{i-1}^n(\mathbf{r}_{u_k f, g}). \end{aligned}$$

Note that $\mathbf{r}_{u_k f, g} = \mathbf{0}$ unless $g = u_k u_l f$ for some $l = 1, \dots, n$. Thus to check the summation in (2–1) equals zero we only need to check the case when $g = u_k u_l f$ for some k, l, and in which case it becomes

$$\begin{aligned} &\mathbf{A}_i^n(\mathbf{r}_{f, u_k f})\mathbf{A}_{i-1}^n(\mathbf{r}_{u_k f, u_k u_l f}) + \mathbf{A}_i^n(\mathbf{r}_{f, u_l f})\mathbf{A}_{i-1}^n(\mathbf{r}_{u_l f, u_k u_l f}) \\ &= \mathbf{A}_i^n(-\mathbf{e}_j)\mathbf{A}_{i-1}^n(-\mathbf{e}_k) + \mathbf{A}_i^n(-\mathbf{e}_k)\mathbf{A}_{i-1}^n(-\mathbf{e}_j) = \mathbf{0} \end{aligned}$$

by Lemma 1.4.

Case 2. suppose $u_k f = f$. Note that $\mathbf{r}_{hg} = \mathbf{0}$ for all $h \in T_{\text{socle}}$. Note as well that f is unrelated to $u_j f$ for all j. Thus the summation in (2–1) becomes

$$\begin{aligned}&\sum_{u_j f \in T} \mathbf{A}_i^n(\mathbf{r}_{f,u_j f})\mathbf{A}_{i-1}^n(\mathbf{r}_{u_j f,g}) + \sum_{h \in T_{\text{socle}}} \mathbf{A}_i^n(\mathbf{r}_{fh})\mathbf{A}_{i-1}^n(\mathbf{r}_{hg})\\ &= \sum_{u_j f \in T} \mathbf{A}_i^n(-\mathbf{e}_j)\mathbf{A}_{i-1}^n(\mathbf{r}_{u_j f,g}) = \mathbf{0}\end{aligned}$$

by the same argument as in Case 1.

Case 3. Suppose $f = u_k u_l f$. In this case the summation in (2–1) equals

$$\begin{aligned}&\sum_{u_j f \in T} \mathbf{A}_i^n(\mathbf{r}_{f,u_j f})\mathbf{A}_{i-1}^n(\mathbf{r}_{u_j f,g})\\ &= \sum_{u_j f \in T} \mathbf{A}_i^n(-\mathbf{e}_j)\mathbf{A}_{i-1}^n(\mathbf{r}_{u_j f,g}).\end{aligned}$$

This summation is $\mathbf{0}$ if $g \notin T_{\text{socle}}$ using the same argument as in Case 1. Now assume $g \in T_{\text{socle}}$. If g is not secondly related to f, then the summation in (2–1) equals

$$\begin{aligned}&\mathbf{A}_i^n(\mathbf{r}_{f,u_k f})\mathbf{A}_{i-1}^n(\mathbf{r}_{u_k f,g}) + \mathbf{A}_i^n(\mathbf{r}_{f,u_l f})\mathbf{A}_{i-1}^n(\mathbf{r}_{u_l f,g})\\ &= \mathbf{A}_i^n(-\mathbf{e}_k)\mathbf{A}_{i-1}^n(\alpha_{f,g}\mathbf{e}_l) + \mathbf{A}_i^n(-\mathbf{e}_l)\mathbf{A}_{i-1}^n(\alpha_{f,g}\mathbf{e}_k) = \mathbf{0}\end{aligned}$$

if $k \neq l$, or

$$\begin{aligned}&\mathbf{A}_i^n(\mathbf{r}_{f,u_k f})\mathbf{A}_{i-1}^n(\mathbf{r}_{u_k f,g})\\ &= \mathbf{A}_i^n(-\mathbf{e}_k)\mathbf{A}_{i-1}^n(\alpha_{fg}\mathbf{e}_k) = \mathbf{0}\end{aligned}$$

if $k = l$.

Now assume that $g = u_{k'} u_{l'} f$. Note that $k \neq k'$ or $l \neq l'$. There are actually many combinations to verify, but they are all straightforward and routine. We will demonstrate by a couple of examples and leave the rest for readers to check. When $k = k'$ but k, l, l' are distinct, the summation becomes

$$\begin{aligned}&\mathbf{A}_i^n(\mathbf{r}_{f,u_k f})\mathbf{A}_{i-1}^n(\mathbf{r}_{u_k f,g}) + \mathbf{A}_i^n(\mathbf{r}_{f,u_l f})\mathbf{A}_{i-1}^n(\mathbf{r}_{u_l f,g}) + \mathbf{A}_i^n(\mathbf{r}_{f,u_{l'} f})\mathbf{A}_{i-1}^n(\mathbf{r}_{u_{l'} f,g})\\ &= \mathbf{A}_i^n(-\mathbf{e}_k)\mathbf{A}_{i-1}^n(-\mathbf{e}_{l'} + \alpha_{fg}\mathbf{e}_l) + \mathbf{A}_i^n(-\mathbf{e}_l)\mathbf{A}_{i-1}^n(\alpha_{fg}\mathbf{e}_k) + \mathbf{A}_i^n(-\mathbf{e}_{l'})\mathbf{A}_{i-1}^n(-\mathbf{e}_k)\\ &= \mathbf{0}.\end{aligned}$$

When $k = l$ but k, k' and l' are distinct, the summation equals

$$\begin{aligned}&\mathbf{A}_i^n(\mathbf{r}_{f,u_k f})\mathbf{A}_{i-1}^n(\mathbf{r}_{u_k f,g}) + \mathbf{A}_i^n(\mathbf{r}_{f,u_{k'} f})\mathbf{A}_{i-1}^n(\mathbf{r}_{u_{k'} f,g}) + \mathbf{A}_i^n(\mathbf{r}_{f,u_{l'} f})\mathbf{A}_{i-1}^n(\mathbf{r}_{u_{l'} f,g})\\ &= \mathbf{A}_i^n(-\mathbf{e}_k)\mathbf{A}_{i-1}^n(\alpha_{fg}\mathbf{e}_k) + \mathbf{A}_i^n(-\mathbf{e}_{k'})\mathbf{A}_{i-1}^n(-\mathbf{e}_{l'}) + \mathbf{A}_i^n(-\mathbf{e}_{l'})\mathbf{A}_{i-1}^n(-\mathbf{e}_{k'})\\ &= \mathbf{0}.\end{aligned}$$

Observe that our argument is independent of the choice of the α_{fg}'s. If we replace the α_{fg}'s and the x_k's by distinct indeterminates, the condition $(*)$ will continue to hold. Since in this case A_i is a matrix over the domain $\mathbb{Z}[\alpha_{fg}, x_k]_{\substack{f,g\in T\\ 1\leq k\leq n}}$, we have that $rkA_i + rkA_{i-1} \leq \binom{n}{i-1}|T|$. This remains true for our original choice of α_{fg}'s and x_k's since the rank of the A_i's cannot increase. By the remark to Proposition 1.5, the complex $K_\bullet$ is a free resolution of M.

We will use the discussion above to compute an actual example.

Example 2.2. Let $\boldsymbol{x} = x_1, \ldots, x_n$ be a minimal set of generators for the maximal ideal m of the regular local ring R. Let $M = R/m^{N+1} + (\sum_1^n x_k^N)$. We want to use the discussion above to make a rough estimation for $r_i(M)$ to verify Horrocks' conjecture.

When $N = 1$, by a change of generators for m the module M becomes $R/m^2 + (x_1)$, which is taken care of in [5]. We may assume $N \geq 2$. We will also assume that $n \geq 5$ and $i \geq 2$ since Horrocks' conjecture is fairly well-known up to $n = 4$ and $i = 1$.

Let $\boldsymbol{u} = u_1, \ldots, u_n$ and let $\mathcal{T} = \{f : f$ is a monomial in $\boldsymbol{u}$ and $degf \leq N\}$. According to the results in [5], the module R/m^{N+1} is isomorphic to the quotient module of the free module on the free generators $\{w_f\}_{f\in\mathcal{T}}$ modulo the submodule generated by

$$\{x_k w_f\}_{\substack{f\in\mathcal{T}\\ \deg f = N}} \cup \{x_k w_f - w_{u_k f}\}_{\substack{f\in\mathcal{T}\\ \deg f < N}}.$$

In fact, the equivalence class of w_f is identified with the class of $f(\boldsymbol{x})$ in R/m^{N+1}. Thus M is isomorphic to the quotient module of the free module on the free generators $\{w_f\}_{f\in\mathcal{T}}$ modulo the submodule generated by

$$\{x_k w_f\}_{\substack{f\in\mathcal{T}\\ degf = N}} \cup \{x_k w_f - w_{u_k f}\}_{\substack{f\in\mathcal{T}\\ degf < N}} \cup \{\sum_{k=1}^{n} w_{u_k^N}\}.$$

Let $T = \mathcal{T}\setminus\{u_1^N\}$. From the discussion above we have that M is isomorphic to the quotient module of the free module on the free generators $\{w_f\}_{f\in T}$ modulo the submodule generated by

$$\{x_k w_f\}_{\substack{f\in T\\ \deg f = N}} \cup \{x_k w_f - w_{u_k f}\}_{\substack{f\in T\\ \deg f < N}} \cup \{x_1 w_{u_1^{N-1}} + \sum_{2}^{n} w_{u_k^N}\}.$$

Thus M is isomorphic to the cokernel of the matrix of the form (1–1) with

$$\mathbf{r}_{fg} = \begin{cases} -\mathbf{e}_k, & \text{if } g = u_k f \in T, \\ \mathbf{e}_1, & \text{if } f = u_1^{N-1} \text{ and } g = u_k^N \text{ for some } k \neq 1, \\ \mathbf{0}, & \text{otherwise.} \end{cases}$$

Let $A_i = (\mathbf{A}_i^n(\mathbf{r}_{fg}))_{f,g\in T}$. For $f, g \in T$, let

$$\mathbf{r}'_{fg} = \begin{cases} \mathbf{e}_k, & \text{if } g = u_k f \in T, \\ \mathbf{0}, & \text{otherwise,} \end{cases}$$

and let $A'_i = (\mathbf{A}_i^n(\mathbf{r}'_{fg}))_{f,g\in T}$. Let $L = R/m^{N+1} + (x_1^N)$. By the results in [5], we have

$$r_i(L) = \binom{n-1}{i-1}|T| - \mathrm{rk}\overline{A}'_i.$$

Note that

$$\overline{A}_i = \overline{A}'_i + \begin{pmatrix} \cdots\cdots\cdots & \mathbf{0} & \cdots\cdots\cdots\cdots\cdots\cdots \\ \mathbf{0} \ \cdots \ \mathbf{0} & \mathbf{A}_i^n(\mathbf{e}_1) & \cdots \ \mathbf{A}_i^n(\mathbf{e}_1) \ \cdots \\ \cdots\cdots\cdots & \mathbf{0} & \cdots\cdots\cdots\cdots\cdots\cdots \end{pmatrix}.$$

Hence $\mathrm{rk}\overline{A}_i \le \mathrm{rk}\overline{A}'_i + \binom{n-1}{i-1}$. Thus

$$\begin{aligned} r_i(M) &= \binom{n-1}{i-1}|T| - \mathrm{rk}\overline{A}_i \\ &\ge \binom{n-1}{i-1}|T| - \mathrm{rk}\overline{A}'_i - \binom{n-1}{i-1} \\ &= r_i(L) - \binom{n-1}{i-1}. \end{aligned}$$

Using the method in [5] we may find $r_i(L)$. Let $T' = \{f \in T : \deg f = N\} \cup \{u_1^{N-1}\}$ and $T'' = \{f \in T : \deg f \le N,\ u_1 \nmid f\}$. Let $\bar{\boldsymbol{x}} = x_2, \ldots, x_n$ and let $\bar{\boldsymbol{u}} = u_2, \ldots, u_n$. Let L' and L'' be the modules associated with T' and T'' with respect to $(\bar{\boldsymbol{u}}, \bar{\boldsymbol{x}})$. The Main Theorem [5, Theorem 0.2] tells us that $r_i(L) = r_{i-1}(L') + r_i(L'')$. Let $J = (x_2, \ldots, x_n)$. Note that L' is actually $R/J^2 \oplus \left(\binom{N+n-1}{N} - n\right)$ copies of R/J while $L'' \simeq R/J^{N+1}$. Hence

$$\begin{aligned} r_{i-1}(L') &= r_{i-1}(R/J^2) + \left(\binom{N+n-1}{N} - n\right)\binom{n-2}{i-2} \\ &\ge \binom{n-2}{i-2} + \left(\binom{n+1}{2} - n\right)\binom{n-2}{i-2} \ge 2\binom{n-2}{i-2} \end{aligned}$$

for $N \ge 2$ and $n \ge 2$. We claim that we have $r_i(L'') \ge 2\binom{n-2}{i-1}$ as well. According to [5, Example 1.2], we have that $r_i(R/(x_{n-i+1}, \ldots, x_n)^{N+1}) = \binom{i+N-1}{N} \ge 2 = 2\binom{i-1}{i-1}$ for $i \ge 2$. Let $K = (x_3, \ldots, x_n)$. According to [5, Example 1.2], we have

$$\begin{aligned} r_i(L'') &= \binom{n+N-2}{N} r_{i-1}(R/K) + r_i(R/K^{N+1}) \\ &\geq 2\binom{n-3}{i-2} + 2\binom{n-3}{i-1}, \qquad \text{by the induction hypothesis,} \\ &= 2\binom{n-2}{i-1} \end{aligned}$$

for $N \geq 1$ and $n \geq 3$. Thus $r_i(L) \geq 2\binom{n-1}{i-1}$ and $r_i(M) \geq \binom{n-1}{i-1}$.

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Dept. of Mathematics, National Chung Cheng University
Minghsiung, Chiayi 621, Taiwan, R.O.C.
E-mail: stchang@math.ccu.edu.tw