

## A SURVEY OF THE COMPRESSIBLE NAVIER-STOKES EQUATIONS

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**Abstract.** This paper presents mathematical properties of solutions to the Navier-Stokes equations for compressible fluids. We first review existence results for the Cauchy problem, and describe some regularity properties of solutions in the presence of possibly vanishing densities. Finally, we address the problem of the low Mach number limit leading to incompressible models.

### 1. INTRODUCTION

Let us consider a gas of density  $\rho \geq 0$ , temperature  $T$ , velocity field  $\mathbf{u}$ , pressure  $p(\rho, T)$ , internal energy density  $e(\rho, T)$  and entropy density  $s(\rho, T)$  contained in a domain  $\Omega$ , which denotes either a bounded, unbounded, or periodic domain of  $\mathbf{R}^N$  ( $N \geq 1$ ). The compressible isentropic Navier-Stokes equations in  $\mathcal{D}'(\mathbf{R}_+ \times \Omega)$  read as follows

$$(1.1) \quad \partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0,$$

$$(1.2) \quad \partial_t(\rho \mathbf{u}) + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) = \rho \mathbf{f} + \operatorname{div} \Sigma,$$

$$(1.3) \quad \partial_t \left( \rho \left( e + \frac{|\mathbf{u}|^2}{2} \right) \right) + \operatorname{div} \left( \rho \mathbf{u} \left( e + \frac{|\mathbf{u}|^2}{2} \right) \right) = \rho \mathbf{u} \cdot \mathbf{f} + \operatorname{div}(\Sigma \cdot \mathbf{u} + k \nabla T),$$

where  $\Sigma = 2\mu D(\mathbf{u}) + (\lambda(\operatorname{div} \mathbf{u}) - p)I$  is the internal stress tensor,  $\mathbf{f}$  the external forces and  $k \geq 0$  the thermal conduction parameter. The viscosity coefficients  $\lambda, \mu$  are assumed to satisfy  $\mu > 0$  and  $\lambda + 2\mu > 0$  which in particular covers

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the case of Stokes law  $N\lambda + 2\mu = 0$ . The strain tensor  $D(\mathbf{u})$  is defined as the symmetric part of the gradient of the velocity  $\mathbf{u}$ :

$$D(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u} + {}^t\nabla\mathbf{u}).$$

When  $\Omega$  is a domain of  $\mathbf{R}^N$ , we enforce the boundary conditions  $\mathbf{u}|_{\partial\Omega} = 0$  and  $\frac{\partial T}{\partial n}|_{\partial\Omega} = 0$ . Even though similar results are available for more general state laws, we will focus in the sequel on the ideal gas case corresponding to  $e = C_v T$ ,  $p = \rho R T$  for some constants  $C_v > 0$ ,  $R > 0$ , and define  $\gamma = 1 + R/C_v$ . Then, equation (1.3) can be replaced by an equation on the entropy  $s = C_v \log(T/\rho^{\gamma-1})$ :

$$(1.4) \quad \begin{aligned} & \partial_t(\rho s) + \operatorname{div}(\rho \mathbf{u} s) \\ &= \operatorname{div}\left(k \frac{\nabla T}{T}\right) + k \frac{|\nabla T|^2}{T^2} + \frac{2\mu D(\mathbf{u}) : D(\mathbf{u}) + \lambda(\operatorname{div} \mathbf{u})^2}{T}. \end{aligned}$$

Here  $A : B$  denotes the matrix product,

$$A : B \equiv \operatorname{tr}(A^t B) = \sum_{i,j} a_{ij} b_{ij}.$$

Finally, the above system (1.1) (1.2) (1.4) is supplemented with initial conditions

$$\rho|_{t=0} = \rho_0 \geq 0, \quad \rho \mathbf{u}|_{t=0} = \mathbf{m}_0, \quad \text{and} \quad \rho s|_{t=0} = S_0.$$

A particular model which was studied by many authors over the past few years is the so-called barotropic ( $p$  only depends on  $\rho$ ) or isentropic model, obtained by assuming that the volumetric entropy  $s$  remains constant as the time evolves. This simplification consists first in neglecting the quadratic term involving the velocity  $\mathbf{u}$  at the right-hand side of (1.4), and setting  $k = 0$ , so that the entropy is purely transported by the fluid

$$(1.5) \quad \partial_t(\rho s) + \operatorname{div}(\rho \mathbf{u} s) = 0,$$

and assuming next that the initial entropy  $s_0 = S_0/\rho_0$  is constant. The system (1.1) (1.2) then reduces to

$$(1.1) \quad \partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0,$$

$$(1.6) \quad \partial_t(\rho \mathbf{u}) + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) = \mu \Delta \mathbf{u} + \nabla(\xi \operatorname{div} \mathbf{u}) - \nabla p(\rho) + \rho \mathbf{f},$$

where  $\xi = \lambda + \mu$  and  $p(\rho) = a\rho^\gamma$  for some positive constant  $a$ . Because the above compressible model is of mixed type, parabolic in  $\mathbf{u}$  and essentially

hyperbolic in  $\rho$ , most of the global-in-time existence results restrict to initial data close to equilibrium, which means that  $\rho_0$  and  $T_0$  are taken close to some positive constants  $\bar{\rho}$  and  $\bar{T}$ , and  $\mathbf{u}_0$  chosen small enough. This allows to derive global-in-time positive lower bounds for the density and parabolic estimates for the velocity  $\mathbf{u}$ . The one-dimensional problem was addressed in the late '70s by Kazhikhov and Shelukhin [19] for smooth enough data, and in the mid '80s by Serre [40] [41] and Hoff [13] for discontinuous initial data, but still bounded away from zero. A similar approach for the multidimensional problem (1.1) (1.2) (1.3) was given in the early '80s by Matsumura and Nishida [34] [35], who proved global existence of smooth solutions for data close to equilibrium, and later by Hoff in the '90s for discontinuous initial data [14] [15] [16]. As far as existence issues are concerned, the major breakthrough is due to P.L. Lions [28] [29] [31] in 1993. The only restriction on initial data is that the initial energy is finite, so that the density is allowed to vanish. He obtains global existence of weak solutions - defined as solutions with finite energy - when the exponent  $\gamma$  is large enough. However, this theorem only applies to the isentropic case (1.1) (1.6), or to the slightly more general model (1.1) (1.2) (1.5) consisting in a pure transport of entropy. Indeed, the existence of global weak solutions to the full Navier-Stokes equations (1.1) (1.2) (1.3) is still open.

The regularity and uniqueness of solutions is also an important issue, when no smallness assumption is made on the initial data. In the isentropic case, the local-in-time problem was first analyzed by Solonnikov in the mid '70s [42]. The question whether the formation of vacuum in some regions, and whether concentration phenomena on the density play a role in the formation of singularity is still widely open. In some two-dimensional simplified models, global well-posedness results are available since the work of Kazhikhov [20] [21]. However, a very interesting spherically symmetric counterexample by Weigant [46] gives some evidence for blow-up of smooth solutions in finite time in any dimension, at least if nonzero external forces are considered. Without external forcing, the long time behavior of smooth solutions is an open problem. In dimension two, partial results obtained by Desjardins in [6] show that the maximal norm of the density controls the breakdown of weak solutions, even if vacuum forms in the fluid. In a recent paper [47], Xin considers the full Navier-Stokes equations (1.1) (1.2) (1.3) and proves that in general, local smooth solutions with compactly supported densities do not persist for all times, which gives an additional evidence for finite time blow-up.

As usual in fluid mechanics problems, a dimensional analysis on the physical quantities in the Navier-Stokes equations gives rise to dimensionless parameters such as the Mach number, Reynolds number, Prandtl number, Peclet

number, etc. One of the fundamental questions of fluid dynamics (e.g. the Navier-Stokes equations and the Euler's equations) is that of dependence on the physical parameters. Typically, problems in this class involve a *singular limit* as such a parameter tends to some value. The incompressible limit of the compressible Navier-Stokes equations is a physical problem involving dissipation when such a singular limiting process is interesting. Here are two of the most important questions of this nature:

- (a) Are the incompressible equations the limit of the slightly compressible equations?
- (b) Are the Euler's equations the inviscid limit of the Navier-Stokes equations?

The first problem is a classical one: for inviscid flow, it was addressed in the early '80s by Klainerman–Majda [22] [23] and Ebin [9], later by Ukai [44] and recently by Beirão da Veiga [1] [2] [3] for both inviscid and viscous flows; these authors were working with classical (smooth) – and therefore local-in-time – solutions. For viscous flows, the incompressible Navier-Stokes equations have global weak solutions constructed by Leray in 1934; generically it remains unknown whether a Leray solution remains regular or is unique. Therefore, the methods used by Klainerman, Majda, Ebin, Ukai and Beirão da Veiga do not apply in such cases. Quite recently, P. L. Lions [28] announced the global existence of weak solutions to the isentropic compressible Navier-Stokes equations and its incompressible limit [29] [32] in 1993. The rigorous proof of the barotropic compressible Navier-Stokes equation (time discretized case) and its incompressible limit is given by C.-K. Lin [25].

Besides including the molecular viscosity self-consistently, the role of entropy in an ideal fluid was also considered by C.-K. Lin in 1997 [26] [27] and recently by P. L. Lions [32] and B. Desjardins and C.-K. Lin [7]. For inviscid fluid it had been studied by Schochet in 1986 [37]. With the inclusion of entropy it is found that two distinct routes to incompressibility are possible, distinguished according to the relative magnitudes of the entropy fluctuation. This leads to two distinct models for pure transport of entropy, nearly incompressible polytropic compressible Navier-Stokes equations. When entropy variations are extremely small the low Mach number limit of the compressible model is the incompressible Navier-Stokes equations. However, when entropy variations are large, the singular limit system for the compressible Navier-Stokes equations as the Mach number vanishes is the nonhomogeneous Navier-Stokes equations.

Regarding to the second question for the incompressible model in the absence of boundaries, the answer is **yes** for short times (Kato, 1972 [17]) or at least as long as the Euler solution is smooth (Constantin, 1986; Constantin

and Foias 1988 [4]). In the presence of boundaries the answer is not clear, because of discrepancies in the boundary behavior – boundary layers – (see the work of Kato [18] and recent progress by Grenier [11]). The central problem of describing the inviscid limit is open in this case for both incompressible and compressible models.

## 2. EXISTENCE OF SOLUTIONS

As emphasized by many authors working on compressible fluid mechanics, vacuum is a major difficulty as far as existence of strong solutions for the Navier-Stokes equations is concerned. One way to avoid possibly vanishing densities is to consider an initial density close enough to a positive constant  $\bar{\rho}$ , an initial temperature close enough to a positive reference temperature  $\bar{T}$  and a small enough initial velocity field. In that case, global existence results can be derived for the full Navier-Stokes equations (1.1) (1.2) (1.3) since positive lower bounds on the density persist for all time. Moreover, uniqueness results can be obtained as long as enough regularity is assumed on the data. The one dimensional problem was first analyzed by Kazhikhov and Shelukhin [19] at the end of the '70s and the case  $N \geq 2$  by Matsumura and Nishida [34] [35], Valli and Zajackowski [45] in the '80s. One way to weaken the assumptions on the data is to allow discontinuous initial conditions, which is very important in the physical theory of nonequilibrium thermodynamics as well as in the mathematical theory of inviscid models for compressible fluids. This question has been studied by Serre [40] [41] in the one-dimensional case, and Hoff [14] [15] [16] for  $N \geq 2$ . In [14], Hoff explains why the effective viscous flux defined by  $F = (\lambda + 2\mu)\operatorname{div} \mathbf{u} - p(\rho, T) + p(\bar{\rho}, \bar{T})$  and the vorticity  $\nabla \times \mathbf{u}$  play a crucial role in the derivation of a priori bounds, since these two quantities gain some Sobolev regularity. Moreover, the effective viscous flux  $F$  reveals the underlying hyperbolic effect on the density, since global  $L^\infty$  bounds on  $\rho$  are obtained writing an evolution equation on  $\rho$  in terms of  $F$ . However, global existence of weak solutions for the full Navier-Stokes equations remains open for large initial conditions.

Another way to allow more general data has been investigated by Kazhikhov [20] [21], who considers barotropic simplified models at low Reynolds numbers, such as the potential flow or the Stokes approximation. In these two cases as well as in the case when the viscosity coefficient  $\lambda$  is a suitably increasing function of the density, he obtains global existence and uniqueness of strong solutions in dimension  $N = 2$  without assuming that the data are close to equilibrium, but still for initial density bounded away from zero.

For the  $N$ -dimensional system ( $N \geq 2$ ), the most natural functional spaces for the Cauchy problem are the energy spaces, which do not require positive

lower bounds on the density. In the case of barotropic fluids (1.1) (1.6) with a  $\gamma$ -type pressure law ( $\gamma > 1$ ) and when the external forces  $\mathbf{f}$  vanish, the energy estimate reads as

$$(2.1) \quad \int_{\Omega} \left( \frac{1}{2} \rho |\mathbf{u}|^2 + \frac{a\rho^\gamma}{\gamma-1} \right) (t) + \int_0^t ds \int_{\Omega} \left( \mu |\nabla \mathbf{u}|^2 + \xi (\operatorname{div} \mathbf{u})^2 \right) \\ \leq \int_{\Omega} \left( \frac{|\mathbf{m}_0|^2}{2\rho_0} + \frac{a\rho_0^\gamma}{\gamma-1} \right).$$

Following the methodology of Leray, weak solutions can be defined as finite energy solutions, namely,  $\rho \in L^\infty(0, T; L^\gamma(\Omega))$ ,  $\sqrt{\rho} \mathbf{u} \in L^\infty(0, T; L^2(\Omega))^N$ , and  $\mathbf{u} \in L^2(0, T; H_0^1(\Omega))^N$ . Assuming that the initial data satisfy

$$\rho_0 \in L^1(\Omega) \cap L^\gamma(\Omega), \quad \rho_0 \geq 0, \quad \mathbf{m}_0 \in L^{\frac{2\gamma}{\gamma+1}}(\Omega)^N \quad \text{and} \quad |\mathbf{m}_0|^2/\rho_0 \in L^1(\Omega),$$

Lions proved in 1993 [28] [31] a global existence and stability theorem for weak solutions  $(\rho, \mathbf{u})$ . Here, we agree that

$$|\mathbf{m}_0|^2/\rho_0 = 0 \quad \text{on} \quad \{\mathbf{x} \in \Omega : \rho_0(\mathbf{x}) = 0\}.$$

Assuming in addition that  $\gamma$  is large enough:  $\gamma \geq 3/2$  if  $N = 2$ ,  $\gamma \geq 9/5$  if  $N = 3$ ,  $\gamma > N/2$  if  $N \geq 4$ , he proved

**Theorem 1.** *There exists  $(\rho, \mathbf{u}) \in L^\infty(0, \infty; L^\gamma(\Omega)) \times L^2(0, \infty; H_0^1(\Omega))^N$  solution of (1.1) (1.6) satisfying in addition:  $\rho \in C([0, \infty); L^p(\Omega))$  if  $1 \leq p < \gamma$ ,  $\rho |\mathbf{u}|^2 \in L^\infty(0, \infty; L^1(\Omega))$ ,  $\rho \in L_{loc}^q([0, \infty); L^q(\Omega))$  for  $1 \leq q \leq \gamma - 1 + 2\gamma/N$ . Moreover, the energy inequality holds for almost all  $t \geq 0$ .*

Let us point out the gain of  $L^p$  regularity for  $t > 0$  on the density, although the system has hyperbolic features in  $\rho$ . Notice also that the same result holds with an external force  $\mathbf{f}$  which is integrable enough in space and time. Moreover, the more general model given by (1.1) (1.2) and the pure transport of entropy (1.5) can also be handled under the same restrictions on  $\gamma$  as soon as the initial entropy  $S_0$  satisfies  $S_0/\rho_0 \in L^\infty(\Omega)$ . However, such a result for the full Navier-Stokes problem (1.1) (1.2) (1.3) seems very difficult to handle, since the energy bounds are not sufficient to define a reasonable notion of weak solutions. Moreover, the range of  $\gamma$  for which Theorem 1 holds does not cover the physically interesting cases  $\gamma = 5/3$  and  $\gamma = 7/5$  corresponding respectively to monoatomic and diatomic gases in dimension  $N = 3$ . Notice however that in the stationary case, as well as in the case of time-discretized models, where all the quantities derived with respect to time are replaced by their finite difference approximations, existence of solutions for barotropic models are proven [31] for  $\gamma > 5/3$  in dimension  $N = 3$  and  $\gamma > N/2$  when  $N \neq 3$ .

## 3. REGULARITY PROBLEMS

We want now to focus on the regularity problem for the barotropic Navier-Stokes equations without smallness assumptions on the data. Starting from initial densities that have positive lower bounds, local existence of smooth solutions can be proven by classical means, since lower bounds on the density persist for small enough time. Considering suitably smooth initial density and velocity, Solonnikov proved in 1976 [42] a local existence and uniqueness theorem in any space dimension  $N \geq 1$ . For  $C^2$  pressure laws, a given  $q > N$ , external forces  $\mathbf{f} \in L^q((0, T) \times \Omega)^N$  for all  $T < +\infty$ , and initial data  $(\rho_0, \mathbf{u}_0)$  satisfying  $0 < m \leq \rho_0 \leq M < +\infty$ ,  $\rho_0 \in W^{1,q}(\Omega)$  and  $\mathbf{u}_0 \in W_0^{2-2/q,q}(\Omega)^N$ , he proved

**Theorem 2.** *There exists a positive time  $T_0 \in (0, \infty]$  and a unique solution  $(\rho, \mathbf{u})$  on  $[0, T_0)$  of (1.1) (1.6) such that for all  $t \in [0, T_0)$ , the density at time  $t$  is also bounded and bounded away from zero and for all  $T < T_0$ ,  $\rho \in L^\infty(0, T; W^{1,q}(\Omega))$ ,  $\partial_t \rho \in L^q((0, T) \times \Omega)$ ,  $0 < m(t) \leq \rho(t, \cdot) \leq M(t) < +\infty$ ,  $\mathbf{u} \in L^q(0, T; W_0^{2,q}(\Omega))^N$ , and  $\partial_t \mathbf{u} \in L^q((0, T) \times \Omega)^N$ .*

In [46], Weigant exhibits a counterexample with spherical symmetry: for  $\gamma \in [1, \frac{N}{N-1})$  and all  $q > N$ , there exist initial data  $(\rho_0, \mathbf{u}_0)$  and external forces  $\mathbf{f}$  satisfying the assumptions of Theorem 2 such that for some  $T_* < +\infty$ , the unique local smooth solution  $(\rho, \mathbf{u})$  blows up in  $L^\infty$  norm

$$\lim_{t \rightarrow T_*^-} |\rho(t, \cdot)|_{L^\infty(\mathbf{T}^N)} = +\infty.$$

The natural question is then to ask whether vacuum plays a role in the formation of singularities. Moreover, it can be asked conversely whether solutions remain smooth if the density  $\rho$  has  $L^\infty$  bounds. These two questions have been investigated by Desjardins in [6] in the bounded two-dimensional case. When  $\gamma > 1$ , it turns out that the maximum norm of the density  $|\rho(t, \cdot)|_{L^\infty}$  controls the breakdown of local solutions. In other words, as long as  $\rho$  is bounded in  $L^\infty((0, T_0) \times \Omega)$ , the solutions keep some smoothness expressed in terms of the effective viscous flux  $F$  and vorticity  $\nabla \times \mathbf{u}$ , no matter how much vacuum appears when the time evolves. Hence, vacuum does not yield additional singularities in two dimensions.

Let us emphasize that the counterexample by Weigant holds in the range  $\gamma \in [3/2, 2)$  in dimension  $N = 2$  for which weak solutions are known to exist for all time in view of Theorem 1. The question whether there exists a critical exponent  $q^* > 2\gamma - 1$  such that  $\rho$  blows up in  $L^{q^*}$  norm is a very challenging open problem in the two-dimensional case.

Finally, let us mention recent progress due to Xin [47], who gives sufficient conditions on the blow-up of smooth solutions to the full Navier-Stokes equations with initial density of compact support. As a consequence of this result, he proves that in the absence of heat conduction ( $k = 0$ ), solutions will blow up in finite time as soon as the initial density has compact support.

#### 4. SINGULAR LIMITS

We will now focus on the incompressible limit problem in this section. For a discussion on the physical motivations and on the mathematical setup, the reader is referred to [22] [23] [32] [33] and references therein. The question of the incompressible limit in fluid dynamics has received considerable attention. Some fundamental facts on this problem have been established by Klainerman and Majda in [22] [23] (see also [33]). The basic result, which has been proven in various contexts, is that *slightly compressible fluid flows are close to incompressible flows* even though the equations for the latter are related to those for the former via a singular limit. This justifies the use of the incompressible flow equations for certain real fluids that are actually slightly compressible (see [22] [23] [33] etc.). In particular, Klainerman and Majda ([22] [23]) studied the convergence of classical solutions of the compressible fluid equations to their incompressible limit as the Mach number becomes small. However, to insure that the solution is close to a solution of the incompressible equation, one has to *prepare* the initial data. The principle is the *initialization* one chooses the initial data so that a certain number of time derivatives at  $t = 0$  remains bounded independently of the compressibility.

In two dimensions and under periodic boundary conditions, the solution of the incompressible Navier-Stokes equations is known to remain smooth for all time. Hagstrom and Lorenz [12] showed a similar result in  $\mathbf{T}^2$  for the isentropic compressible Navier-Stokes equations if the Mach number is sufficiently small and the initial data are almost incompressible. It is not assumed that the initial data are small and the solution, to the leading order, consists of the corresponding incompressible flow plus a highly oscillatory part describing the acoustic waves.

For the ill-prepared case, the presence of rapidly oscillating waves makes the passage to the limit in the nonlinear term  $\rho_\epsilon \mathbf{u}_\epsilon \otimes \mathbf{u}_\epsilon$  rather difficult. The inviscid model was studied by Ukai [44] in the whole space. In the case of the Navier-Stokes equations, due to the lack of a priori estimates of the global weak solutions the limiting process for the pressure is much more involved and the oscillations on the initial data generate an additional pressure term at the incompressible limit (which may be called eddy pressure term). Adapting the method introduced by Schochet [37] and Grenier [10], P. L. Lions proved in [29]



and [32] that when the density becomes constant and the Mach number goes to zero the global weak solutions built in [28] [31] tend to the Leray-Hopf's global weak solutions of the incompressible Navier-Stokes equations. Various asymptotic results concerning the global weak solutions of the compressible isentropic Navier-Stokes equations are also obtained (see [29] and [32]).

The compressible isentropic Navier-Stokes equations in appropriate nondimensional form are defined by the equations:

$$(4.1) \quad \partial_t \rho_\epsilon + \operatorname{div} (\rho_\epsilon \mathbf{u}_\epsilon) = 0,$$

$$(4.2) \quad \partial_t (\rho_\epsilon \mathbf{u}_\epsilon) + \operatorname{div} (\rho_\epsilon \mathbf{u}_\epsilon \otimes \mathbf{u}_\epsilon) + \frac{a}{\epsilon^2} \nabla \rho_\epsilon^\gamma = \mu_\epsilon \Delta \mathbf{u}_\epsilon + \xi_\epsilon \nabla \operatorname{div} \mathbf{u}_\epsilon,$$

$$(4.3) \quad (\rho_\epsilon, \rho_\epsilon \mathbf{u}_\epsilon)|_{t=0} = (\rho_\epsilon^0, \mathbf{m}_\epsilon^0), \quad \rho_\epsilon^0 \geq 0.$$

On the other hand, the incompressible Navier-Stokes equations describing the evolution of the velocity field  $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$  of an ideal fluid over a given domain  $\Omega$  in  $\mathbf{R}^N$  are given by

$$(4.4) \quad \partial_t \mathbf{u} + \operatorname{div} (\mathbf{u} \otimes \mathbf{u}) - \mu \Delta \mathbf{u} + \nabla \pi = 0, \quad \operatorname{div} \mathbf{u} = 0, \quad \mathbf{u}|_{t=0} = \mathbf{u}_0(\mathbf{x}).$$

J. Leray published in 1934 his famous result establishing the existence of global weak solutions of (4.4) for arbitrary initial data with finite energy. It is not known if the initial data determine the Leray solutions uniquely. Thus, different constructive process procedure might conceivably lead to different solutions. The basic property of these weak solutions is that they satisfy the energy inequality

$$(4.5) \quad \int_\Omega \frac{1}{2} |\mathbf{u}|^2(t) + \int_0^t \int_\Omega \mu |\nabla \mathbf{u}|^2 \leq \int_\Omega \frac{1}{2} |\mathbf{u}_0|^2.$$

Let us consider a sequence of solutions  $(\rho_\epsilon, \mathbf{u}_\epsilon)$  of the compressible Navier-Stokes equations (4.1) (4.2) (4.3) with periodic boundary conditions and assume that

$$(4.6) \quad \begin{aligned} \rho_\epsilon &\in L^\infty(0, \infty; L^\gamma(\mathbf{T}^N)) \wedge C(0, \infty; L^p(\mathbf{T}^N)), \quad \forall 1 \leq p \leq \gamma, \\ \mathbf{u}_\epsilon &\in L^2(0, \infty; H^1(\mathbf{T}^N))^N, \quad \rho_\epsilon |\mathbf{u}_\epsilon|^2 \in L^\infty(0, \infty; L^1(\mathbf{T}^N)), \end{aligned}$$

and

$$(4.7) \quad \rho_\epsilon \mathbf{u}_\epsilon \in C([0, \infty); L^{2\gamma/(\gamma+1)}(\mathbf{T}^N) - w)^N,$$

i.e.,  $\rho_\epsilon \mathbf{u}_\epsilon$  is continuous with respect to  $t \geq 0$  with values in  $L^{2\gamma/(\gamma+1)}(\mathbf{T}^N)$  endowed with the weak topology. To obtain the incompressible limit the initial data are required to satisfy

$$\begin{aligned} \rho_\epsilon^0 &\in L^\gamma(\mathbf{T}^N), \quad \mathbf{m}_\epsilon^0 \in L^{2\gamma/(\gamma+1)}(\mathbf{T}^N), \quad \rho_0 |\mathbf{u}_\epsilon^0|^2 \in L^1(\mathbf{T}^N), \\ \mathbf{m}_\epsilon^0 &= 0 \text{ a.e. on } \{\rho_\epsilon^0 = 0\} \text{ and } \mathbf{u}_\epsilon^0 = 0 \text{ a.e. on } \{\rho_\epsilon^0 = 0\}. \end{aligned}$$

Furthermore, let us assume that  $\sqrt{\rho_\epsilon^0} \mathbf{u}_\epsilon^0$  converges weakly in  $L^2(\mathbf{T}^N)$  to some  $\tilde{\mathbf{u}}^0$  and

$$(4.8) \quad \int_{\mathbf{T}^N} \rho_\epsilon^0 |\mathbf{u}_\epsilon^0|^2 + \frac{1}{\epsilon^2} \int_{\mathbf{T}^N} \left( (\rho_\epsilon^0)^\gamma - \gamma \rho_\epsilon^0 (\bar{\rho}_\epsilon)^{\gamma-1} + (\gamma-1) (\bar{\rho}_\epsilon)^\gamma \right) \leq C,$$

where

$$\bar{\rho}_\epsilon = \frac{1}{(2\pi)^N} \int_{\mathbf{T}^N} \rho_\epsilon^0 \rightarrow 1.$$

Under the above assumptions, P.L. Lions proved in 1993 [29] and 1998 [32] the following convergence result concerning the incompressible limit of the Navier-Stokes equations.

**Theorem 3.** *Assume  $\gamma > \frac{N}{2}$ . Then  $\rho_\epsilon$  converges to 1 in  $C([0, T]; L^\gamma)$  and  $\mathbf{u}_\epsilon$  is bounded in  $L^2(0, T; H^1)$  for all  $T \in (0, \infty)$ . In addition, for any subsequence of  $\mathbf{u}_\epsilon$  (still denoted by  $\mathbf{u}_\epsilon$ ) weakly converging in  $L^2(0, T; H^1)$  ( $\forall T \in (0, \infty)$ ) to some  $\mathbf{u}$ ,  $\mathbf{u}$  is a solution of the incompressible Navier-Stokes equation corresponding to the initial condition  $\mathbf{u}^0 = P\tilde{\mathbf{u}}^0$ , where  $P$  is the orthogonal projection onto incompressible vector fields.*

Notice that similar results hold in the whole space case and in bounded domains if the boundary conditions are slightly modified [32]. Indeed, the same analysis cannot be carried out for Dirichlet boundary conditions. Let us also observe that Theorem 3 assumes the existence of global weak solutions to the Navier-Stokes equations, which is provided by Theorem 2, at least for large enough  $\gamma$ .

In many respects, Lions' global existence theory of the isentropic compressible Navier-Stokes equations is analogous to the Leray's theory for the incompressible Navier-Stokes equations. This theorem shows how the Leray solutions can be understood as an appropriate singular limit (incompressible limit) of a sequence of Lions' solutions.

Majda [33] has drawn attention to the importance of understanding both viscous hydrodynamics and the effects of thermal conduction in the nearly incompressible limit. P.L. Lions also proved in [31] the more general model given by (1.1) (1.2) and the pure transport of entropy (1.5) under the same restrictions on  $\gamma$ , as soon as the initial entropy  $S_0$  satisfies  $S_0/\rho_0 \in L^\infty(\Omega)$ .

Let us then consider the following system of nondimensional equation in a periodic domain  $\mathbf{T}^N$ :

$$(4.9) \quad \partial_t \rho_\epsilon + \operatorname{div}(\rho_\epsilon \mathbf{u}_\epsilon) = 0, \quad \partial_t(\rho_\epsilon a_\epsilon) + \operatorname{div}(\rho_\epsilon a_\epsilon \mathbf{u}_\epsilon) = 0,$$

$$(4.10) \quad \partial_t(\rho_\epsilon \mathbf{u}_\epsilon) + \operatorname{div}(\rho_\epsilon \mathbf{u}_\epsilon \otimes \mathbf{u}_\epsilon) + \frac{1}{\epsilon^2} \nabla(\rho_\epsilon a_\epsilon)^\gamma = \mu_\epsilon \Delta \mathbf{u}_\epsilon + \xi_\epsilon \nabla \operatorname{div} \mathbf{u}_\epsilon,$$

$$(4.11) \quad \begin{aligned} (\rho_\epsilon, \rho_\epsilon a_\epsilon, \rho_\epsilon \mathbf{u}_\epsilon)|_{t=0} &= (\rho_\epsilon^0, \rho_\epsilon^0 a_\epsilon^0, \mathbf{m}_\epsilon^0), \\ \rho_\epsilon^0 &\geq 0, \quad 0 < a_{\min} \leq a_\epsilon^0 \leq a_{\max} < +\infty. \end{aligned}$$

In addition, we assume that the viscosity coefficients  $\xi_\epsilon = 0$ ,  $\mathbf{u}_\epsilon = \mathbf{u}$  is a fixed constant and

$$\frac{1}{(2\pi)^N} \int_{\mathbf{T}^N} \rho_\epsilon^0 a_\epsilon^0 d\mathbf{x} = 1 \quad \forall \epsilon > 0, \quad |a_\epsilon^0 - \bar{a}|_{L^\infty(\mathbf{T}^N)} \leq C\epsilon,$$

$$\frac{1}{\epsilon^2} \left( (a_\epsilon^0 \rho_\epsilon^0)^\gamma - 1 - \gamma (a_\epsilon^0 \rho_\epsilon^0 - 1) \right) \quad \text{is bounded in } L^1(\mathbf{T}^N) \text{ uniformly in } \epsilon.$$

In this case the energy inequality reads as

$$(4.12) \quad \begin{aligned} & \int_{\mathbf{T}^N} \left( \frac{1}{2} \rho_\epsilon |\mathbf{u}_\epsilon|^2 + \frac{(a_\epsilon \rho_\epsilon)^\gamma - 1 - \gamma(a_\epsilon \rho_\epsilon - 1)}{\epsilon^2(\gamma - 1)} \right) d\mathbf{x} + \int_0^t \int_{\mathbf{T}^N} \mu |\nabla \mathbf{u}_\epsilon|^2 dx ds \\ & \leq \int_{\mathbf{T}^N} \left( \frac{1}{2} \rho_\epsilon^0 |\mathbf{u}_\epsilon^0|^2 + \frac{(a_\epsilon^0 \rho_\epsilon^0)^\gamma - 1 - \gamma(a_\epsilon^0 \rho_\epsilon^0 - 1)}{\epsilon^2(\gamma - 1)} \right) d\mathbf{x}. \end{aligned}$$

The density variations in real fluids are related to both pressure and entropy variations, even in the incompressible limit. The behavior of density variations depends crucially on the relative size of the pressure, temperature, and entropy fluctuations. Indeed, for this model we have

**Theorem 4.** *Assume either that  $\gamma$  is large enough as in Theorem 1, or  $\gamma > \frac{N}{2}$  and weak solutions exist for all time. Then  $\rho_\epsilon$  converges to  $\bar{\rho} = \bar{a}^{-1}$  in  $C([0, T]; L^\gamma(\mathbf{T}^N))$  and  $\mathbf{u}_\epsilon$  is bounded in  $L^2(0, T; H^1(\mathbf{T}^N))$  for all  $T \in (0, \infty)$ . In addition, for any subsequence of  $\mathbf{u}_\epsilon$  (still denoted by  $\mathbf{u}_\epsilon$ ) weakly converging in  $L^2(0, T; H^1(\mathbf{T}^N))$  ( $\forall T \in (0, \infty)$ ) to some  $\bar{\mathbf{u}}$ , the weak limit  $\bar{\mathbf{u}}$  satisfies*

$$(4.13) \quad \partial_t(\bar{\rho} \bar{\mathbf{u}}) + \operatorname{div}(\bar{\rho} \bar{\mathbf{u}} \otimes \bar{\mathbf{u}}) + \nabla \pi = \mu \Delta \bar{\mathbf{u}}, \quad \operatorname{div} \bar{\mathbf{u}} = 0, \quad \bar{\mathbf{u}}|_{t=0} = P\bar{\mathbf{u}}.$$

Moreover, if in addition  $\sigma_\epsilon^0 = (a_\epsilon^0 - \bar{a})/\epsilon$  converges strongly in  $L^1(\mathbf{T}^N)$  to some  $\bar{\sigma}_0$ , then  $\sigma_\epsilon \equiv (a_\epsilon - \bar{a})/\epsilon$  converges to some  $\bar{\sigma}$  in  $C([0, T]; L^q(\mathbf{T}^N)) \quad \forall q < \infty$ , where  $\bar{\sigma}$  is the unique solution of the transport equation

$$(4.14) \quad \partial_t \bar{\sigma} + \operatorname{div}(\bar{\sigma} \bar{\mathbf{u}}) = 0, \quad \bar{\sigma}|_{t=0} = \bar{\sigma}_0.$$

The details of the theorem and its proof are given by Desjardins and C.-K. Lin in [7]. Finally let us mention some interesting open problems. For sufficiently small initial data, Nishida and Matsumura [34] [35] have proved the

global existence of classical solutions for the full compressible Navier-Stokes equations at a fixed Mach number. Recently, Hoff [16] proved the global existence of weak solutions to the Navier-Stokes equations for compressible, heat-conducting flow when the initial density and temperature are close to constants. The global convergence as the Mach number tends to zero to classical or weak solutions of the incompressible Navier-Stokes equations is a very challenging problem. Moreover, large fluctuations of entropy seem to yield at the limit nonhomogeneous incompressible models, but this question is still far from being understood. Besides the discussion of incompressible limits above there are other very interesting physical systems of equations with a myriad of singular limits. The equations of meteorology and magnetohydrodynamics are typical examples.

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